CHARACTERIZATIONS FOR HARDY'S INEQUALITY

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1. Introduction

We discuss necessary and sufficient conditions for validity of the following multidimensional version of Hardy's inequality. Let Ω is a open subset of $\mathbb{R}^n$ and $1 < p < \infty$. We say that p-Hardy's inequality holds in Ω, if there is a uniform constant c_H such that

$$\int_\Omega \left(\frac{|u(x)|}{\delta(x)} \right)^p dx \le c_H \int_\Omega |\nabla u(x)|^p \, dx \tag{1.1}$$

for all $u \in W_0^{1,p}(\Omega)$, where we denote

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

By a density argument, it is enough to consider (1.1) for compactly supported smooth functions $u \in C_0^\infty(\Omega)$.

Several sufficient Lipschitz and Hölder type boundary conditions for which Hardy's inequality holds have been given by Nečas [41], Kufner and Opic [26], [42]. Maz'ya has given capacitary characterizations of Hardy's inequality in Chapter 2 of [38]. We return to this in Section 2. Ancona [2] (for $p = 2$, $n \ge 2$), Lewis [31], and Wannebo [49] (for $p \ge 1$, $n \ge 2$) proved that Hardy's inequality holds under the assumption that the complement of Ω satisfies a uniform capacity density condition

$$\operatorname{cap}_p\big((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x,r), B(x,2r)\big) \ge c_T \operatorname{cap}_p\big(\overline{B}(x,r), B(x,2r)\big) \tag{1.2}$$

for every $x \in \mathbb{R}^n \setminus \Omega$ and for all radii $r > 0$. The definition and properties of variational capacity can be found in Chapter 2 of [38] or Chapter 2 of [16]. If (1.2) holds we say that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick (or p-fat).

The class of open sets whose complement satisfies the uniform capacity density condition is relatively large. Every nonempty $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick for $p > n$, and hence the condition is nontrivial only when $p \le n$. In particular, this implies that when $p > n$ Hardy's inequality holds for every proper open subset of $\mathbb{R}^n$. Hence we only consider the case $1 < p \le n$ here.

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The capacity density condition has several applications in the theory of partial differential equations. It is stronger than the Wiener criterion

$$\int_0^1 \left(\frac{\operatorname{cap}_p\big((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x,r), B(x,2r)\big)}{\operatorname{cap}_p\big(\overline{B}(x,r), B(x,2r)\big)} \right)^{1/(p-1)} \frac{dr}{r} = \infty,$$

which characterizes regular boundary points for the Dirichlet problem for the p-Laplace equation. The sufficiency was shown by Maz'ya in [36], necessity for $p > n - 1$ by Lindqvist and Martio in [32] and for $1 < p \leq n$ by Kilpeläinen and Malý in [19].

In [13] Hajłasz showed that the capacity density condition is sufficient for a pointwise version of Hardy's inequality in terms of the Hardy-Littlewood maximal function. Similar result has been also obtained in [21]. Recently in [28] Lehrbäck showed that the pointwise Hardy's inequality is equivalent to uniform thickness of the complement. See also [22]. In this work we also discuss a characterization through a boundary Poincaré inequality.

In the bordeline case $p = n$, there are several characterizations of Hardy's inequality. In this case, rather surprisingly, certain analytic, metric and geometric conditions turn out to be equivalent. Ancona proved in [2] that the uniform p-thickness is also necessary for the validity of Hardy's inequality when $p = n = 2$ and in [31] Lewis generalized this result for $p = n \geq 2$. On the other hand, Sugawa proved in [46] that, when $p = n = 2$, Hardy's inequality is equivalent to the uniform perfectness of the complement. Recently this result has been generalized in [23] for other values of n. We outline the main points of the argument in this work and discuss other characterizations of Hardy's inequalities in the borderline case.

The following variational problem is naturally related to p-Hardy's inequality. Consider the Rayleigh quotient

$$\lambda_p = \lambda_p(\Omega) = \inf \frac{\displaystyle\int_\Omega |\nabla u(x)|^p \, dx}{\displaystyle\int_\Omega \left(\frac{|u(x)|}{\delta(x)} \right)^p dx}, \tag{1.3}$$

where the infimum is taken over all $u \in W_0^{1,p}(\Omega)$. Observe, that $\lambda_p(\Omega) > 0$ if and only if p-Hardy's inequality holds in Ω.

This approach has obtained lot of attention, see, for example, Barbatis, Filippas and Tertikas [3], [4], [5], Brezis, Marcus, Mizel, Pinchover and Shafrir [7], [8], [33], [34], Davies [12], Matskewich and Sobolevskii [35], Pichover and Tintarev [43], [44], [45] and Tidblom [47], [48].

