Appeared in Manuscripta Math. **105** (2001), 401–423.

REGULARITY OF QUASI-MINIMIZERS ON METRIC SPACES

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Abstract. Using the theory of Sobolev spaces on a metric measure space we are able to apply calculus of variations and define p-harmonic functions as minimizers of the p-Dirichlet integral. More generally, we study regularity properties of quasi-minimizers of p-Dirichlet integrals in a metric measure space. Applying the De Giorgi method we show that quasi-minimizers, and in particular p-harmonic functions, satisfy Harnack's inequality, the strong maximum principle, and are locally Hölder continuous, if the space is doubling and supports a Poincaré inequality.

1. Introduction

The classical Dirichlet problem is to find a harmonic function with given boundary values. An alternative variational formulation of this problem is to minimize the Dirichlet integral

$$\int |Du|^2 \, dx$$

among all functions which have required boundary values. A more general nonlinear variation of the classical Dirichlet problem is to study minimizers of the p-Dirichlet integral

$$\int |Du|^p \, dx,$$

with $1 < p < \infty$. The minimizers are solutions to the corresponding Euler-Lagrange equation, which in this case is the p-Laplace equation

$$\operatorname{div}(|Du|^{p-2} Du) = 0,$$

1991 *Mathematics Subject Classification.* 49N60, 35J60.
Key words and phrases. Quasi-minima, variational integrals, Harnack inequality.

and continuous solutions are called p-harmonic functions.

It is not clear what the counterpart for the p-Laplace equation is in a general metric measure space, but the variational approach is available; it is possible to define p-harmonic functions as minimizers of p-Dirichlet integral in a metric measure space. The basic reason is that Sobolev spaces on a metric measure space can be defined without the notion of partial derivatives; see [C], [H], [HeK] and [Sh2]. The definitions in these references are different but by [Sh2] they give the same Sobolev space under mild assumptions. Cheeger's goal in [C] is to study differentiability of Lipschitz functions on metric measure spaces. His definition of Sobolev spaces is tailored to make lower semicontinuity of the Sobolev norm under L^p convergence a virtual tautology. This leads to the existence of a differential as a measurable section of a finite dimensional cotangent bundle from which the reflexivity of the Sobolev space follows. Hence direct methods in the calculus of variations can be easily applied to prove the existence for the p-Dirichlet problem; see section 7 in [C] and [Sh1]. In this work we study the regularity properties of p-harmonic functions on a metric measure space.

In the Euclidean case minimizers of the p-Dirichlet integral are known to be locally Hölder continuous. There are at least two ways of seeing this. One possible approach is to use Moser's iteration technique (see [Mo1] and [Mo2]), which gives Harnack's inequality and then Hölder continuity follows from this in a standard way. From our point of view there is a drawback in Moser's argument; it is based on the differential equation and it seems to us that it cannot be applied in the general metric setting. However, there is another approach by De Giorgi [DeG], which relies only on the minimization property. In contrast with Moser's technique, De Giorgi's method gives Hölder continuity and then Harnack's inequality can be obtained as in [DT]. One of the advantages of De Giorgi's method is that it is applicaple to quasi-minimizers as well. We recall that a quasi-minimizer minimizes the p-Dirichlet integral up to a multiplicative constant; see [GG1] and [GG2]. Hence, in particular, p-harmonic functions are quasi-minimizers. We have chosen this more general approach to emphazise the fact that the obtained properties hold in a very general context and are very robust. For example, they are preserved under bi-Lipschitz perturbations of the metric.

The purpose of this note is to adapt De Giorgi's method to the metric setting. We show that if the space is doubling in measure and supports a $(1, q)$-Poincaré inequality, then quasi-minimizers, and in particular p-harmonic functions, satisfy Harnack's inequality, the strong maximum principle, and are locally Hölder continuous. We note that Harnack's inequality is the strongest claim and all other properties follow from it in a standard way. However, these claims are closely related to each other, and for expository purposes we first prove Hölder continuity, then the strong maximum principle, and finally Harnack's inequality, since the proofs of these properties are based on estimates which are needed in the proof of Harnack's inequality. De Giorgi's method is based on two ingredients: Sobolev and Caccioppoli type estimates. We observe that these estimates are available under our assumptions. Here we use results of [HaK], which show that the Poincaré inequality implies a Sobolev type estimate. Then we very closely follow the presentation of [Gia] and [Giu] and show that De Giorgi's method applies. However, there are a few delicate points in the argument and hence we are somewhat careful in details. For example, the doubling condition comes into play in several occasions. In addition, we do not have exactly the same exponents in the Sobolev type estimate as in the Euclidean case. Finally, the proof of [DT] is based on the Krylov-Safonov covering argument which is originally stated in terms of dyadic cubes. Instead of dyadic cubes we use balls, doubling property and a simple maxoimal function argument.

Our work is closely related to the paper [C] of Cheeger. As he points out in Remark 7.19 of [C], Moser's iteration scheme can be used if the p-harmonic functions are defined with respect to an L^∞ Riemannian metric and the unit sphere is smooth and strictly convex. Our approach shows that these additional assumptions are not needed for local Hölder continuity and Harnack's inequality. There exists a remarkable literature on Harnack's inequalities under various circumstances; see for example [AC], [CDG], [FL], [FKS], [HS], [JX], [LU], [Ma], [SC1], and [SC2]. Finally we note that boundary continuity for quasi-minimizers on metric measure spaces have recently been studied in [B].

This note is organized as follows. The second section focuses on the preliminary notation and definitions needed in the rest of the paper. There we also fix the general setup and conventions used later in the paper without further notice. In addition, we prove a Sobolev type inequality for functions which vanish on a large set. The third section explores the relationship between quasi-minimizers and the De Giorgi class of functions. In particular, there we prove a Caccioppoli type estimate. In the next two sections local boundedness and local Hölder continuity properties of the De Giorgi class are studied. In section 6 we prove the strong maximum principle and in section 7 the Harnack inequality for quasi-minimizers.

Acknowledgements. We are grateful to Piotr Hajłasz and Ilkka Holopainen for helpful conversations on graphs, and Juha Heinonen and Pekka Koskela for their encouragement. We also thank Jana Björn, Olli Martio and Seppo Rickman for helpful suggestions on improving the paper. This research was completed while the authors visited the Mittag-Leffler Institute; we wish to thank the Institute for the support.

2. Preliminaries

In this section we recall basic definitions and describe the general setup of our study.

We assume that X is a metric measure space equipped with a Borel regular measure μ. Throughout the paper we assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. Later we impose further requirements on the space and on the measure; see subsection 2.13.

2.1. Upper gradients. Let $u\colon X \to [-\infty, \infty]$ be a function. A non-negative Borel measurable function $g\colon X \to [0, \infty]$ is said to be an *upper gradient* of u if for all compact rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds. \tag{2.2}$$

If $u(x) = u(y) = \infty$ or $u(x) = u(y) = -\infty$, we define the left side of (2.2) to be ∞. See [C], [HeK], [KoM] and [Sh2] for a discussion of upper gradients. Observe that upper gradients are called very weak gradients in [HeK], but we use different terminology here.

Let $1 \le p < \infty$. The *p-modulus* of a family of paths Γ in X is the number

$$\inf_{\rho} \int_X \rho^p \, d\mu,$$

where the infimum is taken over all non-negative Borel measurable functions ρ such that for all rectifiable paths γ which belong to Γ we have

$$\int_\gamma \rho \, ds \ge 1.$$

It is known that the p-modulus is an outer measure on the collection of all paths in X. A property is said to hold for *p-almost all* paths, if the set of non-constant paths for which the property fails is of zero p-modulus. If (2.2) holds for p-almost all paths γ in X, then g is said to be a *p-weak upper gradient* of u.

