Papers on Analysis:
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MINIMAL, MAXIMAL AND REVERSE HÖLDER INEQUALITIES

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1. Introduction. Let (X,d) be a metric space and suppose that μ is a Borel measure on X. We assume that the measure of every nonempty open set is strictly positive and that $\mu(X)<\infty$. In addition we assume that μ satisfies the doubling condition

$$\mu(2B)\le c\mu(B),\qquad B\subset X, \tag{1.1}$$

for some $c\ge 1$ independent of the open ball B. Here $2B$ denotes the ball with the same center as B but the radius doubled. We also make a technical assumption that $0<\operatorname{diam}(X)<\infty$. Let $f\colon X\to[0,\infty]$ be a $\mu-$measurable function. Hölder's inequality implies that

$$\Big(\fint_B f^t\,d\mu\Big)^{1/t}\le\Big(\fint_B f^s\,d\mu\Big)^{1/s},\qquad B\subset X,$$

whenever $-\infty<t<s<\infty$ and $ts\ne 0$. Here we use the notation

$$\fint_B f\,d\mu=\frac{1}{\mu(B)}\int_B f\,d\mu.$$

We call the inequality above Hölder's inequality even though in this generality it is a special case of Jensen's inequality. We are interested in functions which satisfy an inequality in the reverse direction uniformly over all balls; by this we mean that there are $c\ge 1$ and $-\infty<t<s<\infty$, $ts\ne 0$, such that

$$\Big(\fint_B f^s\,d\mu\Big)^{1/s}\le c\Big(\fint_B f^t\,d\mu\Big)^{1/t},\qquad B\subset X. \tag{1.2}$$

It is crucial for us that (1.2) holds for every ball B with the same constant c. Note that we allow also negative powers in (1.2). Replacing f by its power, we may suppose that the reverse Hölder inequality is of the form

$$\fint_B f^t\,d\mu\le c\Big(\fint_B f\,d\mu\Big)^t,\qquad B\subset X, \tag{1.3}$$

where $t > 1$ or $t < 0$. If f is locally integrable and satisfies (1.3), we denote $f \in \mathcal{RH}_t$. If $f \in \mathcal{RH}_t$, $t > 1$, then it is well-known that f is locally integrable to a slightly greater power than t. In the Euclidean case this is a result of Gehring [Ge]. On the other hand, if $f \in \mathcal{RH}_t$, $t < 0$, a theorem of Muckenhoupt implies that f is locally integrable to a power which is strictly smaller than t. In the limiting case $t = -\infty$, inequality (1.3) reads

$$\fint_B f\,d\mu \le c \operatorname*{ess\,inf}_B f, \qquad B \subset X, \tag{1.4}$$

and if $t = \infty$ it is

$$\operatorname*{ess\,sup}_B f \le c \fint_B f\,d\mu, \qquad B \subset X. \tag{1.5}$$

The ultimate limit is, of course, a Harnack type inequality

$$\operatorname*{ess\,sup}_B f \le c \operatorname*{ess\,inf}_B f, \qquad B \subset X.$$

These inequalities imply higher integrability properties as well: (1.4) implies that f is locally integrable to a strictly greater power than one and (1.5) implies that f is locally integrable to a negative power. If $p > 1$ and $t = 1/(1-p)$, then (1.3) is Muckenhoupt's $\mathcal{A}_p$−condition. Limiting inequality (1.4) is Muckenhoupt's $\mathcal{A}_1$−condition [M1]. If a locally integrable function f satisfies (1.5), we denote $f \in \mathcal{RH}_\infty$. For the class $\mathcal{RH}_\infty$, we refer to Andersen and Young [AY]. See also [CSN1], [CSN2], [F] and [M2].

We shall show that all the mentioned higher integrability results are variations on the same theme: minimal and maximal function inequalities. Our proofs are elementary but they give sharp results and lead to results which are of independent interest. The basic method was developed in [Ki] and some of the results appeared already there, although in a slightly less general form. A general principle is that if we want to prove that the function is locally integrable to a negative power, we should use the minimal function inequalities; if we want to prove that the function is locally integrable to a positive power, we should use the maximal function inequalities. This resembles the fact that the minimal function contains information of the function in the set where the function is small whereas the maximal function obeys the function closely in the set where the function is large. Some sharp results have been previously obtained in [AS], [BSW], [I], [Ki1], [Ki2], [Ko1], [Ko2], [N], [R] and [W].