In the one-dimensional case, already Hardy observed that

$$\lambda_p = \left(1 - \frac{1}{p}\right)^p$$

and that the minimum is not attained in (1.3), see [14] and [15]. In higher dimensions, the constant λ_p generally depends on p and on the domain Ω.

Note that $u \in W_0^{1,p}(\Omega)$ is a minimizer of (1.3) if and only if it is a weak solution to the following nonlinear eigenvalue problem

$$\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + \lambda_p \frac{|u(x)|^{p-2}u(x)}{\delta(x)^p} = 0. \tag{1.4}$$

In [2] Ancona charaterized Hardy's inequality for $p = 2$ with supersolutions called strong barriers of (1.4). In this work we generalize this characterization for other values of p.

We also consider the self-improving phenomena related to Hardy's inequalities. It is not difficult to see, that if $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick, then it is uniformly q-thick for every $q > p$ as well. Lewis showed in [31] that p-thickness has a deep self-improving property: p-thickness implies the same condition for some smaller value of p. For another proof, we refer to Mikkonen [40].

Hardy's inequality is self-improving as well. Indeed, Koskela and Zhong showed in [25] that if Hardy's inequality holds for some value of p, then it also holds for other values of p that are close enough. In contrast with the capacity density condition, it may happen that Hardy's inequality fails for some particular values of p. Indeed, for a punctured ball p-Hardy's inequality holds when $p \neq n$ and it does not hold when $p = n$. More generally, it has been shown in [25] that Hardy's inequality cannot hold if the boundary contains $(n - p)$-dimensional parts. Roughly speaking Hardy's inequality may hold if the complement of the domain is either large or small in the neighbourhood of each boudary point.

Many arguments related to Hardy's inequality are based on general principles and some of them apply even on metric measure spaces, see Björn, MacManus and Shanmugalingam [6], Kinnunen, Kilpeläinen and Martio [18], Korte and Shanmugalingam [23] and Korte, Lehrbäck and Tuominen [22]. In [11] Danielli, Garofalo and Phuc studied Hardy's inequalities in Carnot-Carathéodory spaces. We refer to Buckley and Koskela [9] and Chianci [10] for studies relevant to Orlicz-Sobolev spaces.

2. Maz'ya type characterization

In this section, we present a characterization of Hardy's inequality in terms of inequalities connecting measures and capacities. To be on the safe side, we recall the definition of the variational capacity here. Let Ω be an open subset of $\mathbb{R}^n$ and let K be a compact subset of Ω. The variational p-capacity of K with respect to Ω is defined to be

$$\operatorname{cap}_p(K,\Omega) = \inf \int_\Omega |\nabla u(x)|^p \, dx,$$

where the infimum is taken over all $u \in C_0^\infty(\Omega)$ such that $u(x) \geq 1$ for every $x \in K$. The same quantity is obtained, if instead of smooth functions the infimum is taken, for example, over compactly supported continuous functions in $W_0^{1,p}(\Omega)$.

The proof the following result is based on an elegant truncation argument that can be found on page 110 of Maz'ya's monograph [38]. For more information about this kind of characterizations, we refer to Chapter 2 in [38] and [39]. For generalizatons, see [37] and [20].

Theorem 2.1. *An open set Ω satisfies p-Hardy's inequality if and only if there is a constant c_M such that*

$$\int_K \delta(x)^{-p} \, dx \leq c_M \operatorname{cap}_p(K,\Omega) \tag{2.2}$$

for every compact subset K of Ω.

Proof. First assume that p-Hardy's inequality holds in Ω. Let $u \in C_0^\infty(\Omega)$ such that $u(x) \geq 1$ for every $x \in K$. By (1.1), we have

$$\int_K \delta(x)^{-p} \, dx \leq \int_\Omega \left(\frac{|u(x)|}{\delta(x)} \right)^p dx \leq c_H \int_\Omega |\nabla u(x)|^p \, dx,$$

and by taking infimum over all such functions u, we obtain (2.2) with $c_M = c_H$.

