LOCAL BEHAVIOUR OF SOLUTIONS TO DOUBLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. We give a relatively simple and transparent proof for Harnack's inequality for certain degenerate doubly nonlinear parabolic equations. We consider the case where the Lebesgue measure is replaced with a doubling Borel measure which supports a Poincaré inequality.

1. Introduction

Our purpose is to study the local behaviour of nonnegative weak solutions to the doubly nonlinear parabolic equation

$$\operatorname{div}(|Du|^{p-2}Du) = \frac{\partial(u^{p-1})}{\partial t}, \qquad 1 < p < \infty. \tag{1.1}$$

When $p = 2$ we have the standard heat equation. Observe that the solutions to (1.1) can be scaled by nonnegative factors, but due to the nonlinearity of the term $(u^{p-1})_t$ we cannot add a constant to a solution. As far as we know, equation (1.1) has first been studied by Trudinger in [Tru], where he proved a Harnack inequality for nonnegative weak solutions. The proof was based on Moser's celebrated work [Mo1] and used a parabolic version of the John-Nirenberg lemma. Twenty years later the proof of the parabolic John-Nirenberg lemma was simplified by Fabes and Garofalo, see [FaGa]. However, the parabolic BMO remains to be technically demanding. Our main objective is to give a relatively simple and transparent proof for Harnack's inequality using the approach of Moser in [Mo2]. In particular, the parabolic John-Nirenberg lemma is replaced with a lemma due to Bombieri in [BoGi] and [Bomb]. Let us point out a slightly unexpected phenomenon related to the parabolic BMO. In the case $p = 2$ it is known that if u is a nonnegative solution, then $\log u$ is a subsolution to the same equation. However, if $p \neq 2$, then $\log u$ is not a subsolution to equation (1.1). Instead it is a subsolution to an equation of the p-parabolic type studied in [DiBe].

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To show that our proof is based on a general principle we consider the case where the Lebesgue measure is replaced with a more general Borel measure. The measure is assumed to be doubling and to support a Poincaré inequality. The precise definitions will be given below. The corresponding result in the elliptic case for measures induced by Muckenhoupt's weights has been studied by Fabes, Kenig and Serapioni in [FKS]. See also [ChFr]. The weighted theory in the parabolic case has been studied by Chiarenza and Serapioni in [ChSe]. However, in their approach the role of the measure is somewhat different compared to ours. For the heat equation Grigor'yan and Saloff-Coste observed that the doubling condition and the Poincaré inequality are not only sufficient but also necessary conditions for a scale invariant parabolic Harnack principle on Riemannian manifolds, see [SaCo1], [SaCo2] and [Gri]. Our contibution is to show the sufficiency for the general $p \neq 2$ in a Euclidean space. It is a very interesting question whether also the necessity holds in this case. Moreover, the doubling condition and the Poincaré inequality are rather standard assumptions in analysis on metric spaces, see for example [HaK] and references therein. It is well known that Moser's technique is essentially based on a combination of a Sobolev and a Caccioppoli type inequalities. We take a full advatange of a metric space result, which states that the doubling property and the Poincaré inequality imply a Sobolev type inequality, see [BCLS], [HaK], [SaCo1], [SaCo2].

Our argument applies to more general equations of the type

$$\mathrm{div} A(x,t,u,Du) = \frac{\partial(u^{p-1})}{\partial t},$$

where A is a Caratheodory function and satisfies the standard structural conditions (see for example [DiBe], [DBUV], [WZYL])

$$A(x,t,u,Du) \cdot Du \geq C_0 |Du|^p$$

and

$$|A(x,t,u,Du)| \leq C_1 |Du|^{p-1},$$

where C_0 and C_1 are positive constants. However, for expository purposes, we only consider equation (1.1).

2. Preliminaries

In this section we describe our assumptions and results more precisely. Let μ be a Borel measure and suppose that Ω is an open set in $\mathbb{R}^n$. The Sobolev space $H^{1,p}(\Omega,\mu)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the Sobolev norm

$$\|u\|_{1,p,\Omega} = \Big(\int_\Omega |u|^p \, d\mu\Big)^{1/p} + \Big(\int_\Omega |Du|^p \, d\mu\Big)^{1/p}.$$

A function belongs to the local Sobolev space $H^{1,p}_{loc}(\Omega,\mu)$ if it belongs to $H^{1,p}(\Omega',\mu)$ for every open subset Ω' of Ω, whose closure is a compact subset of Ω. The Sobolev space with zero boundary values $H^{1,p}_0(\Omega,\mu)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm. For the basic properties of weighted Sobolev spaces we refer to [HKM].

We denote by $L^p(t_1,t_2;H^{1,p}(\Omega))$, $t_1<t_2$, the space of functions such that for almost every t, $t_1<t<t_2$, the function $x\mapsto u(x,t)$ belongs to $H^{1,p}(\Omega,\mu)$ and

$$\int_{t_1}^{t_2}\int_\Omega \big(|u(x,t)|^p+|Du(x,t)|^p\big)\,d\mu(x)\,dt<\infty.$$

Notice that the time derivative u_t is deliberately avoided. The definition for the space $L^p_{loc}(t_1,t_2;H^{1,p}_{loc}(\Omega,\mu))$ is clear.

Let $t_1<t_2$ and $1<p<\infty$. A nonnegative function u which belongs to $L^p_{loc}(t_1,t_2;H^{1,p}_{loc}(\Omega,\mu))$ is a weak solution to (1.1) in $\Omega\times(t_1,t_2)$ if

$$\int_{t_1}^{t_2}\int_\Omega \Big(|Du|^{p-2}Du\cdot D\eta-u^{p-1}\frac{\partial\eta}{\partial t}\Big)\,d\mu\,dt=0 \tag{2.1}$$

for all $\eta\in C_0^\infty(\Omega\times(t_1,t_2))$. Further, we say that u is a supersolution to (1.1), if the integral (2.1) is nonnegative for all $\eta\in C_0^\infty(\Omega\times(t_1,t_2))$ with $\eta\ge 0$. If this integral is nonpositive, we say that u is a subsolution.

Let $t_1<\tau_1<\tau_2<t_2$. If the test function η vanishes only on the lateral boundary $\partial\Omega\times(\tau_1,\tau_2)$, then the boundary terms

$$\int_\Omega u(x,\tau_1)^{p-1}\eta(x,\tau_1)\,d\mu=\lim_{\sigma\to 0}\frac{1}{\sigma}\int_{\tau_1}^{\tau_1+\sigma}\int_\Omega u(x,t)^{p-1}\eta(x,t)\,d\mu\,dt$$

and

$$\int_\Omega u(x,\tau_2)^{p-1}\eta(x,\tau_2)\,d\mu=\lim_{\sigma\to 0}\frac{1}{\sigma}\int_{\tau_2-\sigma}^{\tau_2}\int_\Omega u(x,t)^{p-1}\eta(x,t)\,d\mu\,dt$$

have to be included. In the case of a supersolution to the doubly nonlinear equation (1.1) the condition becomes

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\int_\Omega |Du|^{p-2}Du\cdot D\eta\,d\mu\,dt\\&\qquad+\Big[\int_\Omega u^{p-1}\eta\,d\mu\Big]_{t=\tau_1}^{\tau_2}-\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}\frac{\partial\eta}{\partial t}\,d\mu\,dt\ge 0\end{aligned} \tag{2.2}$$

for almost every τ_1,τ_2 with $t_1<\tau_1<\tau_2<t_2$.