2.3. Newtonian spaces. The upper gradient is a substitute for the modulus of a gradient in a metric space, but in order to be able to do calculus of variations we need a concept of Sobolev spaces in a metric measure space. Let $1 \le p < \infty$. We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p-integrable functions u that have a p-integrable p-weak upper gradient g. This space is equipped with a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients of u. When $p > 1$, by the uniform convexity of $L^p(X)$ we have that whenever $u \in N^{1,p}(X)$ there is a function g_u in $L^p(X)$-convex hull formed by the set of all p-weak upper gradients of u, called the *minimal p-weak upper gradient of* u, so that g_u is a p-weak upper gradient of u and

$$\|g_u\|_{L^p(X)} = \inf \|g\|_{L^p(X)}$$

where the infimum is over all p-weak upper gradients g of u; see [Sh1] or [C].

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of the Newtonian spaces we refer to [Sh2]. We recall here some facts for future reference. It can be shown that $N^{1,p}(X)$ is a Banach space. It is also very useful to know that if $1 < p < \infty$ every function u that has a p-integrable upper gradient has a minimal p-integrable p-weak upper gradient, denoted g_u, in the sense that if g is another p-weak upper gradient of u, then $g_u \le g$ μ-almost everywhere. The functions in $N^{1,p}(X)$ are absolutely continuous on p-almost every path, which means that $u\circ\gamma$ is absolutely continuous on $[0, \text{length}(\gamma)]$ for p-almost every rectifiable arc-length parametrized path γ in X.

The *p-capacity* of a set $E\subset X$ is defined by

$$\mathrm{C}_p(E)=\inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u\in N^{1,p}(X)$, whose restriction to a neighbourhood of E is bounded below by 1. Capacity is the natural measure for exceptional sets of Sobolev functions. It is easy to see that sets of zero capacity are also of measure zero, but the converse is not true in general. See [KM] for more properties of the capacity.

In order to be able to compare the boundary values of Sobolev functions we need a notion of Sobolev spaces with zero boundary values in a metric measure space. Let E be an arbitrary subset of X. Following the method of [KKM], we define $\widetilde{N}^{1,p}_0(E)$ to be the set of functions $u\colon E\to[-\infty,\infty]$ for which there exists a function $\widetilde{u}\in\widetilde{N}^{1,p}(X)$ such that $\widetilde{u}=u$ μ-almost everywhere in E and

$$\mathrm{C}_p(\{x\in X\setminus E\colon \widetilde{u}(x)\neq 0\})=0.$$

Next we define an equivalence relation on $\widetilde{N}^{1,p}_0(E)$ by saying that $u\sim v$ if $u=v$ μ-almost everywhere on E. Finally we let $N^{1,p}_0(E)=\widetilde{N}^{1,p}_0(E)/\sim$, equipped with the norm

$$\|u\|_{N^{1,p}_0(E)}=\|\widetilde{u}\|_{\widetilde{N}^{1,p}(X)},$$

be the *Newtonian space with zero boundary values.* The norm is unambiguously defined by [Sh1] and the obtained space is a Banach space.

We use the following observation several times: suppose that there is a Borel set $A\subset X$ so that u is constant μ-almost everywhere in $X\setminus A$. Then if g is an upper gradient of u, then $g\chi_A$ is a p-weak upper gradient of u, and hence the minimal p-weak upper gradient $g_u=0$ μ-almost everywhere on $X\setminus A$. Here χ_A is the characteristic function of A. For open sets A this has been proved in [Sh1] and the general claim follows from fact that a locally finite Borel measure is a Radon measure, and hence the measure of a Borel set can be approximated by measures of open sets containing the set. It follows from this that if u and v are functions such that $u=v$ μ-almost everywhere on a Borel set A, then $g_u=g_v$ μ-almost everywhere; see Corollary 2.25 in [C].

2.4. Poincaré inequalities. A metric measure space X is said to be *doubling* if there is a constant $c_d\geq 1$ so that

$$\mu(B(z,2r))\leq c_d\mu(B(z,r)) \tag{2.5}$$

for every open ball $B(z,r)$ in X. Throughout the work we use the convention that $B(z,r)$ refers to an open ball. The constant c_d in (2.5) is called the doubling constant of μ. Note that by the doubling property, if $B(y,R)$ is a ball in X, $z\in B(y,R)$ and $0<r\leq R<\infty$, then

$$\frac{\mu(B(z,r))}{\mu(B(y,R))}\geq c\Big(\frac{r}{R}\Big)^Q \tag{2.6}$$

for some c and Q depending only on the doubling constant.

Let $1 \le q < \infty$. The space X is said to *support a weak* $(1,q)$*-Poincaré inequality* if there are constants $c_0 > 0$ and $\tau \ge 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}|\, d\mu \le c_0 r \Big(\fint_{B(z,\tau r)} g^q \, d\mu \Big)^{1/q} \tag{2.7}$$

for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,r)$ and for all upper gradients g of u. The word weak refers to the possibility that $\tau > 1$. If $\tau = 1$, the space is said to support a $(1,q)$*-Poincaré inequality.* A result of [HaK] shows that in a doubling measure space a weak $(1,q)$-Poincaré inequality implies a weak (t,q)-Poincaré inequality for some $t > q$ possibly with a different τ. More precisely, there are $c > 0$ and $\tau' \ge 1$ such that

$$\Big(\fint_{B(z,r)} |u - u_{B(z,r)}|^t \, d\mu \Big)^{1/t} \le cr \Big(\fint_{B(z,\tau' r)} g^q \, d\mu \Big)^{1/q}, \tag{2.8}$$

where $1 \le t < Qq/(Q-q)$ if $q < Q$ and $t \ge 1$ if $q \ge Q$, for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,r)$ and for all upper gradients g of u. Conversely, by the Hölder inequality we see that a (t,q)-Poincaré inequality implies the same inequality for smaller values of t and larger values of q. In particular, if the space supports a weak (t,q)-Poincaré inequality, then it also supports a weak $(1,q)$-Poincaré inequality. It can also be shown that in a space supporting a weak $(1,q)$-Poincaré inequality, every ball, whose complement is non-empty, has a non-empty boundary. This is a strengthening of the topological notion of uniform perfectness; see the comments following inequality (2.11).

Moreover, in inequalities (2.7) and (2.8) we can replace the upper gradient g with any p-weak upper gradient in $L^p_{\mathrm{loc}}(X)$, because of the result in [KoM] which states that every p-weak upper gradient in $L^p_{\mathrm{loc}}(X)$ can be approximated in $L^p(X)$ by an upper gradient in $L^p_{\mathrm{loc}}(X)$. Indeed, given any p-weak upper gradient $g_1 \in L^p_{loc}(X)$ of u and any $\varepsilon > 0$ we can find an upper gradient g_2 so that $\|g_2 - g_1\|_{L^p(X)} < \varepsilon$. Here $L^p_{\mathrm{loc}}(X)$ is the space of all measurable functions that are p-integrable on bounded subsets of X.

2.9. Sobolev inequalities. Next we prove a Sobolev type inequality for functions which vanish on a large set. The paper [B] has a better capacitary version of this inequality, but for our purposes it suffices to consider the more easily proved version below.

2.10. Lemma. *Let X be a doubling metric measure space supporting a weak $(1,q)$-Poincaré inequality for some $1 < q < p$. Suppose that $u \in N^{1,p}(X)$ and let $A = \{x \in B(z,R) \colon |u(x)| > 0\}$. If $\mu(A) \le \gamma \mu(B(z,R))$ for some γ with $0 < \gamma < 1$, then there is a constant $c > 0$ so that*

$$\Big(\fint_{B(z,R)} |u|^t \, d\mu \Big)^{1/t} \le cR \Big(\fint_{B(z,\tau' R)} g_u^q \, d\mu \Big)^{1/q},$$

where t and τ' are as in (2.8). The constant c depends only on γ and the constants c and τ' of (2.8).