Then assume that (2.2) holds. By a density argument it is enough to prove (1.1) for compactly supported smooth functions in Ω. Let $u \in C_0^\infty(\Omega)$ and for $k \in \mathbb{Z}$ denote

$$E_k = \{x \in \Omega : |u(x)| > 2^k\}.$$

By (2.2), we have

$$\begin{aligned}\int_\Omega \left(\frac{|u(x)|}{\delta(x)}\right)^p dx &\le \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \int_{E_k\setminus E_{k+1}} \delta(x)^{-p}\,dx\\ &\le c_M \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \operatorname{cap}_p(\overline{E}_k,\Omega)\\ &\le c_M 2^p \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \operatorname{cap}_p(\overline{E}_{k+1},E_k).\end{aligned}$$

Define $u_k : \Omega \to [0,1]$ by

$$u_k(x) = \begin{cases} 1, & \text{if} \quad |u(x)| \ge 2^{k+1},\\ \dfrac{|u(x)|}{2^k} - 1, & \text{if} \quad 2^k < |u(x)| < 2^{k+1},\\ 0, & \text{if} \quad |u(x)| \le 2^k. \end{cases}$$

Then $u_k \in W_0^{1,p}(\Omega)$ is a continuous function, $u_k = 1$ in $\overline{E}_{k+1}$ and $u_k = 0$ in $\mathbb{R}^n \setminus E_k$. Therefore, we may apply it as a test function for the capacity and obtain

$$\operatorname{cap}_p(\overline{E}_{k+1},E_k) \le \int_{E_k\setminus E_{k+1}} |\nabla u_k(x)|^p\,dx \le 2^{-pk} \int_{E_k\setminus E_{k+1}} |\nabla u(x)|^p\,dx.$$

Consequently, we arrive at

$$\begin{aligned}\sum_{k=-\infty}^{\infty} 2^{(k+1)p} \operatorname{cap}_p(\overline{E}_{k+1},E_k) &\le 2^p \sum_{k=-\infty}^{\infty} \int_{E_k\setminus E_{k+1}} |\nabla u(x)|^p\,dx\\ &= 2^p \int_\Omega |\nabla u(x)|^p\,dx,\end{aligned}$$

and the claim follows with $c_H = 2^{2p} c_M$. □

Remark 2.3. A result of Koskela and Zhong [25] shows that Hardy's inequality is an open ended condition in the following sense: If Hardy's inequality holds in Ω for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that Hardy's inequality holds in Ω for every q with $p-\varepsilon < q < p+\varepsilon$. For a weighted result, see [24]. This implies that the Maz'ya type condition (2.2) is an open ended condition as well. Indeed, if (2.2) holds, then there are $c > 0$ and $\varepsilon > 0$ for which

$$\int_K \delta(x)^{-q}\,dx \le c \operatorname{cap}_q(K,\Omega)$$

for all q with $p - \varepsilon < q < p + \varepsilon$.

3. The capacity density condition

In this section we consider a sufficient condition for Hardy's inequality in terms of uniform thickness of the complement, see (1.2). The capacity density condition has a deep self-improving property, which is essential in many questions. The following result is due to Lewis (Theorem 1 in [31]). See also Ancona [2] and Section 8 of Mikkonen [40].

Theorem 3.1. *If $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick, then there is $q < p$ such that $\mathbb{R}^n \setminus \Omega$ is uniformly q-thick.*

Assume that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick, let $u \in C_0^\infty(\Omega)$ and denote

$$A = \big\{y \in B(x,r) : u(y) = 0\big\}.$$

By a capacitary version of a Poincaré type inequality, there is $c = c(n,p)$ such that

$$\begin{aligned}
&\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|^p\,dy\Big)^{1/p} \\
&\le \left(\frac{c}{\operatorname{cap}_p\big(A\cap \overline{B}(x,r), B(x,2r)\big)}\int_{B(x,r)} |\nabla u(y)|^p\,dy\right)^{1/p} \qquad (3.2)\\
&\le c\,r\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |\nabla u(y)|^p\,dy\right)^{1/p}.
\end{aligned}$$

For a systematic study of such inequalities we refer to Chapter 10 in Maz'ya's monograph [38]. See also Chapter 8 in [1]. Here we also used the fact that

$$\operatorname{cap}_p\big(\overline{B}(x,r), B(x,2r)\big) = c\,r^{n-p},$$

where $c = c(n,p)$, see 2.2.4 in [38].

Inequality (3.2) implies the pointwise estimate

$$|u(x)| \le c\,\delta(x)\big(M_\Omega(|\nabla u|^p)(x)\big)^{1/p}, \qquad (3.3)$$

for every $x \in \Omega$ with $c = c(n,p)$. The restricted Hardy-Littlewood maximal function is defined as

$$M_\Omega f(x) = \sup \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy,$$

where the supremum is taken over radii $r > 0$ for which $r \le 2\delta(x)$. This kind of pointwise Hardy's inequalities have been considered in [13] and [21]. See also [11] and [24]. By Theorem 3.1, we have pointwise Hardy's inequality (3.3) also for some $q < p$. Integrating this inequality over Ω and using the maximal function theorem we arrive at

$$\int_\Omega \Big(\frac{|u(x)|}{\delta(x)}\Big)^p\,dx \le c\int_\Omega \big(M_\Omega(|\nabla u|^q)(x)\big)^{p/q}\,dx \le c\int_\Omega |\nabla u(x)|^p\,dx$$

for every $u \in C_0^\infty(\Omega)$ with $c = c(n,p,q)$.