The measure μ is doubling, if there exists a universal constant $D_0\ge 1$ such that

$$\mu(B(z,2R))\le D_0\mu(B(z,R)) \tag{2.3}$$

for every $z\in\mathbb{R}^n$ and $R>0$. Here $B(z,R)$ denotes the open ball with center z and radius R. The dimension related to a doubling measure is defined as $d_\mu = \log_2 D_0$. Note that in the case of the Lebesgue measure the dimension is n.

The measure is said to support a weak $(1,p)$-Poincaré inequality if there exist constants $P_0 > 0$ and $\tau \geq 1$ such that

$$\fint_{B(z,R)} |v - v_{B(z,R)}| \, d\mu \leq P_0 R \Big(\fint_{B(z,\tau R)} |Dv|^p \, d\mu \Big)^{1/p}, \tag{2.4}$$

for every $v \in H^{1,p}_{loc}(\mathbb{R}^n, \mu)$, $z \in \mathbb{R}^n$ and $R > 0$. Here we use the notation

$$v_{B(z,R)} = \fint_{B(z,R)} v \, d\mu = \frac{1}{\mu(B(z,R))} \int_{B(z,R)} v \, d\mu.$$

The word weak refers to the possibility that $\tau > 1$. If $\tau = 1$, the space is said to support a $(1,p)$-Poincaré inequality.

From now on we assume that the measure μ is doubling and supports a weak $(1,p)$-Poincaré inequality. Moreover, we assume that the measure is nontrivial in the sense that the measure of every nonempty open set is strictly positive and measure of every bounded set is finite.

Let $0 < \sigma \leq 1$, $\tau \in \mathbb{R}$ and $B(z,r)$ be a ball in $\mathbb{R}^n$. We denote

$$U = B(z,r) \times (\tau - r^p, \tau + r^p),$$

$$\sigma U^+ = B(z,\sigma r) \times \Big(\tau + \frac{1}{2} r^p - \frac{1}{2} (\sigma r)^p, \tau + \frac{1}{2} r^p + \frac{1}{2} (\sigma r)^p \Big)$$

and

$$\sigma U^- = B(z,\sigma r) \times \Big(\tau - \frac{1}{2} r^p - \frac{1}{2} (\sigma r)^p, \tau - \frac{1}{2} r^p + \frac{1}{2} (\sigma r)^p \Big).$$

We give a proof for the following scale invariant parabolic Harnack inequality.

Theorem 2.5. *Let $1 < p < \infty$ and assume that the measure μ is doubling and supports a weak $(1,p)$-Poincaré inequality. Let $u \geq \rho > 0$ be a weak solution to (1.1) in U and let $0 < \sigma < 1$. Then we have*

$$\operatorname*{ess\,sup}_{\sigma U^-} u \leq C \operatorname*{ess\,inf}_{\sigma U^+} u, \tag{2.6}$$

where the constant C depends only on p, D_0, P_0 and σ.

Note carefully that the constant in (2.6) is independent of ρ. A modification of the proof shows that the technical assumption $u \geq \rho$ can be removed and that the result holds for all nonnegative solutions.

The original proof with the Lebesgue measure is due to Trudinger [Tru]. For a different approach we refer to a recent work by Gianazza and Vespri [GiVe].

It is well-known that the local Hölder continuity of a weak solution is a consequence of the Harnack inequality when $p = 2$, see [Mo1]. However, due to the nonlinearity of the term $(u^{p-1})_t$ when $p \neq 2$, it is not clear how to modify the same proof for the doubly nonlinear equation (1.1). The local Hölder continuity of the solution has been proved in [Ve] using a different method.

In the Euclidean space the weak $(1,p)$-Poincaré inequality implies the $(1,p)$-Poincaré inequality, if the measure is doubling, see Theorem 3.4 in [HaK]. Thus we may assume that $\tau = 1$ in (2.4). On the other hand, these assumptions imply a weak (κ,p)-Sobolev-Poincaré inequality with

$$\kappa = \begin{cases} \dfrac{d_\mu p}{d_\mu - p}, & 1 < p < d_\mu, \\ 2, & p \geq d_\mu, \end{cases} \tag{2.7}$$

where d_μ is the dimension related to the measure. More precisely, there are constants $C > 0$ and $\tau' \geq 1$ such that

$$\Big(⨍_{B(z,R)} |v - v_{B(z,R)}|^\kappa \, d\mu \Big)^{1/\kappa} \leq CR \Big(⨍_{B(z,\tau' R)} |Dv|^p \, d\mu \Big)^{1/p}, \tag{2.8}$$

for every $z \in \mathbb{R}^n$ and $R > 0$. The constant C depends only on p, D_0 and P_0. For the proof, we refer to [BCLS] and [HaK]. Again, by Theorem 3.4 in [HaK] we may take $\tau' = 1$ in (2.8).

For Sobolev functions with the zero boundary values we have the following version of Sobolev's inequality. Suppose that $v \in H_0^{1,p}(B(z,R),\mu)$. Then

$$\Big(⨍_{B(z,R)} |v|^\kappa \, d\mu \Big)^{1/\kappa} \leq CR \Big(⨍_{B(z,R)} |Dv|^p \, d\mu \Big)^{1/p}. \tag{2.9}$$

For the proof we refer, for example, to [KS].

The following weighted Poincaré inequality is a consequence of the doubling property (2.3) and the $(1,p)$-Poincaré inequality (2.4). We refer to Theorem 5.3.4 in [SaCo1].

Theorem 2.10. *Suppose that $u \in H^{1,p}(B(z,R),\mu)$. Let*

$$\phi(x) = \Big(1 - \frac{|x-z|}{R} \Big)_+^\theta,$$

where $\theta > 0$. Then there exists a constant $C = C(p, D_0, P_0, \theta)$ such that for all $0 < r < R$

$$⨍_{B(z,r)} |u - u_\phi|^p \phi \, d\mu \leq C r^p ⨍_{B(z,r)} |Du|^p \phi \, d\mu,$$

where

$$u_\phi = \frac{\int_{B(z,r)} u\phi \, d\mu}{\int_{B(z,r)} \phi \, d\mu}.$$

We use the following modification of an abstract lemma originally due to Bombieri [BoGi] and [Bomb]. Our proof follows closely [Mo2] and Lemma 2.2.6 in [SaCo1].

Lemma 2.11. *Let ν be a Borel measure and θ, A and γ be positive constants, $0 < \delta < 1$ and $0 < q \leq \infty$. Let U_σ be bounded measurable sets with $U_{\sigma'} \subset U_\sigma$ for $0 < \delta \leq \sigma' < \sigma \leq 1$. Moreover, if $q < \infty$, we assume that the doubling condition $\nu(U_1) \leq A\nu(U_\delta)$ holds. Let f be a positive measurable function on U_1 which satisfies the reverse Hölder inequality*

$$\Big(\fint_{U_{\sigma'}} f^q \, d\nu\Big)^{1/q} \leq \Big(\frac{A}{(\sigma-\sigma')^\theta} \fint_{U_\sigma} f^s \, d\nu\Big)^{1/s}$$

with $0 < s < q$. Assume further that f satisfies

$$\nu(\{x \in U_1 | \log f > \lambda\}) \leq \frac{A\nu(U_\delta)}{\lambda^\gamma}$$

for all $\lambda > 0$. Then

$$\Big(\fint_{U_\delta} f^q \, d\nu\Big)^{1/q} \leq C,$$

where C depends only on θ, δ, γ, q and A.