Proof. By the Minkowski inequality and (2.8) we have

$$\Big(\fint_{B(z,R)} |u|^t \, d\mu\Big)^{1/t} \le \Big(\fint_{B(z,R)} |u - u_{B(z,R)}|^t \, d\mu\Big)^{1/t} + |u_{B(z,R)}|$$
$$\le cR\Big(\fint_{B(z,\tau' R)} g_u^q \, d\mu\Big)^{1/q} + |u_{B(x,R)}|.$$

The Hölder inequality implies that

$$|u_{B(z,R)}| \le \Big(\frac{\mu(A)}{\mu(B(z,R))}\Big)^{1-1/t} \Big(\fint_{B(z,R)} |u|^t \, d\mu\Big)^{1/t}$$
$$\le \gamma^{1-1/t}\Big(\fint_{B(z,R)} |u|^t \, d\mu\Big)^{1/t}.$$

Hence we obtain

$$(1-\gamma^{1-1/t})\Big(\fint_{B(z,R)} |u|^t \, d\mu\Big)^{1/t} \le cR\Big(\fint_{B(z,\tau' R)} g_u^q \, d\mu\Big)^{1/q},$$

from which the claim follows since $0 < \gamma < 1$. □

We observe that Lemma 2.10 gives a Sobolev inequality for Sobolev functions with zero boundary values. To be more precise, there exists $c > 0$ so that for every ball $B(z,R)$ with $0 < R \le \operatorname{diam}(X)/3$ and every $u \in N_0^{1,p}(B(z,R))$ we have

$$\Big(\fint_{B(z,R)} |u|^t \, d\mu\Big)^{1/t} \le cR\Big(\fint_{B(z,R)} g_u^q \, d\mu\Big)^{1/q}. \tag{2.11}$$

This follows easily from Lemma 2.10 after noting that $g_u = 0$ almost everywhere on $X \backslash B(z,R)$ and by observing that there must be a point on the sphere $\partial B(z,2R)$. If there is no such point, then it is easy to construct a function which violates the Poincaré inequality.

2.12. Quasi-minimizers. Now we are ready to formulate the minimization problem for the p-Dirichlet integral in a metric measure space. By $N^{1,p}_{\mathrm{loc}}(\Omega)$ we mean the space of all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ that have an upper gradient in $L^p_{\mathrm{loc}}(\Omega)$, where $L^p_{\mathrm{loc}}(\Omega)$ is the space of all measurable functions that are p-integrable on bounded subsets of X.

Suppose that $\Omega \subset X$ is open. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called *p-harmonic* on Ω, if for every bounded open subset Ω' of Ω with $\overline{\Omega}' \subset \Omega$ and $v \in N^{1,p}(\Omega')$ with $u - v \in N_0^{1,p}(\Omega')$ we have

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu,$$

where g_u and g_v are the minimal weak upper gradients of u and v respectively.

A function u is said to be a *quasi-minimizer* on Ω if there is a constant $K > 0$ so that for all bounded open subsets Ω' of Ω with $\overline{\Omega}' \subset \Omega$ and for all functions $v \in N^{1,p}(\Omega')$ with $u - v \in N_0^{1,p}(\Omega')$ the inequality

$$\int_{\Omega' \cap \{u \ne v\}} g_u^p \, d\mu \le K \int_{\Omega' \cap \{u \ne v\}} g_v^p \, d\mu$$

is satisfied.

In particular, every p-harmonic function is a quasi-minimizer with $K = 1$. p-harmonic functions on metric measure spaces have been studied in [C] and [Sh1].

2.13. General setup. A very interesting fact for us is that if the metric measure space is doubling and supports the $(1,p)$-Poincaré inequality with $1 < p < \infty$, then $N^{1,p}(X)$ is reflexive. This result has been proved in [C] by Cheeger. He employs a different definition of Sobolev spaces on a metric measure space using only upper gradients and a concept of generalized upper gradients and bypassing the notions of moduli of path families and weak upper gradients. However, our definition gives rise to the same space as his when $1 < p < \infty$; see [Sh2]. Since the notion of p-weak upper gradients provides insight into the geometric aspect of this function space, we use the definition developed in [Sh1] in the De Giorgi method given here, which itself is a geometric argument. Cheeger has also shown that the minimal upper gradient of a locally Lipschitz function can be obtained as the pointwise Lipschitz constant μ-almost everywhere provided the space is doubling and supports a Poincaré inequality; see section 6 of [C]. There is yet another definition of Sobolev spaces on a metric measure spaces given by Hajłasz [H] based on a maximal function inequality. If the measure is doubling and the space supports a weak $(1,q)$-Poincaré inequality for some q with $1 < q < p$, then all three definitions yield the same space. Therefore doubling and Poincaré type assumptions seem to form a natural context for us to work with.

From now on we assume without further notice that the metric measure space X is equipped with a doubling Borel regular measure for which the measure of every nonempty open set is positive and the measure of every bounded set is finite. Furthermore we assume that the space supports a weak $(1,q)$-Poincaré inequality for some q with $1 < q < p$.

3. Quasi-minimizers and De Giorgi class

In this section we show that quasi-minimizers, and in particular p-harmonic functions, satisfy a Caccioppoli type estimate on level sets.

3.1. Definition. Let Ω be an open subset of X. The function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ belongs to the *De Giorgi class $DG_p(\Omega)$,* if there exists a constant $c > 0$ such that for all $k \in \mathbf{R}$, $z \in \Omega$, and $0 < \rho < R \leq \operatorname{diam}(X)/3$ so that $B(z,R) \subset \Omega$, we have

$$\int_{A_z(k,\rho)} g_u^p \, d\mu \leq \frac{c}{(R-\rho)^p} \int_{A_z(k,R)} (u-k)^p \, d\mu, \tag{3.2}$$

where $A_z(k,r) = \{x \in B(z,r) \colon u(x) > k\}$. In the rest of the discussion we drop the subscript z from $A_z(k,r)$ as $z \in \Omega$ is fixed. Observe that (3.2) is equivalent to

$$\int_{B(z,\rho)} g_{(u-k)_+}^p \, d\mu \leq \frac{c}{(R-\rho)^p} \int_{B(z,R)} (u-k)_+^p \, d\mu, \tag{3.3}$$

where we denote by $(u-k)_+$ the function $\max\{u-k, 0\}$.

To prove the local Hölder continuity, the strong maximum principle and Harnack's inequality for a quasi-minimizer u, we show that u and $-u$ belong to the De Giorgi class and in the subsequent sections we prove that such functions satisfy the corresponding property.