This proof relies heavily on a rather deep Theorem 3.1. Wannebo has given a more direct proof of Hardy's inequality if the complement of the domain is uniformly thick in [49]. See also [50], [51] and [52]. The idea in Wannebo's proof is to first use a Poincaré type inequality (3.8) and a Whitney type covering argument to show that

$$\int_{\{x\in\Omega:2^{-k-1}<\delta(x)<2^{-k}\}} |u(x)|^p\,dx \le c2^{-kp}\int_{\{x\in\Omega:\delta(x)\le 2^{-k+1}\}} |\nabla u(x)|^p\,dx$$

for every $k \in \mathbb{Z}$. Multiplying both sides by $\delta(x)^{-p-\beta}$, with $\beta > 0$, and summing up over k leads to weighted Hardy's inequality

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{p+\beta}}\,dx \le \frac{c}{\beta}\int_\Omega \frac{|\nabla u(x)|^p}{\delta(x)^\beta}\,dx. \tag{3.4}$$

An application of this inequality to

$$u(x)\delta(x)^{\beta/p}$$

for $\beta > 0$ is small enough, gives unweighted p-Hardy's inequality. Weighted Hardy's inequalities have also been studied in [27], [29] and [30].

The pointwise Hardy inequality is not equivalent to Hardy's inequality. For example, the punctured ball satisfies pointwise Hardy's inequality only in the case when $p > n$ but usual Hardy's inequality also holds when $1 < p < n$. A recent result of Lehrbäck [28] shows that the uniform thickness is not only sufficient but also necessary condition for pointwise Hardy's inequality. Before giving the statement of the result here we recall a definition from [28]. An open set Ω in $\mathbb{R}^n$ satisfies an inner boundary density condition with exponent α, if there exists a constant $c > 0$ such that

$$\mathcal{H}^\alpha_\infty\big(B(x,2\delta(x))\cap\partial\Omega\big) \ge c\delta(x)^\alpha$$

for every $x \in \Omega$. Here

$$\mathcal{H}^\alpha_\infty(E) = \left\{\sum_{i=1}^\infty r_i^\alpha : E \subset \bigcup_{i=1}^\infty B(x_i,r_i)\right\}$$

is the spherical Hausdorff content of the set E. Let us now formulate the main result of [28].

Theorem 3.5. *The following conditions are equivalent:*

(1) *The set $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick,*
(2) *the set Ω satisfies pointwise Hardy's inequality with some $q < p$,*
(3) *there exists α with $n - p < \alpha \le n$ so that Ω satisfies the inner boundary density condition with the exponent α.*

Observe that all these conditions have the property that if they hold true for some parameter, then they also hold for all larger parameters. However, Hardy's inequality does not share this property with them.

Remark 3.6. A recent result of [22] shows that condition (2) in Theorem 3.5 can be replaced with pointwise p-Hardy's inequality. By Theorem 3.1 we conclude that pointwise p-Hardy's inequality implies the pointwise q-Hardy inequality for some $q < p$. Hence pointwise Hardy's inequality is a self improving property. It would be interesting to obtain a direct proof, without Theorem 3.1, for this self improving result.

We present yet another characterization of uniform thickness through a boundary Poincaré inequality of type (3.2).

Theorem 3.7. *The set $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick if and only if*

$$\int_{B(x,r)} |u(y)|^p \, dy \le c\, r^p \int_{B(x,r)} |\nabla u(y)|^p \, dy \tag{3.8}$$

for every $x \in \mathbb{R}^n \setminus \Omega$ and $u \in C_0^\infty(\Omega)$.

Proof. The uniform p-thickness implies (3.8) by the capacitary version of the Poincaré inequality (3.2).