Proof. We denote

$$\psi = \psi(\sigma) = \log\Big(\fint_{U_\sigma} f^q \, d\nu\Big)^{1/q}.$$

Hölder's inequality gives

$$\begin{aligned}\fint_{U_\sigma} f^s \, d\nu =& \frac{1}{\nu(U_\sigma)} \int_{\log f \leq \psi/2} f^s \, d\nu + \frac{1}{\nu(U_\sigma)} \int_{\log f > \psi/2} f^s \, d\nu \\ \leq& \exp(\psi s/2) + \Big(\fint_{U_\sigma} f^q \, d\nu\Big)^{s/q} \Big(\frac{\nu(\{\log f > \psi/2\})}{\nu(U_\sigma)}\Big)^{(q-s)/q} \\ \leq& \exp(\psi s/2) + \exp(\psi s)\left(\frac{A}{(\psi/2)^\gamma}\right)^{(q-s)/q}.\end{aligned}$$

Let ψ be so large that

$$0 < \log\big(\psi^\gamma / A 2^\gamma\big) \leq q\psi.$$

The obtained lower bound on ψ depends on A, γ and q. We call it A_1. If we choose

$$s = \frac{2}{3}\psi^{-1} \log\big(\psi^\gamma / A 2^\gamma\big),$$

then $0 < s < q$ and we have

$$\fint_{U_\sigma} f^s \, d\nu \leq 2\exp(\psi s/2).$$

Next, we take a logarithm of the reverse Hölder inequality and use the estimate above. This implies that

$$\begin{aligned}\psi(\sigma') \leq& \frac{1}{s}\left(\log\Big(\frac{2A}{(\sigma-\sigma')^\theta}\Big) + \psi(\sigma)s/2\right)\\ =&\frac{\psi(\sigma)}{2}\left(3\log\Big(\frac{2A}{(\sigma-\sigma')^\theta}\Big)/\log\big(\psi^\gamma/A2^\gamma\big)+1\right).\end{aligned}$$

Suppose that

$$\psi^\gamma/A2^\gamma \geq \Big(\frac{2A}{(\sigma-\sigma')^\theta}\Big)^6$$

from which it follows that

$$\psi \geq \frac{A_2}{(\sigma-\sigma')^{6\theta/\gamma}},$$

where A_2 depends only on A and γ. For $\psi(\sigma)$ large we have

$$\psi(\sigma') \leq \frac{3}{4}\psi(\sigma).$$

On the other hand, if

$$\psi(\sigma) \leq \min\Big(A_1, \frac{A_2}{(\sigma-\sigma')^{6\theta/\gamma}}\Big),$$

then the doubling condition implies that

$$\psi(\sigma') \leq \log\Big(\frac{\nu(U_\sigma)}{\nu(U_{\sigma'})}\Big) + \psi(\sigma) \leq \log A + \min\Big(A_1, \frac{A_2}{(\sigma-\sigma')^{6\theta/\gamma}}\Big).$$

From this we conclude that here exists a constant C depending only on A, γ and q such that

$$\psi(\sigma') \leq \frac{3}{4}\psi(\sigma) + C\Big(1 + \frac{1}{(\sigma-\sigma')^{6\theta/\gamma}}\Big).$$

The claim follows by a standard iteration argument (see for example Lemma 5.1 in [Giaq]). □

3. Estimates for super- and subsolutions

The following four Caccioppoli type estimates are essentially consequences of choosing a correct test function in (2.1). There is a well-recognized difficulty with the test functions. Namely, in proving estimates we usually need a test function which depends on the solution itself. Then we cannot avoid that the "forbidden quantity" u_t shows up in the calculation of η_t. In most cases one can easily overcome this difficulty by using an equivalent definition in terms of Steklov averages, as on pages 18 and 25 in [DiBe] and in Chapter 2 of [WZYL]. Alternatively, one can proceed using convolutions with smooth mollifiers as on pages 199–121 in [AS]. Observe that the mollification is taken with respect to the time variable only.

Lemma 3.1. *Suppose that $u \geq \rho > 0$ is a supersolution in $\Omega \times (t_1, t_2)$. Then $v = u^{-1}$ is a subsolution.*

Proof. Let $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$. Formally we choose the test function $\eta = u^{2(1-p)}\varphi$. Then

$$D\eta = -2(p-1)u^{1-2p}\varphi Du + u^{2(1-p)}D\varphi$$

and

$$\frac{\partial \eta}{\partial t} = -2(p-1)u^{1-2p}\varphi\frac{\partial u}{\partial t} + u^{2(1-p)}\frac{\partial \varphi}{\partial t}.$$

A substitution in (2.2) leads to

$$\begin{aligned} 0 \leq &- 2(p-1)\int_{t_1}^{t_2}\int_\Omega |Du|^p u^{1-2p}\varphi\, d\mu\, dt \\ &+ \int_{t_1}^{t_2}\int_\Omega u^{2(1-p)}|Du|^{p-2}Du \cdot D\varphi\ d\mu\, dt \\ &+ 2(p-1)\int_{t_1}^{t_2}\int_\Omega u^{-p}\varphi\frac{\partial u}{\partial t}\, d\mu\, dt - \int_{t_1}^{t_2}\int_\Omega u^{1-p}\frac{\partial \varphi}{\partial t}\, d\mu\, dt. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} \int_{t_1}^{t_2}\int_\Omega u^{-p}\varphi\frac{\partial u}{\partial t}\, d\mu\, dt &= -\frac{1}{p-1}\int_{t_1}^{t_2}\int_\Omega \frac{\partial(u^{1-p})}{\partial t}\varphi\, d\mu\, dt \\ &= \frac{1}{p-1}\int_{t_1}^{t_2}\int_\Omega u^{1-p}\frac{\partial \varphi}{\partial t}\, d\mu\, dt. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} 0 &\leq \int_{t_1}^{t_2}\int_\Omega |Du|^{p-2}Du \cdot D\varphi\, u^{2(1-p)}\, d\mu\, dt + \int_{t_1}^{t_2}\int_\Omega u^{1-p}\frac{\partial \varphi}{\partial t}\, d\mu\, dt \\ &= -\int_{t_1}^{t_2}\int_\Omega \Big(|Dv|^{p-2}Dv \cdot D\varphi - v^{p-1}\frac{\partial \varphi}{\partial t}\Big)\, d\mu\, dt. \end{aligned}$$

Here we used the fact that $Du = -v^{-2}Dv$. □

Lemma 3.2. *Suppose that $u \geq \rho > 0$ is a supersolution in $\Omega \times (t_1, t_2)$ and let $\varepsilon > 0$ with $\varepsilon \neq p-1$. Then there exists a constant $C = C(p, \varepsilon)$ such that*

$$\begin{aligned} &\int_{t_1}^{t_2}\int_\Omega |Du|^p u^{-\varepsilon-1}\varphi^p\, d\mu\, dt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\, d\mu \\ &\qquad \leq C\int_{t_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}|D\varphi|^p\, d\mu\, dt + C\int_{t_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^{p-1}\Big|\frac{\partial \varphi}{\partial t}\Big|\, d\mu\, dt \end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$.