Suppose that u is a quasi-minimizer on Ω. We show that $u \in DG_p(\Omega)$. Let $B(z,R) \subset \Omega$ and $0 < \rho < R \le \operatorname{diam}(X)/3$. Let η be a $c/(R-\rho)$-Lipschitz cutoff function so that $0 \le \eta \le 1$, $\eta = 1$ on $B(z,\rho)$ and the support of η is contained in $B(z,R)$. Set

$$v = u - \eta \max(u-k, 0).$$

Then $u - v \in N_0^{1,p}(A(k,R))$. By the energy quasi-minimizing property of u employed on the subdomain $B(z,R)$ (see section 2.12), we have

$$\int_{A(k,\rho)} g_u^p \, d\mu \le \int_{A(k,R)} g_u^p \, d\mu \le K \int_{A(k,R)} g_v^p \, d\mu.$$

Note that $v = u - \eta(u-k) = (1-\eta)(u-k) + k$ on $A(k,R)$. Hence μ-almost everywhere on this set

$$g_v \le (u-k) g_\eta + (1-\eta) g_u;$$

see Lemma 2.4 in [Sh1] or [C]. Since $g_\eta \le c/(R-\rho)$, we get

$$\begin{aligned} \int_{A(k,\rho)} g_u^p \, d\mu \le & c \int_{A(k,R)} \big((u-k)^p g_\eta^p + (1-\eta)^p g_u^p \big) \, d\mu \\ \le & \frac{c}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu + c \int_{A(k,R)\setminus A(k,\rho)} g_u^p \, d\mu. \end{aligned}$$

Here we used the fact that $1 - \eta = 0$ on $A(k,\rho)$. Adding the term $c\int_{A(k,\rho)} g_u^p$ to the left and right hand sides of the inequality above, we see that

$$(1+c) \int_{A(k,\rho)} g_u^p \, d\mu \le c \int_{A(k,R)} g_u^p \, d\mu + \frac{c}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu.$$

This implies that

$$\int_{A(k,\rho)} g_u^p \, d\mu \le \theta \int_{A(k,R)} g_u^p \, d\mu + \frac{c}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu,$$

where $\theta = c/(c+1) < 1$. Hence, if $0 < \rho < r \le R$, then

$$\int_{A(k,\rho)} g_u^p \, d\mu \le \theta \int_{A(k,r)} g_u^p \, d\mu + \frac{c}{(r-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu. \tag{3.4}$$

Now we recall a technical lemma; see Lemma 5.1 in [Gia].

3.5. Lemma. *Let $R > 0$ and $f\colon (0,R] \to [0,\infty)$ be a bounded function. Suppose that for $0 < \rho < r \le R < \infty$ we have*

$$f(\rho) \le \gamma (r-\rho)^{-\alpha} + \theta f(r) \tag{3.6}$$

with $\alpha > 0$, $0 \le \theta < 1$, and $\gamma \ge 0$. Then there is a constant $c = c(\alpha, \theta)$ so that

$$f(\rho) \le c\gamma (r-\rho)^{-\alpha}. \tag{3.7}$$

for $0 < \rho < r \le R$.

From (3.4) and Lemma 3.5 we conclude that there is a constant c depending only on p and the quasi-minimizer constant K so that

$$\int_{A(k,\rho)} g_u^p \, d\mu \le \frac{c}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu, \tag{3.8}$$

and hence u is in the De Giorgi class. Finally we observe that if u is a quasi-minimizer, then so is $-u$. Thus we have proved the following result.

3.9. Proposition. *Let Ω be an open subset of X. If u is a quasi-minimizer in Ω, then $u \in DG_p(\Omega)$ and $-u \in DG_p(\Omega)$.*

4. De Giorgi class and boundedness

Suppose that a function u is in $DG_p(\Omega)$. Let $0 < R/2 < \rho < R \le \operatorname{diam}(X)/3$ such that $B(z,R) \subset \Omega$. Then $g_{(u-k)_+} \le g_u \chi_{A(k,R)}$ in $B(z,R)$ and by inequality (3.3) we see that

$$\int_{B(z,(R+\rho)/2)} g^p_{(u-k)_+}\,d\mu \le \frac{c}{(R-\rho)^p}\int_{B(z,R)} (u-k)^p_+\,d\mu. \tag{4.1}$$

Here we use the product rule again; see Lemma 2.4 in [Sh1] or [C]. Let η be a $c/(R-\rho)$-Lipschitz cutoff function so that $0 \le \eta \le 1$, the support of η is contained in $B(z,(R+\rho)/2)$, and $\eta = 1$ on $B(z,\rho)$. Then, letting $v = \eta(u-k)_+$, we have

$$g_v \le g_{(u-k)_+}\eta + (u-k)_+ g_\eta \le g_{(u-k)_+} + \frac{c}{R-\rho}(u-k)_+.$$

Inequality (4.1) implies that

$$\begin{aligned}
&\int_{B(z,(R+\rho)/2)} g_v^p\,d\mu \\
&\le c\int_{B(z,(R+\rho)/2)} g^p_{(u-k)_+}\,d\mu + \frac{c}{(R-\rho)^p}\int_{B(z,(R+\rho)/2)} (u-k)^p_+\,d\mu \\
&\le \frac{c}{(R-\rho)^p}\int_{B(z,R)} (u-k)^p_+\,d\mu.
\end{aligned}$$

Since the space supports a weak $(1,p)$-Poincaré inequality, by inequality (2.11) we get $t > p$ (see the discussion after (2.8)) so that

$$\begin{aligned}
\Big(\fint_{B(z,\rho)} (u-k)^t_+\,d\mu\Big)^{p/t} &\le c\Big(\fint_{B(z,(R+\rho)/2)} |v|^t\,d\mu\Big)^{p/t} \\
&\le cR^p\Big(\fint_{B(z,(R+\rho)/2)} g_v^q\,d\mu\Big)^{p/q} \le c\frac{R^p}{(R-\rho)^p}\fint_{B(z,R)} (u-k)^p_+\,d\mu.
\end{aligned} \tag{4.2}$$

The Hölder inequality implies that

$$\fint_{B(z,\rho)} (u-k)^p_+\,d\mu \le \Big(\fint_{B(z,\rho)} (u-k)^t_+\,d\mu\Big)^{p/t}\Big(\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))}\Big)^{1-p/t}.$$

Therefore, inequality (4.2) gives us

$$\begin{aligned}
&\fint_{B(z,\rho)} (u-k)^p_+\,d\mu \\
&\qquad\le c\frac{R^p}{(R-\rho)^p}\Big(\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))}\Big)^{1-p/t}\fint_{B(z,R)} (u-k)^p_+\,d\mu.
\end{aligned} \tag{4.3}$$

Let $h < k$. Then

$$(4.4)\qquad \begin{aligned}(k-h)^p \mu(A(k,\rho)) &= \int_{A(k,\rho)} (k-h)^p \, d\mu \\ &\le \int_{A(k,\rho)} (u-h)^p \, d\mu \le \int_{A(h,\rho)} (u-h)^p \, d\mu.\end{aligned}$$

Let

$$u(h,\rho) = \Big(\fint_{B(z,\rho)} (u-h)_+^p \, d\mu \Big)^{1/p}.$$

Then, by inequality (4.4) and the doubling condition, we have

$$\mu(A(k,\rho)) \le \frac{\mu(B(z,\rho))}{(k-h)^p} u(h,\rho)^p \le c \frac{\mu(B(z,R))}{(k-h)^p} u(h,R)^p,$$

and by inequality (4.3) we obtain

$$(4.5)\qquad \begin{aligned}u(k,\rho) &\le c \frac{R}{R-\rho} \Big(\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))} \Big)^{1/p-1/t} u(k,R) \\ &\le c \frac{R}{R-\rho} (k-h)^{-\theta} u(h,R)^{1+\theta},\end{aligned}$$

where $\theta = 1 - p/t > 0$.

The following proposition is a modification of Proposition 5.1 in [Gia].

4.6. Proposition. *For any number $k_0 \in \mathbf{R}$ we have $u(k_0 + d, R/2) = 0$, where*

$$(4.7)\qquad d^\theta = c\, 2^{(1+\theta)^2/\theta+1} u(k_0,R)^\theta.$$

Here c and θ are as in (4.5).

Proof. Let $k_n = k_0 + d(1 - 2^{-n})$ and $\rho_n = R/2 + 2^{-n-1}R$, $n = 0,1,2,\dots$ Then $\rho_0 = R$, $\rho_n \searrow R/2$, and $k_n \nearrow k_0 + d$ as $n \to \infty$. Next we show that for every $n = 0,1,2,\dots$ we have

$$(4.8)\qquad u(k_n,\rho_n) \le 2^{-\mu n} u(k_0,R),$$

where $\mu = (1+\theta)/\theta$.