To prove the reverse implication, let $u \in C_0^\infty(B(x,2r))$ be such that $u(x) = 1$ for every $x \in (\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x,r)$. If

$$\frac{1}{|B(x,r/2)|} \int_{B(x,r/2)} |u(y)|^p \, dy \ge \frac{1}{2^p},$$

then by the standard Poincaré inequality we have

$$\frac{1}{2^p} |B(x,r/2)| \le c \int_{B(x,2r)} |u(y)|^p \, dy \le c\, r^p \int_{B(x,2r)} |\nabla u(y)|^p \, dy,$$

from which it follows that

$$\int_{B(x,2r)} |\nabla u(y)|^p \, dy \ge c\, r^{n-p}.$$

Assume then that

$$\frac{1}{|B(x,r/2)|} \int_{B(x,r/2)} |u(y)|^p \, dy < \frac{1}{2^p}.$$

Clearly

$$|B(x,r/2)| \le 2^{p-1} \left(\int_{B(x,r/2)} |u(y)|^p \, dy + \int_{B(x,r/2)} |1 - u(y)|^p \, dy \right)$$

and consequently

$$\int_{B(x,r/2)} |1 - u(y)|^p \, dy \ge 2^{1-p} |B(x,r/2)| - \int_{B(x,r/2)} |u(y)|^p \, dy \ge c r^n.$$

Let $v = (1-u)\varphi$, where $\varphi \in C_0^\infty(B(x,r))$ is a cutoff function such that $\varphi = 1$ in $B(x,r/2)$. Then $v \in C_0^\infty(\Omega)$ and by (3.8) we have

$$\begin{aligned}\int_{B(x,r/2)} |1-u(y)|^p\,dy &= \int_{B(x,r/2)} |v(y)|^p\,dy \\ &\le c\,r^p \int_{B(x,r/2)} |\nabla v(y)|^p\,dy \\ &\le c\,r^p \int_{B(x,2r)} |\nabla u(y)|^p\,dy.\end{aligned}$$

It follows that

$$\int_{B(x,2r)} |\nabla u(y)|^p\,dy \ge c\,r^{n-p}.$$

Taking infimum over all such functions u, we conclude that

$$\operatorname{cap}_p\big((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x,r), B(x,2r)\big) \ge c\,r^{n-p}$$

and hence Ω is uniformly p-thick. □

Remark 3.9. Again, by Theorem 3.1 we conclude that the boundary Poincaré inequality 3.8 implies the same inequality for some $q < p$. Hence the the boundary Poincaré inequality is a self-improving property.

4. Characterizations in the borderline case

In the borderline case when $p = n$ there are several characterizations available for Hardy's inequality. We begin with the following metric definition. The set $\mathbb{R}^n \setminus \Omega$ is uniformly perfect if it contains more than one point, and there is a constant $c_P \ge 1$ so that for each $x \in \mathbb{R}^n \setminus \Omega$ and $r > 0$ we have

$$(\mathbb{R}^n \setminus \Omega) \cap \big(B(x, c_P r) \setminus B(x,r)\big) \ne \emptyset$$

whenever $(\mathbb{R}^n \setminus \Omega) \setminus B(x, c_P r) \ne \emptyset$. For more information about uniform perfectness, see [17] and [46].

Next we give four characterzations for Hardy's inequality in the borderline case. Needless to say that by Theorem 2.1, Theorem 3.5 and Theorem 3.7 we have four more characterizations.

Theorem 4.1. *The following conditions are quantitatively equivalent:*

(1) Ω *satisfies* n*-Hardy's inequality,*
(2) $\mathbb{R}^n \setminus \Omega$ *is uniformly perfect,*
(3) $\mathbb{R}^n \setminus \Omega$ *is uniformly* n*-thick,*
(4) $\mathbb{R}^n \setminus \Omega$ *is uniformly* $(n-\varepsilon)$*-thick for some* $\varepsilon > 0$*.*

Proof. Let us comment the architecture of the proof. Conditions (3) and (4) are equivalent by Theorem 3.1. The fact that (3) and (4) imply (1) can be shown as above. The equivalence of conditions (1) and (3) has been proved by Ancona [2] for $n = 2$ and by Lewis [31] for $n \geq 2$. Sugawa has proved in [46] that conditions (1)–(4) are equivalent for $n = 2$. Recently this result has been generalized for $n \geq 2$ in [23]. We recall the crux of the proof here.

The first step is to show that n-Hardy's inequality implies uniform perfectness (and unboundedness) of the complement. The method is indirect: First, assume that $\mathbb{R}^n \setminus \Omega$ is not uniformly perfect with some large constant $M > 1$. This means that there exists $x_0 \in \mathbb{R}^n \setminus \Omega$ and $r_0 > 0$ such that $B(x_0, Mr_0) \setminus B(x_0, r_0)$ is contained in Ω. The test function

$$u(x) = \begin{cases} \left(\dfrac{|x - x_0|}{r_0} - 1\right)_+, & \text{if } \ |x - x_0| \leq 2r_0, \\ 1, & \text{if } \ 2r_0 < |x - x_0| < \dfrac{Mr_0}{2}, \\ \left(2 - 2\dfrac{|x - x_0|}{Mr_0}\right)_+, & \text{if } \ |x - x_0| \geq \dfrac{Mr_0}{2}. \end{cases}$$

shows that if Ω satisfies n-Hardy's inequality with some constant c_H, then $c_H \geq c \log M$.