1.6. Notation. Our notation is standard. However, some comments are due. All functions are supposed to be μ−measurable and non-negative. Lebesgue spaces $L^p(X)$ consist of equivalence classes of functions modulo sets of measure zero for which

$$\Big(\int_X f^p\,d\mu\Big)^{1/p} < \infty.$$

For short, the distribution sets $\{x \in X\colon f(x) > \lambda\}$ are denoted by $\{f > \lambda\}$.

2. Maximal functions. The Hardy–Littlewood maximal function $\mathcal{M}f\colon X \to [0,\infty]$ of a locally integrable function f is defined by

$$\mathcal{M}f(x) = \sup_B \fint_B f\,d\mu, \tag{2.1}$$

where the supremum is taken over all open balls B containing x. This definition gives the non-centered maximal function but is also possible to define the centered maximal function by taking the supremum over all balls centered at x. Since the measure is doubling, we see that the non-centered and the centered maximal functions are comparable and it does not matter which one we choose, but we state our theorems for the non-centered maximal function only. The term maximal function is due to the fact that Lebesgue's differentiation theorem gives $f \le \mathcal{M}f$ μ-almost everywhere. Because $\{\mathcal{M}f > \lambda\}$, $\lambda > 0$, is open, the maximal function is lower semicontinuous and hence μ-measurable. Our technique is based on the estimation of the measures of distribution sets. We begin with the standard weak type estimates, see [S]. We recall the proofs here for reference, because we use a similar argument in a slightly different context later.

Fix $\lambda > 0$. Then for every $x \in \{\mathcal{M}f > \lambda\}$ there is a ball B_x containing x such that

$$\fint_{B_x} f\,d\mu > \lambda.$$

Using Vitali's covering theorem [CW, Theorem 2.1] we get countably many pairwise disjoint balls B_i, $i = 1, 2, \dots$, and $\sigma \ge 1$, so that

$$\{\mathcal{M}f > \lambda\} \subset \bigcup_{i=1}^{\infty} \sigma B_i.$$

Using the doubling property we find

$$\begin{aligned}\mu(\{\mathcal{M}f > \lambda\}) &\le \sum_{i=1}^{\infty} \mu(\sigma B_i) \le c \sum_{i=1}^{\infty} \mu(B_i) \\ &\le \frac{c}{\lambda} \sum_{i=1}^{\infty} \int_{B_i} f\,d\mu \le \frac{c}{\lambda} \int_{\{\mathcal{M}f>\lambda\}} f\,d\mu, \qquad \lambda > 0,\end{aligned} \tag{2.2}$$

where c depends only on the doubling constant. This is the standard weak type (1,1)-inequality, but there is also an estimate in the reverse direction. To this end, we fix $\lambda > \operatorname{ess\,inf}_X \mathcal{M}f$, then $\{\mathcal{M}f \le \lambda\}$ has positive measure. For every $x \in \{\mathcal{M}f > \lambda\}$ we take the ball $B(x, r_x)$, where r_x is the distance from x to the set $\{\mathcal{M}f \le \lambda\}$. By Vitali's covering theorem, we get a countable subcollection of pairwise disjoint balls $B_i = B(x_i, r_{x_i})$, $i = 1, 2, \dots$, so that

$$\{\mathcal{M}f > \lambda\} \subset \bigcup_{i=1}^{\infty} \sigma B_i.$$

The balls σB_i, $i = 1, 2, \dots$, intersect the set $\{\mathcal{M}f \le \lambda\}$ and therefore we have

$$\fint_{\sigma B_i} f\,d\mu \le \lambda, \qquad i = 1, 2, \dots$$

By summing up we get

$$\begin{aligned}\int_{\{\mathcal{M}f>\lambda\}} f\,d\mu &\le \sum_{i=1}^{\infty} \int_{\sigma B_i} f\,d\mu \le \lambda \sum_{i=1}^{\infty} \mu(\sigma B_i) \\ &\le c\lambda \sum_{i=1}^{\infty} \mu(B_i) \le c\lambda\mu(\{\mathcal{M}f > \lambda\}), \qquad \lambda > \operatorname*{ess\,inf}_X \mathcal{M}f.\end{aligned} \tag{2.3}$$