Proof. Formally we choose the test function $\eta = u^{-\varepsilon}\varphi^p$ so that

$$D\eta = -\varepsilon u^{-\varepsilon-1}\varphi^p Du + u^{-\varepsilon}D(\varphi^p)$$

and

$$\frac{\partial \eta}{\partial t} = -\varepsilon u^{-\varepsilon-1}\varphi^p \frac{\partial u}{\partial t} + u^{-\varepsilon}\frac{\partial(\varphi^p)}{\partial t},$$

where $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$. Let $t_1 < \tau_1 < \tau_2 < t_2$. We integrate by parts and get

$$\begin{aligned}
-\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}\frac{\partial \eta}{\partial t}\,d\mu\,dt + \left[\int_\Omega u^{p-1}\eta\,d\mu\right]_{t=\tau_1}^{\tau_2} \\
=&\frac{\varepsilon}{p-1-\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega \frac{\partial(u^{p-1-\varepsilon})}{\partial t}\varphi^p\,d\mu\,dt - \int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1-\varepsilon}\frac{\partial(\varphi^p)}{\partial t}\,d\mu\,dt \\
&+ \left[\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} \\
\leq&\frac{p(p-1)}{|p-1-\varepsilon|}\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial \varphi}{\partial t}\right|\varphi^{p-1}\,d\mu\,dt \\
&+ \frac{p-1}{p-1-\varepsilon}\left[\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\,dt\right]_{t=\tau_1}^{\tau_2}.
\end{aligned}$$

Hence a substitution of η in (2.2) gives

$$\begin{aligned}
0 \leq& -\varepsilon\int_{\tau_1}^{\tau_2}\int_\Omega |Du|^p u^{-\varepsilon-1}\varphi^p\,d\mu\,dt \\
&+ p\int_{\tau_1}^{\tau_2}\int_\Omega |Du|^{p-1}\varphi^{p-1}|D\varphi|u^{-\varepsilon}\,d\mu\,dt \\
&+ \frac{p(p-1)}{|p-1-\varepsilon|}\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial \varphi}{\partial t}\right|\varphi^{p-1}\,d\mu\,dt \\
&+ \frac{p-1}{p-1-\varepsilon}\left[\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\,dt\right]_{t=\tau_1}^{\tau_2} \\
=& -\varepsilon I_1 + pI_2 + \frac{p(p-1)}{|p-1-\varepsilon|}I_3 + \frac{p-1}{p-1-\varepsilon}I_4.
\end{aligned}$$

Young's inequality implies

$$\begin{aligned}
I_2 &= \int_{\tau_1}^{\tau_2}\int_\Omega \big(|Du|\varphi u^{-(\varepsilon+1)/p}\big)^{p-1}\big(|D\varphi|u^{-\varepsilon+(\varepsilon+1)(p-1)/p}\big)\,d\mu\,dt \\
&\leq \gamma I_1 + c(\gamma)\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p u^{-\varepsilon p+(\varepsilon+1)(p-1)}\,d\mu\,dt \\
&= \gamma I_1 + c(\gamma)\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\,d\mu\,dt,
\end{aligned}$$

where $\gamma > 0$. Thus we have

$$\begin{aligned}
I_1 &- \frac{2(p-1)}{\varepsilon(p-1-\varepsilon)}I_4 \\
&\leq \frac{2p\,c(\varepsilon/2)}{\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\,d\mu\,dt + \frac{2p(p-1)}{\varepsilon|p-1-\varepsilon|}I_3,
\end{aligned}$$

where we have chosen $\gamma = \varepsilon/2p$. Furthermore, if $\varepsilon < p-1$ by choosing $\tau_2 = t_2$ and $\tau_1 = \tau > t_1$ such that

$$\int_\Omega u^{p-1-\varepsilon}(x,\tau)\varphi^p(x,\tau)\,d\mu \geq \frac{1}{2}\operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu$$

we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu \\
&\leq C\int_{\tau_1}^{t_2}\int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\,d\mu\,dt + C\int_{\tau_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}\Big|\frac{\partial\varphi}{\partial t}\Big|\varphi^{p-1}\,d\mu\,dt \\
&\leq C\int_{t_1}^{t_2}\int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\,d\mu\,dt + C\int_{t_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}\Big|\frac{\partial\varphi}{\partial t}\Big|\varphi^{p-1}\,d\mu\,dt.
\end{aligned}$$

We conclude the same estimate for $\varepsilon > p-1$, if we choose $\tau_1 = t_1$ and $\tau_2 = \tau$. Now the result follows with the constant C depending on ε and p. Remark that the constant blows up as ε tends to 0 or $p-1$. □

Next, we show a corresponding result for a subsolution. Observe that in the following lemma we may have quantities which are not a priori finite. Nevertheless, we can make our calculations with a truncated test function. Finally, we obtain the result by letting the level of truncation go to infinity. In fact, this also justifies the formal calculations made in the proof of Lemma 5.1.

Lemma 3.3. *Suppose that $u \geq \rho > 0$ is a subsolution in $\Omega \times (t_1,t_2)$ and let $\varepsilon > 0$. Then there exists a constant $C = C(\varepsilon,p)$ such that*

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega |Du|^p u^{\varepsilon-1}\varphi^p\,d\mu\,dt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1+\varepsilon}\varphi^p\,d\mu \\
&\leq C\int_{t_1}^{t_2}\int_\Omega u^{p-1+\varepsilon}|D\varphi|^p\,d\mu\,dt + C\int_{t_1}^{t_2}\int_\Omega u^{p-1+\varepsilon}\varphi^{p-1}\Big|\frac{\partial\varphi}{\partial t}\Big|\,d\mu\,dt
\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1,t_2))$ with $\varphi \geq 0$.

Proof. This time we formally choose the test function $\eta = u^\varepsilon\varphi^p$. Otherwise the assertion follows as in the proof of Lemma 3.2. The constant C blows up as ε tends to 0. □

Finally, we show a Caccioppoli type estimate for the logarithm of a supersolution.

Lemma 3.4. *Suppose that $u \geq \rho > 0$ is a supersolution in $\Omega \times (t_1,t_2)$. Then there exists a constant $C = C(p)$ such that*

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega |D(\log u)|^p\varphi^p\,d\mu\,dt + \operatorname*{ess\,sup}_{t_1<t<t_2}\Big|\int_\Omega \log u\,\varphi^p\,d\mu\Big| \\
&\qquad\leq C\int_{t_1}^{t_2}\int_\Omega |D\varphi|^p\,d\mu\,dt + C\int_{t_1}^{t_2}\int_\Omega |\log u|\varphi^{p-1}\Big|\frac{\partial\varphi}{\partial t}\Big|\,d\mu\,dt
\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ *with* $\varphi \geq 0$.