The following step is to show that the uniform perfectness of the complement further implies a boundary density condition similar to condition (3) in Theorem 3.5. More precisely, we show that there exists $\alpha > 0$ such that

$$\mathcal{H}^\alpha_\infty(\overline{B}(x_0, r_0) \setminus \Omega) \geq cr_0^\alpha \tag{4.2}$$

for every $x_0 \in \mathbb{R}^n \setminus \Omega$ and $r_0 > 0$. For the argument to work it is essential that $\overline{B}(x_0, r_0) \setminus \Omega$ is compact. Indeed, there are uniformly perfect countable sets, whose Hausdorff-dimension is zero. To estimate the Hausdorff content of the set, we take a cover $\mathcal{F}$ for $\overline{B}(x_0, r_0) \setminus \Omega$ with balls $B(x, r)$. By compactness, we may choose the cover $\mathcal{F}$ to be finite. We may also assume that the balls in $\mathcal{F}$ are centered in $\overline{B}(x_0, r_0) \setminus \Omega$. This may increase (4.2) at most by factor 2^α. Next we reduce the number of balls in $\mathcal{F}$ in such a way that the sum

$$\sum_{B(x,r)\in\mathcal{F}} r^\alpha \tag{4.3}$$

does not increase: Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly perfect with constant M. Then, if $\alpha > 0$ is small enough the elementary inequality

$$r^\alpha + s^\alpha \geq (r + s + 2M \min\{r, s\})^\alpha$$

holds for all $r, s > 0$. Now, if there exists balls $B(x,r)$ and $B(y,s)$ in $\mathcal{F}$ such that $r \le 2s$ and

$$B(x, Mr) \cap B(y,s) \neq \emptyset$$

then

$$B(x,r) \cup B(y,s) \subset B(z, r+s+2M\min\{r,s\})$$

for some $z \in \{x,y\}$. Thus we can replace the original balls $B(x,r)$ and $B(y,s)$ by one with larger radius so that the sum (4.3) does not increase. We continue this replacement procedure until there are no balls satisfying the condition left. As $\mathcal{F}$ is finite, the process ends after a finite number of steps.

Now let $B(x_1,r_1) \in \mathcal{F}$ be the ball that contains x_0. By the uniform perfectness of $\mathbb{R}^n \setminus \Omega$, the set

$$A_1 \setminus \Omega = \big(B(x_1, Mr_1) \setminus B(x_1,r_1)\big) \setminus \Omega$$

is not empty. Now there are two possibilities: Either A_1 intersects the complement of $\overline{B}(x_0,r_0)$ or it intersects some ball $B(x_2,r_2) \in \mathcal{F}$. In the first case $r_1 \ge r_0/(M+1)$. In the second case we know that $r_2 \le r_1/2$, since otherwise the balls $B(x_1,r_1)$ and $B(x_2,r_2)$ would have been replaced by a single ball in the iteration above. We continue in the same way: For a ball $B(x_k,r_k)$, either

$$A_k = B(x_k, Mr_k) \setminus B(x_k,r_k)$$

intersects the complement of $B(x_0,r_0)$ or some ball $B(x_{k+1},r_{k+1}) \in \mathcal{F}$ with radius $r_{k+1} \le r_k/2$. This procedure stops when the first alternative occurs. This happens after a finite number of steps since $\mathcal{F}$ is finite. Let K be the index where the iteration stops. Now since $x_0 \in B(x_1,r_1)$ and $B(x_K, Mr_K)$ intersects the complement of $B(x_0,r_0)$, we have

$$r_0 \le \sum_{i=1}^{K}(M+1)r_i \le (M+1)\sum_{i=1}^{K} 2^{1-i} r_1 \le 2(M+1)r_1.$$

Thus $r_1 \ge r_0/(2(M+1))$ and it follows that

$$\sum_{B(x,r)\in\mathcal{F}} r^\alpha \ge r_1^\alpha \ge \frac{r_0^\alpha}{2(M+1)}.$$

Since this holds for all covers of $\overline{B}(x_0,r_0) \setminus \Omega$, we have obtained a lowerbound for its Hausdorff α-content depending only on the uniform perfectness constant M. Uniform estimate for Hausdorff α-content further implies uniform p-thickness for every $p > n-\alpha$, see for example Lemma 2.31 in [16]. This completes the proof. □

Remark 4.4. Theorem 4.1 gives a relatively elementary proof for Theorem 3.1 with $p = n$. It would be interesting to obtain an elementary proof for other values of p as well.

