Choquet property for the Sobolev capacity in metric spaces

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ABSTRACT. We discuss definitions of first order Sobolev spaces and related capacities on a metric measure space. We show that the natural Sobolev capacity is a Choquet capacity.

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1. Introduction

There are several seemingly different definitions available for first order Sobolev spaces in a metric measure space, see for example [Che], [H], [KoM] and [N]. The definition of [H] is based on a maximal derivative approach, whence [Che], [KoM] and [N] use an upper gradient definition. In general these definitions lead to different spaces. This is the case, for example, if there are too few rectifiable curves in the space. However, these definitions coincide under certain conditions, see [N]. For every space there is a capacity which is the natural measure for exceptional sets for Sobolev functions. The rudiments of capacity theory on metric measure spaces were established in [KiM]. We complete the picture here by showing that the Sobolev capacity is a Choquet capacity. Hence the capacity of an arbitrary Borel set can be estimated by capacities of compact sets from inside and by capacities open sets from outside.

There are somewhat unexpected difficulties in proving the Choquet property in a metric measure space. Classically it is obtained as a consequence of strong subadditivity of the capacity or reflexivity of the Sobolev space, see for example Theorem 2 on page 151 of [EG] and Theorem 2.3 in [MZ]. We use neither of these properties here. A recent result of [Che] shows that an upper gradient definition implies reflexivity. Moreover capacities related to the upper gradient definitions satisfy the strong subadditivity property. Hence in this case the Choquet property follows easily from the classical arguments. However, if we consider the maximal derivative definition, then we do not have reflexivity nor strong subadditivity available. The two fundamental problems are that the operation of taking the maximal derivative is not linear and that the maximal derivative need not vanish on the set where the function is constant. We overcome these obstacles by a direct method which is based on the definitions.

2. Sobolev space

In this section we recall the definition due to Hajłasz of the first order Sobolev spaces on an arbitrary metric space. The details can be found in [H]. Let (X,d) be a metric space and let μ be a non-negative Borel regular outer measure on X. In the following, we keep the triple (X,d,μ) fixed, and for short, we denote it by X. For $1<p<\infty$, $L^p(X)$ is the Banach space of all μ-a.e. defined μ-measurable functions $u\colon X\to[-\infty,\infty]$ for which the norm

$$\|u\|_{L^p(X)}=\Big(\int_X |u|^p\,d\mu\Big)^{1/p}$$

is finite. Suppose that $u\colon X\to[-\infty,\infty]$ is μ-measurable. We denote by $D(u)$ the set of all μ-measurable functions $g\colon X\to[0,\infty]$ such that

$$|u(x)-u(y)|\leqslant d(x,y)\big(g(x)+g(y)\big) \tag{2.1}$$

for every $x, y \in X \setminus N$, $x \neq y$, with $\mu(N) = 0$. The space $L^{1,p}(X)$ consists of all μ-measurable functions u with $D(u) \cap L^p(X) \neq \emptyset$; the space $L^{1,p}(X)$ is endowed with the seminorm

$$\|u\|_{L^{1,p}(X)} = \inf\big\{\|g\|_{L^p(X)} \colon g \in D(u) \cap L^p(X)\big\}. \tag{2.2}$$

An application of the uniform convexity of $L^p(X)$ implies that there is a unique minimizer of (2.2). The *Sobolev space* is

$$M^{1,p}(X) = L^p(X) \cap L^{1,p}(X)$$

equipped with the norm

$$\|u\|_{M^{1,p}(X)} = \big(\|u\|^p_{L^p(X)} + \|u\|^p_{L^{1,p}(X)}\big)^{1/p}.$$

Then the Sobolev space $M^{1,p}(X)$ is a Banach space.

If $X = \mathbf{R}^n$ with the Euclidean metric and the Lebesgue measure, then

$$M^{1,p}(\mathbf{R}^n) = W^{1,p}(\mathbf{R}^n), \qquad 1 < p < \infty.$$

Moreover, the norms are comparable (see [H]). Here $W^{1,p}(\mathbf{R}^n)$ is the classical Sobolev space, that is, the space of functions in $L^p(\mathbf{R}^n)$ whose first distributional derivatives belong to $L^p(\mathbf{R}^n)$ with the norm

$$\|u\|_{W^{1,p}(\mathbf{R}^n)} = \big(\|u\|^p_{L^p(\mathbf{R}^n)} + \|Du\|^p_{L^p(\mathbf{R}^n)}\big)^{1/p}.$$

The following simple lemma is useful in studying the capacity.