Proof. Let $\eta = u^{1-p}\varphi^p$, where $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ and $\varphi \geq 0$. We integrate by parts and obtain

$$\begin{aligned}
-\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}\frac{\partial \eta}{\partial t}\,d\mu\,dt + \left[\int_\Omega u^{p-1}\eta\,d\mu\right]_{t=\tau_1}^{\tau_2}& \\
=&(p-1)\int_{\tau_1}^{\tau_2}\int_\Omega \frac{\partial \log u}{\partial t}\varphi^p\,d\mu\,dt - \int_{\tau_1}^{\tau_2}\int_\Omega \frac{\partial(\varphi^p)}{\partial t}\,d\mu\,dt \\
&+ \left[\int_\Omega \varphi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} \\
=&-p(p-1)\int_{\tau_1}^{\tau_2}\int_\Omega \log u\,\varphi^{p-1}\frac{\partial\varphi}{\partial t}\,d\mu\,dt \\
&+(p-1)\left[\int_\Omega \log u\,\varphi^p\,d\mu\right]_{t=\tau_1}^{\tau_2},
\end{aligned}$$

where $t_1 < \tau_1 < \tau_2 < t_2$. We denote $v = \log u$ and substitute η in (2.2) and get

$$\begin{aligned}
0 \leq& -\int_{\tau_1}^{\tau_2}\int_\Omega |Dv|^p\varphi^p\,d\mu\,dt + \frac{p}{p-1}\int_{\tau_1}^{\tau_2}\int_\Omega |Dv|^{p-1}|D\varphi|\varphi^{p-1}\,d\mu\,dt \\
&+\left[\int_\Omega v\varphi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} + p\int_{\tau_1}^{\tau_2}\int_\Omega |v|\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,d\mu\,dt.
\end{aligned}$$

We apply Young's inequality for the second term on the right-hand side and obtain

$$\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_\Omega (|Dv|\varphi)^{p-1}|D\varphi|\,d\mu\,dt& \\
\leq \frac{p-1}{2p}\int_{\tau_1}^{\tau_2}&\int_\Omega |Dv|^p\varphi^p\,d\mu\,dt + C\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p\,d\mu\,dt.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_\Omega |Dv|^p\varphi^p\,d\mu\,dt - \left[\int_\Omega v\varphi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} \\
&\leq C\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p\,d\mu\,dt + C\int_{\tau_1}^{\tau_2}\int_\Omega |v|\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,d\mu\,dt.
\end{aligned}\tag{3.5}$$

Now the claim follows in the standard way as in the proof of Lemma 3.2. □

Remark. In (3.5) the test function φ does not need to have a compact support in time. We will use this fact later.

4. Reverse Hölder inequality for a supersolution

Let $0 < \sigma \le 1$, $\tau \in \mathbb{R}$, $T > 0$ and $B(z,r)$ be a ball in $\mathbb{R}^n$. We denote

$$Q = B(z,r) \times (\tau - Tr^p, \tau + Tr^p)$$

and

$$\sigma Q = B(z,\sigma r) \times \big(\tau - T(\sigma r)^p, \tau + T(\sigma r)^p\big).$$

The parameter T is going to be chosen so that the time intervals in various lemmas are compatible.

In the following lemma our goal is to obtain a constant which is independent of the parameter s. In the standard approach of Moser [Mo1] only a finite iteration is needed. In that case there is no need to control the asymptotic behaviour of the constant. In our approach the number of iterations is not bounded and we make a geometrically convergent partition of the cylinder Q in order to obtain a uniform bound for the constant.

Lemma 4.1. *Suppose that $u \ge \rho > 0$ is a supersolution in Q and let $0 < \delta < 1$. Then there exist positive constants $C = C(p, q, D_0, P_0, T, \delta)$ and $\theta = \theta(p, D_0)$ such that*

$$\Big(\fint_{\sigma' Q} u^q \, d\mu \, dt\Big)^{1/q} \le \Big(\frac{C}{(\sigma - \sigma')^\theta}\Big)^{1/s} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/s}$$

for all $0 < \delta \le \sigma' < \sigma \le 1$ and for all $0 < s < q < q_0$, where $q_0 = (p-1)(2 - p/\kappa)$ and $\kappa > p$ is as in (2.7).

Proof. The proof is based on the successive use of Sobolev's inequality and Caccioppoli's estimate. Let $\gamma = 2 - p/\kappa$. We fix σ and divide the interval (σ', σ) into k parts by setting

$$\sigma_0 = \sigma, \quad \sigma_k = \sigma', \quad \sigma_j = \sigma - (\sigma - \sigma')\frac{1 - \gamma^{-j}}{1 - \gamma^{-k}}.$$

We shall fix k later. We denote $Q_j = \sigma_j Q = B_j \times T_j$. We also choose test functions with the following properties:

$$\operatorname{supp}(\varphi_j) \subset Q_j,$$

$$0 \le \varphi_j \le 1 \ \text{ in } \ Q_j, \quad \varphi_j = 1 \ \text{ in } \ Q_{j+1},$$

$$|D\varphi_j| \le C\frac{\gamma^j}{r(\sigma - \sigma')}, \quad \Big|\frac{\partial \varphi_j}{\partial t}\Big| \le \frac{C}{T}\Big(\frac{\gamma^j}{r(\sigma - \sigma')}\Big)^p \ \text{ in } \ Q_j.$$

Furthermore, let $\alpha = p-1-\varepsilon$ and $0<\varepsilon<p-1$. An application of Hölder's inequality yields

$$
\begin{aligned}
&\fint_{Q_{j+1}} u^{\gamma\alpha}\,d\mu\,dt \\
&\quad\le \fint_{T_{j+1}} \Big(\fint_{B_{j+1}} u^{\alpha}\varphi_j^p\,d\mu\Big)^{(\kappa-p)/\kappa} \Big(\fint_{B_{j+1}} \big(u^{\alpha/p}\varphi_j\big)^{\kappa}\,d\mu\Big)^{p/\kappa}\,dt \\
&\quad\le \frac{|T_j|\mu(B_j)}{|T_{j+1}|\mu(B_{j+1})}\Big(\operatorname*{ess\,sup}_{T_j}\fint_{B_j} u^{\alpha}\varphi_j^p\,d\mu\Big)^{(\kappa-p)/\kappa} \\
&\qquad\times \fint_{T_j}\Big(\fint_{B_j}\big(u^{\alpha/p}\varphi_j\big)^{\kappa}\,d\mu\Big)^{p/\kappa}\,dt.
\end{aligned}
$$