It follows from this that (2.3) is true for every $\lambda \ge \operatorname{ess\,inf}_X \mathcal{M}f$. If $\lambda < \operatorname{ess\,inf}_X \mathcal{M}f$, then $\mu(\{\mathcal{M}f > \lambda\}) = \mu(X)$ and inequality (2.3) is true whenever

$$\lambda \ge \frac{1}{c} \fint_X f\,d\mu, \tag{2.4}$$

where c is the same constant as in (2.3).

3. Minimal functions. The minimal function is defined in a similar way as the maximal function, but instead of a supremum we take an infimum in definition (2.1). It is clear that the same method used in proving the weak type and the reverse weak type inequalities for the maximal function applies to the minimal function as well. However, there are some drawbacks due to the fact that the centered and the non-centered minimal functions are not comparable.

Let $f \colon X \to [0,\infty]$ be a locally integrable function. The minimal function $mf \colon X \to [0,\infty]$ of f is defined by

$$mf(x) = \inf_B \fint_B f\,d\mu, \tag{3.1}$$

where the infimum is taken over all open balls containing x. Again, there is a centered version of the definition, where the infimum is taken over all balls centered at x. Because these minimal functions are not comparable, in this case it really matters which one we choose. The minimal functions in the one-dimensional case have been recently studied in [CSN2]. Lebesgue's differentiation theorem implies that $mf \le f$ μ-almost everywhere. The set $\{mf < \lambda\}$, $\lambda > 0$, is open and hence the minimal function is upper semicontinuous. Observe, that if $\mu(X) = \infty$ and f is integrable to a power greater than one, then $mf = 0$. We shall make an additional assumption that the measure given by $f\,d\mu$ is doubling, which means that there is $c \ge 1$ so that

$$\int_{2B} f\,d\mu \le c \int_B f\,d\mu, \qquad B \subset X. \tag{3.2}$$

If $f \in \mathcal{RH}_{-t}$, $t > 0$, then (3.2) holds because

$$\begin{aligned}\fint_{2B} f\,d\mu &\le c\Big(\fint_{2B} f^{-t}\,d\mu\Big)^{-1/t}\\ &\le c\Big(\fint_{B} f^{-t}\,d\mu\Big)^{-1/t} \le c\fint_{B} f\,d\mu, \qquad B \subset X.\end{aligned}$$

Here we also used the fact that measure μ is doubling. The last constant c depends only on the constants in (1.3) and (1.1). Fix $\lambda > 0$. Then for every $x \in \{mf < \lambda\}$ there is a ball B_x such that

$$\fint_{B_x} f\,d\mu < \lambda.$$

Using Vitali's covering theorem we get countably many pairwise disjoint balls B_i, $i = 1, 2, \dots$, so that

$$\{mf < \lambda\} \subset \bigcup_{i=1}^{\infty} \sigma B_i.$$

By (3.2) we have

$$\begin{aligned}\int_{\{mf<\lambda\}} f\,d\mu &\le \sum_{i=1}^{\infty}\int_{\sigma B_i} f\,d\mu \le c\sum_{i=1}^{\infty}\int_{B_i} f\,d\mu\\ &\le c\lambda\sum_{i=1}^{\infty}\mu(B_i) \le c\lambda\mu(\{mf < \lambda\}), \qquad \lambda > 0.\end{aligned} \tag{3.3}$$

This corresponds to the weak type (1,1)-inequality for the minimal operator under additional assumption (3.2). For the centered minimal operator in $\mathbf{R}^n$ estimate (3.3) can be proved using Besicovitch's covering theorem without assumption (3.2), but it turns out that the non-centered minimal function is the right tool in studying higher integrability properties.