It is clear that (4.8) holds when $n = 0$. Suppose then that (4.8) holds for some n. Then, by inequality (4.5), we obtain

$$\begin{aligned}u(k_{n+1},\rho_{n+1}) \le& c \frac{\rho_n}{\rho_n - \rho_{n+1}} (k_{n+1} - k_n)^{-\theta} u(k_n,\rho_n)^{1+\theta} \\ \le& c \frac{R}{2^{-n-2}R} (2^{-n-1}d)^{-\theta} \frac{u(k_0,R)^{1+\theta}}{2^{\mu n(1+\theta)}} = 2^{-\mu(n+1)} u(k_0,R).\end{aligned}$$

Thus (4.8) is proved by induction.

Hence $\lim_{n\to\infty} u(k_n,\rho_n) = 0$. As $k_n \le k_0 + d$ and $R/2 \le \rho_n \le R$ for every $n = 0,1,2,\dots$, using the doubling property we conclude that

$$0 \le u(k_0 + d, R/2) \le c\, u(k_n,\rho_n).$$

The claim follows by letting $k \to \infty$. $\square$

Now we are ready to prove the following weak Harnack inequality; see Theorem 5.1 in [Gia]. It implies that functions in the De Giorgi class are locally bounded and the obtained estimate is a basis of our work.

4.9. Theorem. *Let Ω be an open subset of X, $B(z,R) \subset \Omega$ with $0 < R \le \operatorname{diam}(X)/3$, and $k_0 \in \mathbf{R}$. If $u \in DG_p(\Omega)$, then there is a constant $c > 0$ such that*

$$\sup_{B(z,R/2)} u \le k_0 + c\Big(\fint_{B(z,R)} (u-k_0)_+^p \, d\mu\Big)^{1/p}.$$

The constant c depends only on the constant in De Giorgi's condition, the doubling constant, p, q, and constants c and τ' from (2.8). In particular, the constant is independent of the ball $B(z,R)$.

Proof. By Proposition 4.6 we have $u(k_0+d, R/2) = 0$, where d is as in (4.7). This implies that

$$\sup_{B(z,R/2)} u \le k_0 + d = k_0 + c\Big(\fint_{B(z,R)} (u-k_0)_+^p \, d\mu\Big)^{1/p}.$$

Thus the result follows. $\square$

Taking $k_0 = 0$ we see that the following theorem is true.

4.10. Theorem. *Suppose that the hypothesis of Theorem 4.9 hold and, in addition, $-u \in DG_p(\Omega)$. Then*

$$\sup_{B(z,R/2)} |u| \le c\Big(\fint_{B(z,R)} |u|^p \, d\mu\Big)^{1/p},$$

where $c > 0$ is as in Theorem 4.9.

4.11. Remarks. (1) It is easy to see that there is nothing particular in the factor $1/2$ in the claims of Theorem 4.10. Indeed, if $0 < \rho < r \le R$, then

$$\sup_{B(z,\rho)} |u| \le \frac{c}{(1-\rho/r)^{Q/p}} \Big(\fint_{B(z,r)} |u|^p \, d\mu\Big)^{1/p}.$$

To see this, let $\varepsilon > 0$ and take $y \in B(z,\rho)$ so that $|u(y)|^p \ge (\sup_{B(z,\rho)} |u|)^p - \varepsilon$. Then by Theorem 4.10 we have

$$\begin{aligned}(\sup_{B(z,\rho)} |u|)^p &\le \varepsilon + |u(y)|^p \le \varepsilon + (\sup_{B(y,(r-\rho)/4)} |u|)^p \\ &\le \varepsilon + c\fint_{B(y,(r-\rho)/2)} |u|^p \, d\mu.\end{aligned}$$

Doubling property (2.6) of the measure μ implies that

$$\mu(B(y,(r-\rho)/2)) \ge c(1-\rho/r)^Q \mu(B(z,r))$$

from which the claim follows.

(2) It is useful to observe that the claim of Theorem 4.10 hold for every exponent $q > 0$. To be more precise, for every $q > 0$ there is constant c such that

$$\sup_{B(z,\rho)} |u| \le \frac{c}{(1-\rho/R)^{Q/q}} \Big(\fint_{B(z,R)} |u|^q \, d\mu \Big)^{1/q}$$

when $0 < \rho < R < \infty$. If $q > p$, the claim follows directly from Hölder's inequality. Suppose then that $0 < q < p$ and let $0 < \rho < r \le R$. Then an application of Young's inequality gives

$$\begin{aligned} \sup_{B(z,\rho)} |u| &\le \frac{c}{(1-\rho/r)^{Q/p}} \Big(\fint_{B(z,r)} |u|^q |u|^{p-q} \, d\mu \Big)^{1/p} \\ &\le \frac{c}{(1-\rho/r)^{Q/p}} \Big(\fint_{B(z,r)} |u|^q \, d\mu \Big)^{1/p} \big(\sup_{B(z,r)} |u| \big)^{1-q/p} \\ &\le \varepsilon \sup_{B(z,r)} |u| + \frac{c(\varepsilon)}{(1-\rho/r)^{Q/q}} \Big(\fint_{B(z,r)} |u|^q \, d\mu \Big)^{1/q} \\ &\le \varepsilon \sup_{B(z,r)} |u| + \frac{c(\varepsilon)}{(r-\rho)^{Q/q}} \Big(R^Q \fint_{B(z,R)} |u|^q \, d\mu \Big)^{1/q}, \end{aligned}$$

where $0 < \varepsilon < 1$. In the last inequality we used doubling property (2.6). The claim follows now from Lemma 3.5

5. De Giorgi class and Hölder continuity

The aim of this section is to prove De Giorgi's theorem [DeG], which states that functions in De Giorgi's class are locally Hölder continuous.

Suppose that $u \in DG_p(\Omega)$ and let $0 < r < R < \operatorname{diam}(X)/(3\tau')$ be such that $B(z, 2\tau' R) \subset \Omega$. Assume that $\mu(A(h,R)) \le \gamma \mu(B(z,R))$ for some γ with $0 < \gamma < 1$. Let $k > h$, and define

$$v(x) = \min\{u(x), k\} - \min\{u(x), h\}.$$

Since $u \in N^{1,p}(\Omega)$, we note that $v \in N^{1,p}(\Omega)$. By hypothesis,

$$\mu(\{x \in B(z,R) \colon v(x) > 0\}) \le \gamma \mu(B(z,R)).$$

Since the space is assumed to support a weak $(1,q)$-Poincaré inequality for some q with $1 < q < p$, we may use Lemma 2.10 with $t = q$ and we obtain

$$\begin{aligned} (k-h)\mu(A(k,R)) &= \int_{A(k,R)} v \, d\mu \le \int_{B(z,R)} |v| \, d\mu \\ &\le c\mu(B(z,R))^{1-1/q} \Big(\int_{B(z,R)} |v|^q \, d\mu \Big)^{1/q} \\ &\le cR\mu(B(z,R))^{1-1/q} \Big(\int_{B(z,\tau' R)} g_v^q \, d\mu \Big)^{1/q} \\ &\le cR\mu(B(z,R))^{1-1/q} \Big(\int_{A(h,\tau' R) \setminus A(k,\tau' R)} g_v^q \, d\mu \Big)^{1/q}, \end{aligned}$$

where the constant $c > 0$ has the same dependencies as the constant in Lemma 2.10. Here we used the fact that $g_v = g_v \chi_{\{h<u\le k\}}$ μ-almost everywhere. Hence, by Hölder's inequality we have

$$\begin{aligned}(k-h)\mu(A(k,R)) &\le cR\mu(B(z,R))^{1-1/q} \\ &\quad\cdot \Big(\int_{A(h,\tau' R)} g_v^p \, d\mu\Big)^{1/p} \big(\mu(A(h,\tau' R)) - \mu(A(k,\tau' R))\big)^{1/q-1/p}.\end{aligned}$$

Since $u \in DG_p(\Omega)$, we conclude that for $R < \operatorname{diam}(X)/(3\tau')$ so that $B(z, 2\tau' R) \subset \Omega$,

$$\begin{aligned}(k-h)\mu(A(k,R)) &\le c\mu(B(z,R))^{1-1/q} \\ &\quad\cdot \Big(\int_{A(h,2\tau' R)} (u-h)^p \, d\mu\Big)^{1/p} \big(\mu(A(h,\tau' R)) - \mu(A(k,\tau' R))\big)^{1/q-1/p}.\end{aligned} \tag{5.1}$$

Here c depends on γ and on other parameters.