Since the measure μ is doubling and $\sigma_{j+1}\ge \min(\delta,(\gamma+1)^{-1})\sigma_j$, the first factor on the right hand side is bounded by a constant independent of j, r, σ and σ'. We now use Sobolev's inequality together with Caccioppoli's estimate and obtain

$$
\begin{aligned}
&\fint_{Q_{j+1}} u^{\gamma\alpha}\,d\mu\,dt \\
&\quad\le C\Big(\operatorname*{ess\,sup}_{T_j}\fint_{B_j} u^{\alpha}\varphi_j^p\,d\mu\Big)^{(\kappa-p)/\kappa} r^p\fint_{T_j}\fint_{B_j}\big|D(u^{\alpha/p}\varphi_j)\big|^p\,d\mu\,dt \\
&\quad\le C\Big(\operatorname*{ess\,sup}_{T_j}\fint_{B_j} u^{\alpha}\varphi_j^p\,d\mu \\
&\qquad + \frac{1}{\delta T}\int_{T_j}\fint_{B_j}\alpha^p|Du|^p u^{-\varepsilon-1}\varphi_j^p + u^{p-1-\varepsilon}|D\varphi_j|^p\,d\mu\,dt\Big)^{\gamma} \\
&\quad\le C\Big(\int_{T_j}\fint_{B_j} u^{p-1-\varepsilon}\Big(|D\varphi_j|^p+\Big|\frac{\partial\varphi_j}{\partial t}\Big|\Big)\,d\mu\,dt\Big)^{\gamma} \\
&\quad\le C\Big(\frac{\gamma^{jp}}{(\sigma-\sigma')^p}\fint_{Q_j} u^{\alpha}\,d\mu\,dt\Big)^{\gamma}.
\end{aligned}
\tag{4.2}
$$

Careful study of the proof of Lemma 3.2 shows that the constant C is indeed independent of α; the term α^p in the inequality above cancels the impact of the singularity of the constant in Lemma 3.2 when ε is close to $p-1$.

The next step in the proof is to iterate (4.2). Observe that the condition $0<\alpha<p-1$ must be satisfied. This gives the upper bound $q_0 = \gamma(p-1)$ for q. For the iteration, we fix q and s with $q>s$, and k such that $s\gamma^{k-1}\le q\le s\gamma^k$. Let ρ_0 such that $\rho_0\le s$ and $q=\gamma^k\rho_0$. Denote

$\rho_j = \gamma^j \rho_0$ for $j = 0, \dots, k$. Then we have

$$\Big(\fint_{Q_k} u^q \, d\mu \, dt \Big)^{1/q} \le \Big(\frac{C\gamma^k}{\sigma - \sigma'} \Big)^{p/\rho_{k-1}} \Big(\fint_{Q_{k-1}} u^{\rho_{k-1}} \, d\mu \, dt \Big)^{1/\rho_{k-1}}$$
$$\le \vdots$$
$$\le \Big(\frac{c_{prod}(k)}{(\sigma - \sigma')^{\gamma^*}} \fint_{\sigma Q} u^{\rho_0} \, d\mu \, dt \Big)^{1/\rho_0},$$

where

$$c_{prod}(k) = C^{\gamma^*} \prod_{j=0}^{k-1} \big(\gamma^{j+1} \big)^{p\gamma^{-j}}$$

and

$$\gamma^* = p \sum_{j=0}^{k-1} \gamma^{-j} = \frac{p\gamma}{\gamma - 1}(1 - \gamma^{-k}).$$

The constant C depends on q since the constant in Lemma 3.2 has a singularity at $\varepsilon = 0$. Obviously $c_{prod}(k)$ is uniformly bounded on k. From Hölder's inequality we obtain

$$\Big(\fint_{\sigma' Q} u^q \, d\mu \, dt \Big)^{1/q} \le \Big(\frac{C}{(\sigma - \sigma')^{\gamma^*}} \Big)^{1/\rho_0} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt \Big)^{1/s}.$$

Furthermore, since $s\gamma^{k-1} \le \rho_0 \gamma^k$, we have $\rho_0 \ge s/\gamma$ and consequently the required estimate follows with $\theta = p\gamma^2/(\gamma - 1)$. □

5. Boundedness of a subsolution

The proof of the following bound for the essential supremum is based on the standard Moser iteration scheme.

Lemma 5.1. *Suppose that $u \ge \rho > 0$ is a subsolution in Q. Let $0 < \delta < 1$. Then there exist positive constants $C = C(p, D_0, P_0, T, \delta)$ and $\theta = \theta(p, D_0)$, such that*

$$\operatorname*{ess\,sup}_{\sigma' Q} u \le \Big(\frac{C}{(\sigma - \sigma')^{\theta}} \Big)^{1/s} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt \Big)^{1/s}$$

for all $0 < \delta \le \sigma' < \sigma \le 1$ and for all $s > 0$.

Proof. Without loss of generality we can choose $T = 1$. Let the choices of test functions and σ_j be the same as in the proof of Lemma 4.1 with the exception that

$$\sigma_j = \sigma - (\sigma - \sigma')(1 - \gamma^{-j}).$$

As in the proof of Lemma 4.1 we obtain from the Sobolev's inequality and from Lemma 3.3 that

$$\fint_{Q_{j+1}} u^{\gamma\alpha}\,d\mu\,dt \le C\Big(\frac{\alpha^p\gamma^{jp}}{(\sigma-\sigma')^p}\fint_{Q_j} u^{\alpha}\,d\mu\,dt\Big)^{\gamma}, \tag{5.2}$$

where

$$\gamma = 2-\frac{p}{\kappa}, \quad \alpha = p-1+\varepsilon, \quad \varepsilon \ge 1.$$

In Lemma 3.3 the constant is singular as ε is close to 0. We deliberately avoid this singularity by choosing $\varepsilon \ge 1$. Moreover, we choose $\alpha_j = p\gamma^j$, $j = 0, 1, \dots$. We iterate the inequality above and obtain

$$\begin{aligned}&\Big(\fint_{Q_0} u^p\,d\mu\,dt\Big)^{1/p}\\ &\ge \Big(\frac{(\sigma-\sigma')}{C}\Big)^{\gamma^{-1}+\gamma^{-2}+\dots+\gamma^{-k+1}}\prod_{j=0}^{k-1}\gamma^{-2j/\gamma^j}\Big(\fint_{Q_k} u^{\gamma^k p}\,d\mu\,dt\Big)^{1/\gamma^k p}.\end{aligned}$$

We let k tend to infinity and get the result for $s \ge p$ from Hölder's inequality.

If $s < p$, then we have

$$\begin{aligned}\operatorname*{ess\,sup}_{\sigma' Q} u &\le \Big(\frac{C}{(\sigma-\sigma')^{\theta}}\Big)^{1/p}\Big(\fint_{\sigma Q} u^p\,d\mu\,dt\Big)^{1/p}\\ &\le \Big(\frac{p-s}{2p}\operatorname*{ess\,sup}_{\sigma Q} u\Big)^{(p-s)/p}\Big(\frac{2p}{p-s}\Big)^{(p-s)/p}\\ &\quad\times\Big(\frac{C}{(\sigma-\sigma')^{\theta}}\Big)^{1/p}\Big(\fint_{\sigma Q} u^s\,d\mu\,dt\Big)^{1/p}\\ &\le \frac{1}{2}\operatorname*{ess\,sup}_{\sigma Q} u + \Big((p-s)^{s-p}\frac{C}{(\sigma-\sigma')^{\theta}}\Big)^{1/s}\Big(\fint_{\sigma Q} u^s\,d\mu\,dt\Big)^{1/s}\\ &\le \frac{1}{2}\operatorname*{ess\,sup}_{\sigma Q} u + \Big(\frac{C}{(\sigma-\sigma')^{\theta}}\Big)^{1/s}\Big(\fint_{\sigma Q} u^s\,d\mu\,dt\Big)^{1/s},\end{aligned}$$

where we used Young's inequality. By a standard iteration argument (see for example Lemma 5.1 in [Giaq]) we obtain the result. □

6. Logarithmic estimate for a supersolution

We already have the reverse Hölder inequalities for both super- and subsolutions. Next we show that the condition for the logarithm in the assumptions of Lemma 2.11 holds.