As the reader may guess, there is also a reverse weak type inequality for the minimal function. To see this, fix λ with $0 < \lambda < \operatorname{ess\,sup}_X mf$. Then the set $\{mf \ge \lambda\}$ has positive μ-measure. For every $x \in \{mf < \lambda\}$ take the ball $B(x, r_x)$, where r_x is the distance of x from the set $\{mf \ge \lambda\}$. By Vitali's covering theorem, there are countably many pairwise disjoint balls B_i, $i = 1, 2, \dots$, so that

$$\{mf < \lambda\} \subset \bigcup_{i=1}^{\infty} \sigma B_i.$$

The balls σB_i, $i = 1, 2, \dots$, intersect the set $\{mf \ge \lambda\}$ and therefore

$$\fint_{\sigma B_i} f\,d\mu \ge \lambda, \qquad i = 1, 2, \dots$$

Summing up we get

$$
\begin{aligned}
\mu(\{mf<\lambda\}) &\le \sum_{i=1}^{\infty}\mu(\sigma B_i) \le c\sum_{i=1}^{\infty}\mu(B_i)\\
&\le \frac{c}{\lambda}\sum_{i=1}^{\infty}\int_{B_i} f\,d\mu \le \frac{c}{\lambda}\int_{\{mf<\lambda\}} f\,d\mu, \qquad 0<\lambda<\operatorname*{ess\,sup}_X mf.
\end{aligned}
\tag{3.4}
$$

Again, it is easy to see that inequality (3.4) holds whenever $0<\lambda\le \operatorname{ess\,sup}_X mf$. The constant in (3.4) equals the doubling constant in (1.1). If $\operatorname{ess\,sup}_X mf<\lambda<\infty$, then $\mu(\{mf<\lambda\})=\mu(X)$ and (3.4) holds whenever

$$
\lambda \le c\fint_X f\,d\mu. \tag{3.5}
$$

Observe that we did not use hypothesis (3.2) in proving (3.4).

4. Basic equalities. In this section we prove a couple of elementary but useful equalities which are simple consequences of Fubini's theorem.

4.1. Lemma. *Let ν be a measure on X. If $f\colon X\to[0,\infty]$ is a ν-measurable function, $0<r<\infty$ and $0\le\alpha<\infty$, then*

$$
\int_{\{f>\alpha\}} f^r\,d\nu = r\int_{\alpha}^{\infty}\lambda^{r-1}\nu(\{f>\lambda\})\,d\lambda + \alpha^r\nu(\{f>\alpha\}). \tag{4.2}
$$

Proof. By Fubini's theorem we get

$$
\begin{aligned}
\int_{\alpha}^{\infty}\lambda^{r-1}\nu(\{f>\lambda\})\,d\lambda &= \int_{\alpha}^{\infty}\int_X \lambda^{r-1}\chi_{\{f>\lambda\}}\,d\nu\,d\lambda\\
&= \int_X\int_{\alpha}^{\infty}\lambda^{r-1}\chi_{\{f>\lambda\}}\,d\lambda\,d\nu = \int_{\{f>\alpha\}}\int_{\alpha}^{f}\lambda^{r-1}\,d\lambda\,d\nu.
\end{aligned}
\tag{4.3}
$$

Since $r>0$, we can integrate to obtain

$$
\begin{aligned}
\int_{\{f>\alpha\}}\int_{\alpha}^{f}\lambda^{r-1}\,d\lambda\,d\nu &= \frac{1}{r}\int_{\{f>\alpha\}}(f^r-\alpha^r)\,d\nu\\
&= \frac{1}{r}\Big(\int_{\{f>\alpha\}} f^r\,d\nu - \alpha^r\nu(\{f>\alpha\})\Big),
\end{aligned}
$$

which is the desired equality. □

There is also a formula corresponding to (4.2) for the negative exponents.

4.4. Lemma. *Let ν be a measure on X. If $f\colon X \to [0,\infty]$ is a ν-measurable function, $0<r<\infty$ and $0<\alpha<\infty$, then*

$$\int_{\{f<\alpha\}} f^{-r}\,d\nu = r\int_0^\alpha \lambda^{-(r+1)}\nu(\{f<\lambda\})\,d\lambda + \alpha^{-r}\nu(\{f<\alpha\}). \tag{4.5}$$

Proof. Using Lemma 4.1 for $1/f$ we get

$$\int_{\{f<1/\alpha\}} f^{-r}\,d\nu = r\int_\alpha^\infty \lambda^{r-1}\nu(\{f<1/\lambda\})\,d\lambda + \alpha^{r}\nu(\{f<1/\alpha\}).$$