The following result is Proposition 5.2 in [Gia]. We denote

$$m(R) = \inf_{B(z,R)} u \qquad \text{and} \qquad M(R) = \sup_{B(z,R)} u.$$

By the results of Section 4, $M(R)$ is finite.

5.2. Proposition. *Suppose that $u \in DG_p(\Omega)$ is locally bounded below and let $M = M(2\tau' R)$, $m = m(2\tau' R)$ and $k_0 = (M+m)/2$. If $\mu(A(k_0,R)) \le \gamma\mu(B(z,R))$ for some $0 < \gamma < 1$, then*

$$\lim_{k\to M} \mu(A(k,R)) = 0.$$

Proof. Let $k_i = M - 2^{-(i+1)}(M-m)$, $i = 0,1,2,\ldots$ Then $k_i \nearrow M$ as $i \to \infty$ and $k_0 = (M+m)/2$. Note that

$$M - k_{i-1} = 2^{-i}(M-m) \qquad \text{and} \qquad k_i - k_{i-1} = 2^{-(i+1)}(M-m).$$

By inequality (5.1) we have

$$\begin{aligned}(k_i - k_{i-1})\mu(A(k_i,R)) &\le c\mu(B(z,R))^{1-1/q}\Big(\int_{A(k_{i-1},2\tau' R)} (u-k_{i-1})^p \, d\mu\Big)^{1/p} \\ &\quad\cdot \big(\mu(A(k_{i-1},\tau' R)) - \mu(A(k_i,\tau' R))\big)^{1/q-1/p}.\end{aligned}$$

Therefore, as $u - k_{i-1} \le M - k_{i-1}$ on $A(k_{i-1}, 2\tau' R)$, we conclude that

$$\begin{aligned}2^{-(i+1)}(M-m)\mu(A(k_i,R)) &\le c\mu(B(z,R))^{1-1/q+1/p} \\ &\quad\cdot 2^{-i}(M-m)\big(\mu(A(k_{i-1},\tau' R)) - \mu(A(k_i,\tau' R))\big)^{1/q-1/p}.\end{aligned}$$

Note that if $\nu \ge i$, then $\mu(A(k_\nu,R)) \le \mu(A(k_i,R))$. Hence

$$\begin{aligned}\mu(A(k_\nu,R)) &\le c\mu(B(z,R))^{1-1/q+1/p} \\ &\quad\cdot \big(\mu(A(k_{i-1},\tau' R)) - \mu(A(k_i,\tau' R))\big)^{1/q-1/p}.\end{aligned}$$

Now summing the above inequality over $i = 1, 2, \ldots, \nu$, and using the doubling property, we get

$$\begin{aligned}
&\nu\mu(A(k_\nu, R))^{pq/(p-q)} \\
&\le c\mu(B(z,R))^{pq/(p-q)-1}\big(\mu(A(k_0, \tau' R)) - \mu(A(k_\nu, \tau' R))\big) \\
&\le c\mu(B(z,R))^{pq/(p-q)}.
\end{aligned} \tag{5.3}$$

Therefore, $\lim_{n\to\infty} \mu(A(k_n, R)) = 0$ and hence the result follows by the fact that $\mu(A(k, R))$ is a monotonic decreasing function of k. $\square$

Now we are ready to prove De Giorgi's theorem; see page 82 in [Gia]. Let $\operatorname{osc}(u, B(z,r)) = M(r) - m(r)$ denote the oscillation of u on $B(z,r)$.

5.4. Theorem. *Suppose that both u and $-u$ are in $DG_p(\Omega)$. If $0 < r < R < \operatorname{diam}(X)/(3\tau')$ are such that $B(z, 2\tau' R) \subset \Omega$, then*

$$\operatorname{osc}(u, B(z, \tau' r)) \le 4^\alpha \Big(\frac{r}{R}\Big)^\alpha \operatorname{osc}(u, B(z, \tau' R)),$$

for some α with $0 < \alpha \le 1$ independent of the function u and the ball $B(z,R)$. In particular, u is locally Hölder continuous on Ω.

Proof. Let $k_0 = (M+m)/2$, where M and m are as in Proposition 5.2. If $\mu(A(k_0, R)) > \mu(B(z,R))/2$, then

$$\mu(\{x \in B(z,R) \colon -u(x) \le -k_0\}) > \mu(B(z,R))/2.$$

Consequently we have

$$\mu(\{x \in B(z,R) \colon -u(x) > -k_0\}) \le \mu(B(z,R))/2,$$

and then we can consider $-u$ rather than u in the following discussion. Therefore, without loss of generality, we assume that $\mu(A(k_0, R)) \le \mu(B(z,R))/2$. By Theorem 4.9 with k_0 replaced by $k_\nu = M - 2^{-\nu-1}(M-m)$, $\nu = 0, 1, 2, \ldots$, and by the doubling property, we get

$$M(\tau' R/2) \le k_\nu + c(M(2\tau' R) - k_\nu)\Big(\frac{\mu(A(k_\nu, R))}{\mu(B(z,R))}\Big)^{1/p}$$

with $c > 0$ as in Theorem 4.9. By Proposition 5.2 it is possible to choose an integer ν large enough so that

$$c\Big(\frac{\mu(A(k_\nu, R))}{\mu(B(z,R))}\Big)^{1/p} < \frac{1}{2}.$$

Here, by inequality (5.3), it is possible to choose ν to be independent of the ball $B(z,R)$ and the function u. Note that here $\gamma = 1/2$. Hence

$$M(\tau' R/2) < M(2\tau' R) - (M(2\tau' R) - m(2\tau' R))2^{-(\nu+2)},$$

and therefore

$$M(\tau' R/2) - m(\tau' R/2) \le M(\tau' R/2) - m(2\tau' R)$$
$$< \big(M(2\tau' R) - m(2\tau' R)\big)(1 - 2^{-(\nu+2)}).$$

By the above inequality,

$$\operatorname{osc}(u, B(z, \tau' R/2)) < \lambda \operatorname{osc}(u, B(z, 2\tau' R)), \tag{5.5}$$

where $\lambda = 1 - 2^{-(\nu+2)} < 1$. To complete the proof we iterate inequality (5.5). We choose an integer $j \ge 1$ so that $4^{j-1} \le R/r < 4^j$. Inequality (5.5) implies that

$$\operatorname{osc}(u, B(z, \tau' r)) \le \lambda^{j-1} \operatorname{osc}(u, B(z, \tau' 4^{j-1} r)) \le \lambda^{j-1} \operatorname{osc}(u, B(z, \tau' R)).$$

By the choice of j we conclude that

$$\lambda^{j-1} = 4^{(j-1)(\log \lambda)/\log 4} \le 4^{\alpha} \Big(\frac{R}{r}\Big)^{-\alpha},$$

where $\alpha = -(\log \lambda)/\log 4 \le 1$. Thus we have

$$\operatorname{osc}(u, B(z, \tau' r)) \le 4^{\alpha} \Big(\frac{r}{R}\Big)^{\alpha} \operatorname{osc}(u, B(z, \tau' R)).$$

□

Combining Proposition 3.9 and Theorem 5.4 we conclude that every quasi-minimizer is locally Hölder continuous. In particular, this holds for p-harmonic functions.