Let $0 < \sigma \le 1$, $\tau \in \mathbb{R}$, $T > 0$ and $B(z, r)$ be a ball in $\mathbb{R}^n$. We set

$$\begin{aligned}Q &= B(z,r)\times(\tau - Tr^p, \tau + Tr^p),\\ \sigma Q^+ &= B(z,\sigma r)\times(\tau, \tau + T(\sigma r)^p)\end{aligned}$$

and

$$\sigma Q^- = B(z,\sigma r)\times(\tau - T(\sigma r)^p,\tau).$$

Let $d\nu = d\mu\,dt$.

Lemma 6.1. *Suppose that $u\ge\rho>0$ is a supersolution in Q and let*

$$\varphi(x,t)=\varphi(x)=\Big(1-2\frac{|x-z|}{(1+\sigma)r}\Big)_+,$$

where $0<\sigma<1$ and $(x,t)\in B(z,r)\times(\tau-(\sigma r)^p,\tau+(\sigma r)^p)$. Let

$$\beta=\int_{B(z,r)}\log u(x,\tau)\varphi^p(x)\,d\mu(x).$$

Then there exist constants $C=C(p,D_0,P_0,\sigma,T)$ and $C'=C'(p,D_0,\sigma,T)$ such that

$$\nu\big(\{(x,t)\in\sigma Q^-\,|\,\log u(x,t)>\lambda+\beta+C'\}\big)\le\frac{C}{\lambda^{p-1}}\nu(\sigma Q^-)$$

and

$$\nu\big(\{(x,t)\in\sigma Q^+\,|\,\log u(x,t)<-\lambda+\beta-C'\}\big)\le\frac{C}{\lambda^{p-1}}\nu(\sigma Q^+).$$

for every $\lambda>0$.

Proof. Let

$$N=\int_{B(z,r)}\varphi^p(x)\,d\mu(x).$$

Then

$$\Big(\frac{1-\sigma}{1+\sigma}\Big)^p\mu(B(z,\sigma r))\le N\le\mu(B(z,r)).$$

We denote

$$v(x,t)=\log u(x,t)-\beta\quad\text{and}\quad V(t)=\frac{1}{N}\int_{B(z,r)}v(x,t)\varphi^p(x)\,d\mu(x).$$

Remark that $V(\tau)=0$. Since φ is independent of t, we obtain from (3.5) that

$$\int_{t_1}^{t_2}\int_{B(z,r)}|Dv|^p\varphi^p\,d\mu\,dt-\Big[\int_{B(z,r)}v\varphi^p\,d\mu\Big]_{t=t_1}^{t_2}$$
$$\le C\int_{t_1}^{t_2}\int_{B(z,r)}|D\varphi|^p\,d\mu\,dt,$$

where $\tau-(\sigma r)^p\le t_1<t_2\le\tau+(\sigma r)^p$. Furthermore, Theorem 2.10 yields

$$\int_{B(z,r)}|Dv|^p\varphi^p\,d\mu\ge\frac{1}{Cr^p}\int_{B(z,r)}|v-V(t)|^p\varphi^p\,d\mu$$
$$\ge\frac{(1-\sigma)^p}{Cr^p}\int_{B(z,\sigma r)}|v-V(t)|^p\,d\mu.$$

It follows that

$$\begin{aligned}\frac{1}{CNr^p}\int_{t_1}^{t_2}\int_{B(z,\sigma r)}|v-V(t)|^p\,d\mu\,dt+V(t_1)-V(t_2)\\ &\le \frac{C(t_2-t_1)}{r^p}\frac{\mu(B(z,r))}{N}\\ &\le C'\frac{t_2-t_1}{T(\sigma r)^p}.\end{aligned}$$

In the last inequality we used the fact that

$$\begin{aligned}N&=\int_{B(z,r)}\varphi^p(x)\,d\mu(x)\ge\int_{B(z,r/4)}\varphi^p(x)\,d\mu(x)\\ &\ge 2^{-p}\mu(B(z,r/4))\ge 2^{-p}D_0^{-2}\mu(B(z,r)).\end{aligned}$$

By denoting

$$w(x,t)=v(x,t)+C'\frac{t-\tau}{T(\sigma r)^p}\quad\text{and}\quad W(t)=V(t)+C'\frac{t-\tau}{T(\sigma r)^p}$$

we obtain

$$\frac{1}{CNr^p}\int_{t_1}^{t_2}\int_{B(z,r)}|w-W(t)|^p\,d\mu\,dt+W(t_1)-W(t_2)\le 0.$$

From this we conclude that $W(t_1)\le W(t_2)$ whenever $\tau-(\sigma r)^p\le t_1<t_2\le\tau+(\sigma r)^p$. Since W is a monotonic function it is differentiable almost everywhere. As a consequence we have

$$\frac{1}{CNr^p}\int_{B(z,r)}|w-W(t)|^p\,d\mu-W'(t)\le 0 \tag{6.2}$$

for almost every t with $t_1<t<t_2$. Let

$$E_\lambda^-(t)=\{(x,t)\in\sigma Q^-\,|\,w(x,t)>\lambda\}.$$

We observe that

$$\int_{B(z,\sigma r)}|w-W(t)|^p\,d\mu\ge(\lambda-W(t))^p\mu(E_\lambda^-(t))\ge\mu(E_\lambda^-(t))\lambda^p$$

because $W(t)\le W(\tau)=0$ as $\tau>t>t-(\sigma r)^p$. Thus we have

$$-\frac{W'(t)}{(\lambda-W(t))^p}+C\frac{\mu(E_\lambda^-(t))}{Nr^p}\le 0$$

for almost every $\tau>t>t-(\sigma r)^p$. We integrate this inequality over the interval $(\tau-(\sigma r)^p,\tau)$ and obtain

$$\frac{\nu(E_\lambda^-)}{Nr^p}\le C\left[(\lambda-W(t))^{-(p-1)}\right]_{t=\tau-(\sigma r)^p}^{\tau}\le\frac{C}{\lambda^{p-1}}.$$

This implies

$$\nu\big(\{(x,t)\in\sigma Q^-\,|\,\log u(x,t)>\lambda+\beta+C'\}\big)\le\frac{C\nu(\sigma Q^-)}{\lambda^{p-1}}.$$

Let

$$E_\lambda^+(t) = \{(x,t) \in \sigma Q^+ | w(x,t) < -\lambda\}.$$

As in the case of Q^- we conclude that

$$\int_{\sigma B(z,r)} |w - W(t)|^p \, d\mu \geq \mu(E_\lambda^+(t))(\lambda + W(t))^p \geq \mu(E_\lambda^-(t))\lambda^p$$

because $W(t) \geq W(\tau) = 0$ as $\tau < t < \tau + (\sigma r^p)$. Thus, from (6.2), we have

$$-\frac{W'(t)}{(\lambda + W(t))^p} + C\frac{\mu(E_\lambda^+(t))}{\nu(Q^+)} \leq 0$$

for almost every $\tau < t < t + (\sigma r)^p$. An integration over the interval $(\tau, \tau + (\sigma r)^p)$ gives

$$\frac{\nu(E_\lambda^+)}{\nu(\sigma Q^+)} \leq -C\Big[(\lambda + W(t))^{-(p-1)}\Big]_{t=\tau}^{\tau+(\sigma r)^p} \leq \frac{C}{\lambda^{p-1}}.$$

Therefore

$$\nu\big(\{(x,t) \in \sigma Q^+ | \log u(x,t) < -\lambda + \beta - C' \ \}\big) \leq \frac{C\nu(\sigma Q^+)}{\lambda^{p-1}}$$

and the claim follows. □

7. Harnack's inequality

First we give a proof for a weak Harnack inequality. We use the same notation as in Theorem 2.5.