5. Eigenvalue problem

This section gives a characterization of p-Hardy's inequality in terms of weak supersolutions to (1.4). This generalizes Proposition 1 of Ancona in [2], where he obtained the result for $p = 2$. We recall that $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of (1.4), if

$$\int_\Omega \left(|\nabla v(x)|^{p-2} \nabla v(x) \cdot \nabla \varphi(x) - \lambda_p \frac{|v(x)|^{p-2} v(x)}{\delta(x)^p} \varphi(x) \right) dx = 0 \quad (5.1)$$

for all $\varphi \in C_0^\infty(\Omega)$. Moreover, the function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak supersolution of (1.4), if the integral in (5.1) is nonnegative for $\varphi \geq 0$.

Theorem 5.2. *Let $1 < p < \infty$. Inequality (1.1) holds with a finite c_H if and only if there is a positive eigenvalue $\lambda_p = \lambda_p(\Omega) > 0$ and a positive weak supersolution $v \in W_0^{1,p}(\Omega)$ of (1.4) in Ω.*

Proof. First, suppose that there exists a positive weak supersolution v of (1.4). Take $u \in C_0^\infty(\Omega)$. We may assume that $u \geq 0$ in Ω. Let $\varepsilon > 0$. We take

$$\varphi(x) = \frac{u(x)^p}{(v(x) + \varepsilon)^{p-1}}$$

as a test function. It follows that

$$\begin{aligned}
&\lambda_p \int_\Omega \frac{u(x)^p v(x)^{p-1}}{\delta(x)^p (v(x)+\varepsilon)^{p-1}}\, dx \\
&\quad \leq \int_\Omega |\nabla v(x)|^{p-2} \nabla v(x) \cdot \nabla \varphi(x)\, dx \\
&\quad = (1-p) \int_\Omega |\nabla v(x)|^p (v(x)+\varepsilon)^{-p} u(x)^p\, dx \\
&\qquad + p \int_\Omega u(x)^{p-1} (v(x)+\varepsilon)^{1-p} |\nabla v(x)|^{p-2} \nabla v(x) \cdot \nabla u(x)\, dx \\
&\quad \leq (1-p) \int_\Omega \left| \frac{u(x) \nabla v(x)}{v(x)+\varepsilon} \right|^p dx + p \int_\Omega \left| \frac{u(x) \nabla v(x)}{v(x)+\varepsilon} \right|^{p-1} |\nabla u(x)|\, dx.
\end{aligned}$$

Then we use Young's inequality to conclude that

$$\begin{aligned}
&p \int_\Omega \left| \frac{u(x) \nabla v(x)}{v(x)+\varepsilon} \right|^{p-1} |\nabla u(x)|\, dx \\
&\qquad \leq (p-1) \int_\Omega \left| \frac{u(x) \nabla v(x)}{v(x)+\varepsilon} \right|^p dx + \int_\Omega |\nabla u(x)|^p\, dx.
\end{aligned}$$

By combining these estimates and passing to the limit as $\varepsilon \to 0$ with the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \left(\frac{u(x)}{\delta(x)} \right)^p dx \leq \frac{1}{\lambda_p} \int_\Omega |\nabla u(x)|^p\, dx.$$

Thus Ω satisfies p-Hardy inequality with constant $c_H = 1/\lambda_p$.

To prove the other direction, assume that Ω satisfies p-Hardy's inequality with some finite constant c_H. Let $\lambda_p < 1/c_H$. We define a function space

$$X = \{f \in L^p_{\mathrm{loc}}(\Omega) : \nabla f \in L^p(\Omega),\ f\delta^{-1} \in L^p(\Omega)\}$$

and

$$\|f\|_X = \left\|\frac{f}{\delta}\right\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Because Ω satisfies p-Hardy's inequality, $X = W^{1,p}_0(\Omega)$ and the norms are equivalent. Therefore X is a reflexive Banach space. We define the operator $T : X \to X^*$ by

$$(Tf, g) = \int_\Omega |\nabla f(x)|^{p-2}\nabla f(x) \cdot \nabla g(x)\, dx - \lambda_p \int_\Omega \frac{|f(x)|^{p-2} f(x)}{\delta(x)^p} g(x)\, dx,$$

where $f, g \in X$. We will apply the following result from functional analysis. If $T : X \to X^*$ satisfies the following boundedness, demicontinuity and coercivity properties

(i) T is bounded;
(ii) if $f_j \to f$ in X, then $(Tf_j, g) \to (Tf, g)$ for all $g \in X$;
(iii) if (f_j) is a sequence in X with $\|f_j\|_X \to \infty$ as $j \to \infty$, then

$$\frac{(Tf_j, f_j)}{\|f_j\|_X} \to \infty.$$

Then for every $f \in X^*$ there is $v \in X$ which satisfies $Tv = f$.