6. De Giorgi class and strong maximum principle

It was shown in Theorem 7.17 of [C] and in [Sh1] that p-harmonic functions satisfy the maximum principle on their domain of harmonicity: they achieve their maxima and minima on the boundary of the domain. In this section we prove that quasi-minimizers, and in particular p-harmonic functions, satisfy the strong maximum principle: they do not achieve their maxima and minima in the interior of the domain of harmonicity.

We denote $D_z(\tau, R) = \{x \in B(z, R) \colon u(x) < \tau\}$ and drop the subscript z since $z \in \Omega$ is fixed.

6.1. Lemma. *Suppose that $u \ge 0$ and $-u \in DG_p(\Omega)$. Let $0 < R \le \operatorname{diam}(X)/3$ be such that $B(z, R) \subset \Omega$ and $\tau > 0$. Then there is a constant γ_0, $0 < \gamma_0 < 1$, such that if $\mu(D(\tau, R)) \le \gamma_0 \mu(B(z, R))$, then*

$$\inf_{B(z, R/2)} u \ge \tau/2.$$

Here γ_0 is independent of the ball $B(z, R)$.

Proof. By Theorem 4.9 applied to $-u$, with $k_0 = -\tau$, we see that

$$\sup_{B(z, R/2)} -u \le -\tau + c\Big(\frac{1}{\mu(B(z, R))} \int_{D(\tau, R)} (-u + \tau)^p \, d\mu\Big)^{1/p}.$$

This implies that

$$\begin{aligned}\inf_{B(z,R/2)} u \geq& \tau - c\Big(\frac{1}{\mu(B(z,R))}\int_{D(\tau,R)} (\tau-u)^p\,d\mu\Big)^{1/p}\\ \geq& \tau - c\tau\Big(\frac{\mu(D(\tau,R))}{\mu(B(z,R))}\Big)^{1/p},\end{aligned}$$

where the last inequality was obtained by noting that $\tau - u \leq \tau$. To obtain the claim we choose $\gamma_0 = (2c)^{-p}$. □

6.2. Lemma. *Suppose that the hypothesis of Lemma 6.1 holds. For every γ with $0 < \gamma < 1$ there is a constant $\lambda > 0$ such that if $\mu(D(\tau,R)) \leq \gamma\mu(B(z,R))$, then*

$$\inf_{B(z,R/2)} u \geq \lambda\tau.$$

The constant λ is independent of the ball $B(z,R)$, and depends only on γ, the constants related to the Poincaré inequality, the doubling property, and the constant in the De Giorgi inequality satisfied by u.

Proof. Let $-k > -h$ with $h,k > 0$. We apply (5.1) with u replaced by $-u$, k by $-k$ and h by $-h$ respectively. This gives us

$$\begin{aligned}(h-k)\mu(D(k,R)) \leq& c\mu(B(z,R))^{1-1/q}\\ &\cdot\Big(\int_{D(h,2\tau' R)} (h-u)^p\,d\mu\Big)^{1/p}\big(\mu(D(h,\tau' R)) - \mu(D(k,\tau' R))\big)^{1/q-1/p}.\end{aligned}$$

Then we follow the proof of Proposition 5.2 with $m = \tau$ and $M = 0$. As in (5.3) we conclude that

$$\nu\mu(D(2^{-(\nu+1)}\tau,R))^{pq/(p-q)} \leq c\mu(B(z,R))^{pq/(p-q)}$$

for $\nu = 1,2,\dots$ Hence we can choose ν large enough so that

$$\mu(D(2^{-(\nu+1)}\tau,R)) \leq \gamma_0\mu(B(z,R)),$$

where γ_0 is as in Lemma 6.1. The exponent ν is independent of the ball $B(z,R)$ and u. Now by Lemma 6.1, with τ replaced by $2^{-(\nu+1)}\tau$, we get

$$\inf_{B(z,R/2)} u \geq 2^{-(\nu+2)}\tau.$$

□

6.3. Remark. Suppose that $B(z,6R) \subset \Omega$ and $0 < R \leq \operatorname{diam}(X)/18$. If there exists δ, $0 < \delta < 1$, so that

$$\mu(\{x \in B(z,R)\colon u(x) \geq \tau\}) \geq \delta\mu(B(z,R)),$$

then by the doubling property we have

$$\mu(\{x \in B(z,6R)\colon u(x) \geq \tau\}) \geq \frac{\delta}{c_d^3}\mu(B(z,6R)),$$

where $c_d \geq 1$ is the doubling constant of μ. Hence by Lemma 6.2 we have

$$\inf_{B(z,3R)} u \geq \lambda\tau, \tag{6.4}$$

where $\lambda > 0$ is as in Lemma 6.2. In particular, λ is independent of the ball $B(z,R)$. Clearly we may assume that $0 < \lambda < 1$.

Note that if u is not a constant, for $\tau = \max_{B(z,R)} u$, then there is a number γ with $0 < \gamma < 1$ so that $\mu(D(\tau,R)) \leq \gamma\mu(B(z,R))$. By Lemma 6.2 we conclude that

$$\inf_{B(z,R/2)} u \geq \lambda\tau.$$

This proves that $u > 0$ on $B(z,R/2)$. Thus we obtain the following strong maximum principle for quasi-minimizers.

6.5. Corollary. *Let Ω be an open subset of X and suppose that u is a non-constant quasi-minimizer in Ω. Then u does not obtain its minimum or maximum in Ω.*

7. De Giorgi class and Harnack's inequality

In this section we prove a weak Harnack inequality as in [DT], which together with Theorem 4.10 implies the Harnack inequality.

7.1. Theorem. *If $-u \in DG_p(\Omega)$, $u > 0$, then there are $\sigma > 0$ and $c > 0$ such that*

$$\inf_{B(z,3R)} u \geq c\Big(\fint_{B(z,R)} u^\sigma \, d\mu\Big)^{1/\sigma} \tag{7.2}$$

for every ball $B(z,R)$ such that $B(z,6R) \subset \Omega$ with $0 < R \leq \operatorname{diam}(X)/18$. The constants σ and c are independent of the ball $B(z,R)$.

We begin by proving the Krylov-Safonov covering theorem [KS] on a doubling metric measure space.

7.3. Lemma. *Let $B(z,R)$ be a ball in X, and $E \subset B(z,R)$ be μ-measurable. Let $0 < \delta < 1$, and define*

$$E_\delta = \bigcup_{\rho>0} \Big\{B(y,3\rho) \cap B(z,R)\colon y \in B(z,R),\ \mu(E \cap B(y,3\rho)) > \delta\mu(B(y,\rho))\Big\}$$

Then, either $E_\delta = B(z,R)$, or else $\mu(E_\delta) \geq (c_d\delta)^{-1}\mu(E)$, where $c_d \geq 1$ is the doubling constant of μ.

Proof. We define a maximal operator $M\colon B(z,R) \to \mathbf{R}$ by setting

$$M(x) = \sup \frac{\mu(E \cap B(y,3\rho))}{\mu(B(y,\rho))},$$

where the supremum is taken over all open balls $B(y,\rho)$, with $y \in B(z,R)$, such that $x \in B(y,3\rho)$.