Theorem 7.1. *Let $u \geq \rho > 0$ be a supersolution in U. Then there exist constants $C = C(p, D_0, P_0, q, \delta)$ and $q_0 = (p-1)(2 - p/\kappa)$, $\kappa > p$ as in (2.7), such that*

$$\left(\fint_{\delta U^-} u^q \, d\mu \, dt\right)^{1/q} \leq C \operatorname*{ess\,inf}_{\delta U^+} u,$$

for $0 < \delta < 1$ and $0 < q < q_0$.

Proof. We fix $0 < \delta < 1$. Let φ be as in the assumptions of Lemma 6.1 and let β and C' be the corresponding constants. We define

$$v^+ = u^{-1}\exp(\beta - C') \quad \text{and} \quad v^- = u\exp(-\beta - C').$$

We apply Lemma 6.1 for the function u and have

$$\nu\Big(\{(x,t) \in \frac{1+\delta}{2}U^+ | \log v^+(x,t) > \lambda\}\Big) \leq \frac{C}{\lambda^{p-1}}\nu\Big(\frac{1+\delta}{2}U^+\Big)$$

and

$$\nu\Big(\{(x,t) \in \frac{1+\delta}{2}U^- | \log v^-(x,t) > \lambda\}\Big) \leq \frac{C}{\lambda^{p-1}}\nu\Big(\frac{1+\delta}{2}U^-\Big).$$

Here we also used a fact that

$$\nu(B(z, \sigma R) \times (\tau, \tau \pm (\sigma R)^p)) \leq C\nu(\delta U^\pm).$$

Lemma 3.1 implies that v^+ is a subsolution in U. Consequently, Lemma 5.1 gives

$$\operatorname*{ess\,sup}_{\sigma' U^+} v^+ \le \Big(\frac{C}{(\sigma - \sigma')^\theta} \fint_{\sigma U^+} (v^+)^s \, d\mu \, dt \Big)^{1/s}$$

whenever $\delta \le \sigma' < \sigma \le (1+\delta)/2$ and $s > 0$. Note that we have chosen a suitable parameter T to match the time scales in various lemmas. We now use Lemma 2.11 and obtain

$$\operatorname*{ess\,sup}_{\delta U^+} v^+ \le C. \tag{7.2}$$

Furthermore, we have from the corollary of Lemma 4.1 for v^- that

$$\Big(\fint_{\sigma' U^-} (v^-)^q \, d\mu \, dt \Big)^{1/q} \le \Big(\frac{C}{(\sigma - \sigma')^\theta} \fint_{\sigma U^-} (v^-)^s \, d\mu \, dt \Big)^{1/s}$$

when $\delta \le \sigma' < \sigma \le (1+\delta)/2$ and $0 < s < q < q_0$. From Lemma 2.11 we obtain

$$\Big(\fint_{\delta U^-} (v^-)^q \, d\mu \, dt \Big)^{1/q} \le C.$$

Multiplying this with (7.2) gives

$$\Big(\fint_{\delta U^-} u^q \, d\mu \, dt \Big)^{1/q} \le C \operatorname*{ess\,inf}_{\delta U^+} u$$

and the result follows. □

Now we are ready to prove the full Harnack inequality.

Proof of theorem 2.5. We apply Lemma 7.1 with $\delta = (1+\sigma)/2$. The result follows now from Lemma 5.1. □

References

[AS] D.G. Aronsson, J. Serrin, Local behaviour of solutions of quasilinear parabolic equations, Arch. Rat. Mech. Anal 25, 81–122 (1967)

[BCLS] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44, 1033–1074 (1995)

[Bomb] E. Bombieri, Theory of minimal surfaces and a counterexample to the bernstein conjecture in high dimension, Mimeographed Notes of Lectures Held at Courant Institute, New York University (1970)

[BoGi] E. Bombieri, E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces, Invent. Math. 15, 24–46 (1972)

[ChFr] F. Chiarenza, M. Frasca, A note on weighted Sobolev inequality, Proc. Amer. Math. Soc. 93, 703–704 (1985)

[ChSe] F. Chiarenza, R. Serapioni, A Harnack inequality for degenerate parabolic equations, Comm. Partial Differential Equations 9(8), 719–749 (1984)

[DiBe] E. DiBenedetto, Degenerate parabolic equations, Springer-Verlag (1993)

[DBUV] E. DiBenedetto, J.M. Urbano, V. Vespri, Current issues on singular and degenerate evolution equations, Handbook of differential equations, Elsevier, 169–286 (2004)

[FaGa] E. Fabes, N. Garofalo, Parabolic B.M.O. and Harnack's inequality, Proc. Amer. Math. Soc. 50, no. 1, 63–69 (1985)

[FKS] E. Fabes, C. Kenig, R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations 7, no. 1, 77–116 (1982)

[GiVe] U. Gianazza, V. Vespri, A Harnack inequality for solutions of doubly nonlinear parabolic equation, J. Appl. Funct. Anal (to appear)

[Giaq] M. Giaquinta, Introduction to regularity theory for nonlinear elliptic systems, Birkhäuser Verlag (1993)

[Gri] A. Grigor'yan, The heat equation on non-compact Riemannian manifolds, Matem. Sbornik 182, 55–87 (1991). Engl. Transl. Math. USSR Sb. 72, 47–77 (1992)

[HaK] P. Hajłasz, P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. 688 (2000)

[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, Nonliear Potential Theory of Degenerate Elliptic Equations, Oxford University Press, Oxford (1993)

[KS] J. Kinnunen and N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, Manuscripta Math. 105, 401–423 (2001)

[Mo1] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17, 101–134 (1964), and correction in Comm. Pure Appl. Math. 20, 231–236 (1967)

[Mo2] J. Moser, On a pointwise estimate for parabolic equations, Comm. Pure Appl. Math. 24, 727–740 (1971)

[SaCo1] L. Saloff-Coste, Aspects of Sobolev-type inequalities, London Mathematical Society Lecture Note Series 289, Cambridge University Press (2002)

[SaCo2] L. Saloff-Coste, A note on Poincaré, Sobolev and Harnack inequalities, Duke Math. J. 65, IMRN 2, 27–38 (1992)

[Tru] N.S. Trudinger, Pointwise estimates and quasilinear parabolic equations, Comm. Pure Appl. Math. 21, 205–226 (1968)

[Ve] V. Vespri, On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations, Manuscripta Math. 75, 65–80 (1992)

[WZYL] Z. Wu, J. Zhao, J. Yin, H.Li, Nonlinear diffusion equations, World Scientific (2001)

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