Next we will check that these conditions are satisfied. The first condition holds, because

$$\begin{aligned}
\|Tf\|_{X^*} &= \sup_{\|g\|_X \le 1} (Tf, g) \\
&= \sup_{\|g\|_X \le 1} \left| \int_\Omega |\nabla f(x)|^{p-2}\nabla f(x) \cdot \nabla g(x)\, dx \right. \\
&\qquad \left. - \lambda_p \int_\Omega \frac{|f(x)|^{p-2} f(x)}{\delta(x)^p} g(x)\, dx \right| \\
&\le \sup_{\|g\|_X \le 1} \left[\|\nabla f\|^{p-1}_{L^p(\Omega)} \|\nabla g\|_{L^p(\Omega)} + \lambda_p \left\|\frac{f}{\delta}\right\|^{p-1}_{L^p(\Omega)} \left\|\frac{g}{\delta}\right\|_{L^p(\Omega)} \right] \\
&\le (1 + \lambda_p)\|f\|^{p-1}_X \|g\|_X.
\end{aligned}$$

Now let $f_j \to f$ in X and $g \in X$. Then by Hölder's inequality,

$$
\begin{aligned}
&|(Tf_j, g) - (Tf, g)| \\
&= \bigg| \int_\Omega (|\nabla f_j(x)|^{p-2} \nabla f_j(x) - \nabla f(x)|^{p-2} \nabla f(x)) \cdot \nabla g(x)\, dx \\
&\qquad - \lambda_p \int_\Omega \frac{|f_j(x)|^{p-2} f_j(x) - |f(x)|^{p-2} f(x)}{\delta(x)^p} g(x)\, dx \bigg| \\
&\leq (p-1) \|f_j - f\|_X \max\{\|f_j\|_X^{p-2}, \|f\|_X^{p-2}\} \|g\|_X \to 0,
\end{aligned}
$$

as $f_j \to f$ in X. Thus the second condition is satisfied. Finally, let $(f_j)_j$ be a sequence in X with $\|f_j\|_X \to \infty$ as $j \to \infty$. By Hardy's inequality, it follows that

$$
\begin{aligned}
(Tf_j, f_j) &= \int_\Omega \left(|\nabla f_j(x)|^p - \lambda_p \frac{|f_j(x)|^p}{\delta(x)^p} \right) dx \\
&\geq (1 - \lambda_p c_H) \int_\Omega |\nabla f_j(x)|^p\, dx \\
&\geq \frac{1 - \lambda_p c_H}{1 + c_H} \|f_j\|_X^p.
\end{aligned}
$$

Hence

$$
\frac{(Tf_j, f_j)}{\|f_j\|_X} \to \infty, \text{ as } j \to \infty.
$$

Thus also the third condition is satisfied.

We fix a function $w \in X^*$ such that $w \geq 0$ and $w \not\equiv 0$. Then there exists $v \in X = W_0^{1,p}(\Omega)$ such that $Tv = w$. This means that v is a weak supersolution of (1.4).

Moreover, v is positive since

$$
\begin{aligned}
0 \leq (Tv, v_-) &= \int_\Omega |\nabla v(x)|^{p-2} \nabla v(x) \cdot \nabla v_-(x)\, dx \\
&\qquad - \lambda_p \int_\Omega |v(x)|^{p-2} v(x) v_-(x) \delta(x)^{-p}\, dx \\
&= - \int_\Omega |\nabla v_-(x)|^p\, dx + \lambda_p \int_\Omega v_-(x)^p \delta(x)^{-p}\, dx \\
&\leq (\lambda_p - 1/c_H) \int_\Omega v_-(x)^p \delta(x)^{-p}\, dx \leq 0.
\end{aligned}
$$

Here $v_-(x) = -\min(v(x), 0)$ is the negative part of v. Now the strict positivity of v follows from a weak Harnack inequality. □

Remark 5.3. The eigenvalue problem (1.4) has the following stability property: If there is a positive eigenvalue $\lambda_p = \lambda_p(\Omega) > 0$ and a positive weak supersolution of (1.4) in Ω, then there is ε such that for every q with $p - \varepsilon < q < p + \varepsilon$ there is an eigenvalue $\lambda_q = \lambda_q(\Omega) > 0$ and a positive weak supersolution of (1.4) with p relaced by q in Ω. This follows directly from the self-improving result for Hardy's inequality, see Remark 2.3.

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