We claim that

$$E_\delta = \{x \in B(z,R) \colon M(x) > \delta\}$$

for every δ with $0 < \delta < 1$. To see this let $x \in B(z,R)$ such that $M(x) > \delta$. Then there is a ball $B(y,\rho)$, $y \in B(z,R)$, such that $\mu(E \cap B(y,3\rho)) > \delta\mu(B(y,\rho))$ and $x \in B(y,3\rho)$. This means that $x \in E_\delta$. On the other hand, if $x \in E_\delta$, there is ball $B(y,\rho)$, $y \in B(z,R)$, such that $\mu(E \cap B(y,3\rho)) > \delta\mu(B(y,\rho))$ and $x \in B(y,3\rho)$. This implies that $M(x) > \delta$.

Suppose that $B(z,R) \setminus E_\delta \neq \emptyset$. The set E_δ is open by definition. We cover E_δ by balls $B(x,r_x)$, where $x \in E_\delta$ and $r_x = \operatorname{dist}(x, B(z,R) \setminus E_\delta)/2$. By the Vitali type covering lemma, see p. 69 in [CW], there are countably many pairwise disjoint balls $B(x_i,r_i)$, where $r_i = r_{x_i}$, $i = 1,2,\ldots$, such that

$$E_\delta \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i).$$

Then $B(x_i,5r_i) \cap (B(z,R) \setminus E_\delta) \neq \emptyset$ for every $i = 1,2,\ldots$ and there is a point $y_i \in B(x_i,5r_i) \cap (B(z,R) \setminus E_\delta)$. In particular, $M(y_i) \leq \delta$, $i = 1,2,\ldots$ Since $x_i \in B(y_i,5r_i)$, we conclude that

$$\mu(E \cap B(x_i,5r_i)) \leq \delta\mu(B(x_i,\frac{5}{3}r_i)) \leq c_d\delta\mu(B(x_i,r_i)),$$

where we also used the doubling property. If y is a density point of E, then

$$\liminf_{\rho\to 0} \frac{\mu(E \cap B(y,3\rho))}{\mu(B(y,\rho))} \geq \lim_{\rho\to 0} \frac{\mu(E \cap B(y,\rho))}{\mu(B(y,\rho))} = 1 > \delta.$$

Since μ-almost every point of E is a density point, we observe that μ-almost every point of E belongs to E_δ for every $0 < \delta < 1$. From this it follows that

$$\begin{aligned}\mu(E) = \mu(E \cap E_\delta) &\leq \sum_{i=1}^{\infty} \mu(E \cap B(x_i,5r_i)) \\ &\leq c_d\delta \sum_{i=1}^{\infty} \mu(B(x_i,r_i)) \leq c_d\delta\mu(E_\delta).\end{aligned}$$

The above inequality yields the desired result. □

Proof of Theorem 7.1. Suppose that $0 < \delta < 1$ and λ, $0 < \lambda < 1$, is the constant in (6.4) corresponding to δ. Let $t > 0$ and denote

$$A_{t,i} = \{x \in B(z,R) \colon u(x) \geq t\lambda^i\}, \qquad i = 0,1,2,\ldots$$

We apply Lemma 7.3 with $E = A_{t,i-1}$. If there is a point $x \in B(z,R)$ and $\rho > 0$ so that

$$\mu(A_{t,i-1} \cap B(x,3\rho)) \geq \delta\mu(B(x,\rho)),$$

then

$$\mu(A_{t,i-1}\cap B(x,6\rho)) \geq \frac{\delta}{c_d^3}\mu(B(x,6\rho)),$$

and by Remark 6.3 we have

$$\inf_{B(x,3\rho)} u \geq t\lambda^i.$$

Hence if $B(x,3\rho)$ is one of the balls going to make up the set E_δ in Lemma 7.3, then $B(x,3\rho)\cap B(z,R)\subset A_{t,i}$. This implies that $E_\delta \subset A_{t,i}$. Hence by the Krylov-Safonov covering theorem we conclude that

$$\frac{1}{c_d\delta}\mu(A_{t,i-1}) \leq \mu(E_\delta) \leq \mu(A_{t,i}) \tag{7.4}$$

or $A_{t,i} = B(z,R)$. Let $0<\delta<1/c_d$. We choose an integer $j\geq 1$ so that

$$(c_d\delta)^j \leq \mu(A_{t,0})/\mu(B(z,R)) \leq (c_d\delta)^{j-1}.$$

Then by (7.4) we obtain

$$\mu(A_{t,j-1}) \geq \frac{1}{c_d\delta}\mu(A_{t,j-2}) \geq \cdots \geq \frac{1}{(c_d\delta)^{j-1}}\mu(A_{t,0}) \geq c_d\delta\mu(B(z,R)).$$

By Remark 6.3 we see that

$$\inf_{B(z,3R)} u \geq ct\lambda^{j-1}.$$

Here c is the constant in (6.4) corresponding the factor $c_d\delta$. This implies that

$$\inf_{B(z,3R)} u \geq ct\lambda^{j-1} = ct(c_d\delta)^{(j-1)(\log\lambda)/\log(c_d\delta)} \geq ct\Big(\frac{\mu(A_{t,0})}{\mu(B(z,R))}\Big)^\gamma,$$

where $\gamma = \log\lambda/\log(c_d\delta)$. Consequently we obtain the estimate

$$\frac{\mu(A_{t,0})}{\mu(B(z,R))} \leq ct^{-1/\gamma}\inf_{B(z,3R)} u^{1/\gamma}.$$

On the other hand, for $\sigma>0$ we compute

$$\begin{aligned}
⨍_{B(z,R)} u^\sigma\,d\mu =& \frac{\sigma}{\mu(B(z,R))}\int_0^\infty t^{\sigma-1}\mu(A_{t,0})\,dt \\
&\leq \frac{\sigma}{\mu(B(z,R))}\int_\xi^\infty t^{\sigma-1}\mu(A_{t,0})\,dt + \sigma\int_0^\xi t^{\sigma-1}\,dt \\
&\leq c\int_\xi^\infty t^{\sigma-1-1/\gamma}\xi^{1/\gamma}\,dt + \xi^\sigma,
\end{aligned}$$

where $\xi = \inf_{B(z,3R)} u$. If $\sigma < 1/\gamma$, then

$$⨍_{B(z,R)} u^\sigma\,d\mu \leq c\xi^{1/\gamma}(1/\gamma-\sigma)^{-1}\xi^{\sigma-1/\gamma} + \xi^\sigma \leq c\xi^\sigma,$$

and hence

$$\inf_{B(z,3R)} u \geq c\Big(⨍_{B(z,R)} u^\sigma\,d\mu\Big)^{1/\sigma}.$$

This completes the proof. □

Combining Theorem 4.10 (with Remark 4.11 (2)) and Theorem 7.1 we obtain Harnack's inequality.

7.5. Corollary. *Suppose that $u > 0$, $u \in DG_p(\Omega)$ and $-u \in DG_p(\Omega)$. Then there exists a constant $c \geq 1$ so that*

$$\sup_{B(z,R)} u \leq c \inf_{B(z,R)} u$$

for every ball $B(z,R)$ for which $B(z,6R) \subset \Omega$ and $0 < R \leq \operatorname{diam}(X)/18$. Here the constant c is independent of the ball $B(z,R)$ and function u.

In particular, by Proposition 3.9 Harnack's inequality holds for nonnegative quasi-minimizers and p-harmonic functions. We obtain Liouville's theorem as a consequence of the Harnack inequality: if X is unbounded and u is a p-harmonic function on all of X, then either u is constant or it is unbounded.

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