2.3. LEMMA. *Suppose that u_i, $i = 1, 2, \dots$, are μ-measurable functions, let $g_i \in D(u_i)$, $i = 1, 2, \dots$, and denote $g = \sup_i g_i$ and $u = \sup_i u_i$. Then $g \in D(u)$ provided $u < \infty$ μ-a.e.*

PROOF. Let $x, y \in X \setminus N$ with $u(y) \leqslant u(x) < \infty$. Here N is the union of exceptional sets for the functions u_i as in (2.1). Let $\varepsilon > 0$ and choose i such that $u(x) < u_i(x) + \varepsilon$. Since $u(y) \geqslant u_i(y)$, we obtain

$$\begin{aligned} |u(x) - u(y)| &= u(x) - u(y) \leqslant u_i(x) + \varepsilon - u_i(y) \\ &\leqslant d(x,y)\big(g_i(x) + g_i(y)\big) + \varepsilon \leqslant d(x,y)\big(g(x) + g(y)\big) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the result. □

3. Capacity

There is a natural capacity in the Sobolev space; this was studied in [KiM]. For $1 < p < \infty$, the *Sobolev p-capacity* of the set $E \subset X$ is the number

$$\mathrm{C}_p(E) = \inf\big\{\|u\|^p_{M^{1,p}(X)} \colon u \in \mathcal{A}(E)\big\},$$

where

$$\mathcal{A}(E) = \big\{u \in M^{1,p}(X) : u \geqslant 1 \text{ on a neighbourhood of } E\big\}.$$

If $\mathcal{A}(E) = \emptyset$, we set $\mathrm{C}_p(E) = \infty$. Functions belonging to $\mathcal{A}(E)$ are called *admissible* functions for E. Since $M^{1,p}(X)$ is closed under truncation, and truncation does not increase the Sobolev norm, we may restrict ourselves to those admissible functions u for which $0 \leqslant u \leqslant 1$. The Sobolev capacity enjoys the following properties:

(3.1) $\mathrm{C}_p(\emptyset) = 0$.

(3.2) If $E_1 \subset E_2$, then $\mathrm{C}_p(E_1) \subset \mathrm{C}_p(E_2)$ (monotonicity).

(3.3) If $E_i \subset X$, $i = 1, 2, \dots$, then

$$\mathrm{C}_p\Big(\bigcup_{i=1}^\infty E_i\Big) \leqslant \sum_{i=1}^\infty \mathrm{C}_p(E_i)$$

(subadditivity).

(3.4) $\mathrm{C}_p(E) = \inf\big\{\mathrm{C}_p(O) \colon O \supset E \text{ open}\big\}$ (C_p is an outer capacity).

(3.5) If C_i, $i = 1, 2, \dots$, are compact sets so that $X \supset C_1 \supset C_2 \supset \dots$, then

$$\mathrm{C}_p\Big(\bigcap_{i=1}^\infty C_i\Big) = \lim_{i \to \infty} \mathrm{C}_p(C_i).$$

(3.6) If O_i, $i = 1, 2, \dots$, are open sets so that $O_1 \subset O_2 \subset \ldots \subset X$, then

$$\mathrm{C}_p\Big(\bigcup_{i=1}^{\infty} O_i\Big) = \lim_{i\to\infty} \mathrm{C}_p(O_i).$$

For the proofs, we refer to [KiM].

4. Choquet property

Finally we show that the Sobolev capacity is a Choquet capacity. To prove this we need a version of (3.6) for arbitrary sets.

4.1. THEOREM. *If E_i, $i = 1, 2, \dots$, are sets so that $E_1 \subset E_2 \subset \cdots \subset X$, then*

$$\mathrm{C}_p\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{i\to\infty} \mathrm{C}_p(E_i). \tag{4.2}$$

PROOF. Let $E = \bigcup_{i=1}^{\infty} E_i$. Monotonicity yields $\lim_{i\to\infty} \mathrm{C}_p(E_i) \leqslant \mathrm{C}_p(E)$. To prove the opposite inequality, we may assume that $\lim_{i\to\infty} \mathrm{C}_p(E_i) < \infty$. Let $\varepsilon > 0$, $u_i \in \mathcal{A}(E_i)$, $0 \leqslant u_i \leqslant 1$, $i = 1, 2, \dots$, and $g_{u_i} \in D(u_i) \cap L^p(X)$ be such that

$$\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)} \leqslant \mathrm{C}_p(E_i) + \varepsilon.$$

Now (u_i) and (g_{u_i}) are bounded sequences in $L^p(X)$ and hence there are subsequences, which we denote again by (u_i) and (g_{u_i}), such that $u_i \to u$ weakly in $L^p(X)$ and $g_{u_i} \to g$ weakly in $L^p(X)$ as $i \to \infty$. Using the Mazur lemma we obtain a sequence (v_i) of convex combinations of u_i's and $g_{v_i} \in D(v_i) \cap L^p(X)$ such that $v_i \in \mathcal{A}(E_i)$, $v_i \to u$ in $L^p(X)$ and μ-a.e. and $g_{v_i} \to g$ in $L^p(X)$. This sequence can be found in the following way. Let i_0 be fixed. Since every subsequence of (u_i) converges to u weakly in $L^p(X)$, we may use the Mazur lemma for the subsequence $(u_i)_{i \geqslant i_0}$. We obtain a convex combination of finitely many u_i's as close to u as we want in $L^p(X)$. For every $i = i_0, i_0+1, \dots$, there is an open set O_i such that $u_i = 1$ μ-a.e. in O_i. Then the intersection of finitely many O_i's is an open neighbourhood of E_{i_0} and the obtained convex combination equals one μ-a.e. in this neighbourhood. Passing to subsequences, if necessary, we may assume that for every $i = 1, 2, \dots$ we have

$$\|v_{i+1} - v_i\|^p_{L^p(X)} \leqslant 2^{-i} \qquad \text{and} \qquad \|g_{v_{i+1}} - g_{v_i}\|^p_{L^p(X)} \leqslant 2^{-i}.$$

For $j = 1, 2, \dots$ set

$$w_j = \sup_{i \geqslant j} v_i.$$

Then

$$w_j \leqslant v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i|$$

for every $j = 1, 2, \dots$, and this implies that $w_j \in L^p(X)$. Since $w_j < \infty$ μ-a.e. by Lemma 2.3 we have

$$g_{w_j} = \sup_{i \geqslant j} g_{v_i} \in D(w_j)$$

for every $j = 1, 2, \dots$. Let

$$g_j = g_{v_j} + \sum_{i=j}^{\infty} |g_{v_{i+1}} - g_{v_i}|$$

for $j = 1, 2, \dots$ Clearly $g_{w_j} \leqslant g_j$ and $g_j \in L^p(X)$ for all $j = 1, 2, \dots$ from which we conclude that $g_j \in D(w_j) \cap L^p(X)$. On the other hand, it is easy to see that $w_j = 1$ μ-a.e. in a neighbourhood of E. This shows that $w_j \in \mathcal{A}(E)$ and hence

$$\mathrm{C}_p(E) \leqslant \|w_j\|^p_{L^p(X)} + \|g_j\|^p_{L^p(X)}.$$

Since this holds for every j, we have

$$\mathrm{C}_p(E) \leqslant \liminf_{j\to\infty} \big(\|w_j\|^p_{L^p(X)} + \|g_j\|^p_{L^p(X)}\big).$$

Since w_j is a decreasing sequence converging to u μ-a.e., the dominated convergence theorem implies that $\|w_j\|_{L^p(X)} \to \|u\|_{L^p(X)}$. On the other hand, we see that $\|g_j\|_{L^p(X)} \to \|g\|_{L^p(X)}$ and hence

$$\mathrm{C}_p(E) \leqslant \|u\|^p_{L^p(X)} + \|g\|^p_{L^p(X)}.$$

Since $u_i \to u$ and $g_i \to g$ weakly in $L^p(X)$, the weak lower semicontinuity of norms implies

$$\|u\|^p_{L^p(X)} + \|g\|^p_{L^p(X)} \leqslant \liminf_{i\to\infty} \left(\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)}\right) \leqslant \lim_{i\to\infty} \mathrm{C}_p(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$ we complete the proof. □

A set function satisfying properties (3.1)–(3.5) and (4.2) in Theorem 4.1 is called a *Choquet capacity.* Hence we have shown that C_p is a Choquet capacity. A set $E \subset X$ is *capacitable* if

$$\begin{aligned}\mathrm{C}_p(E) &= \sup \{ \mathrm{C}_p(C) \colon C \subset E,\, C \text{ compact}\} \\ &= \inf \{ \mathrm{C}_p(O) \colon O \supset E,\, O \text{ open}\}.\end{aligned}$$

All K-analytic sets are capacitable by Choquet's theorem [Cho]. In particular all K-Borel sets are capacitable. Recall that the σ-algebra of K-Borel sets in X is the smallest σ-algebra that contains all compact subsets of X.

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