Replacing α by $1/\alpha$ and changing variables we obtain

$$\begin{aligned}\int_{\{f<\alpha\}} f^{-r}\,d\nu &= r\int_{1/\alpha}^\infty \lambda^{r-1}\nu(\{f<1/\lambda\})\,d\lambda + \alpha^{-r}\nu(\{f<\alpha\})\\ &= r\int_0^\alpha \lambda^{-(r+1)}\nu(\{f<\lambda\})\,d\lambda + \alpha^{-r}\nu(\{f<\alpha\}).\end{aligned}$$

□

5. Muckenhoupt's condition $\mathcal{A}_1$. In this section we show that functions satisfying Muckenhoupt's condition $\mathcal{A}_1$ are locally integrable to a power greater than one. This section is essentially from [Ki], but we present it here to show the the analogy between classes $\mathcal{A}_1$ and $\mathcal{RH}_\infty$. It is easy to see that $\mathcal{A}_1$-condition

$$⨍_B f\,d\mu \le c\operatorname*{ess\,inf}_B f, \qquad B\subset X, \tag{5.1}$$

is equivalent to

$$\mathcal{M}f \le c\,f \tag{5.2}$$

and hence by (2.3) we get

$$\int_{\{\mathcal{M}f>\lambda\}} \mathcal{M}f\,d\mu \le c\lambda\mu(\{\mathcal{M}f>\lambda\}), \qquad \operatorname*{ess\,inf}_X \mathcal{M}f \le \lambda < \infty. \tag{5.3}$$

The constant in (5.3) depends on the $\mathcal{A}_1$-constant in (5.1) and the doubling constant in (1.1). Observe that the constants in (5.1) and (5.2) are the same. This indicates that the non-centered maximal function is the right tool in studying the $\mathcal{A}_1$-condition. The crucial difference between the centered and the non-centered maximal functions in the Euclidean case when μ equals the Lebesgue measure is that for the non-centered maximal function $\mathcal{M}f=f$ is equivalent to the fact that f is constant, but in the centered case it is equivalent to the fact that f is superharmonic, see [KiM].

Any μ-measurable function $f\colon X\to[0,\infty]$ satisfies Chebyshev's inequality

$$\mu(\{f>\lambda\}) \le \frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu, \qquad 0<\lambda<\infty. \tag{5.4}$$

If we replace the maximal function in (5.3) by any μ-measurable function $f\colon X\to[0,\infty]$ we see that (5.3) is a reverse Chebyshev's inequality.

5.5. Theorem. *If there are $\alpha \geq 0$ and $c > 1$ such that*

$$\int_{\{f>\lambda\}} f\,d\mu \leq c\lambda\mu(\{f>\lambda\}), \qquad \alpha \leq \lambda < \infty, \tag{5.6}$$

then for every r, $1 < r < c/(c-1)$, we have

$$\int_{\{f>\alpha\}} f^r\,d\mu \leq \frac{c}{c-r(c-1)}\alpha^r\mu(\{f>\alpha\}). \tag{5.7}$$

5.8. Example. The upper bound for exponent r and the constant in (5.7) are optimal. To see this, fix $c > 1$, take $X = (0,1)^n$, let μ be the Lebesgue measure and define

$$f\colon X \to [0,\infty], \quad f(x) = x_1^{1/c-1}.$$

A direct calculation shows that f satisfies (5.6) with constant c for every $\lambda \geq 1$. However, the function f is not integrable to any power $r \geq c/(c-1)$. This example also shows that the constant in (5.7) is sharp.

Proof of Theorem 5.5. Let $\beta > \alpha$ and denote $f_\beta = \min(f,\beta)$. Then

$$\int_{\{f_\beta>\lambda\}} f\,d\mu \leq c\lambda\mu(\{f_\beta>\lambda\}), \qquad \alpha \leq \lambda < \infty,$$

and we apply (4.2) with $d\nu = f\,d\mu$ and r replaced by $r-1$ and get

$$\begin{aligned}\int_{\{f>\alpha\}} f_\beta^r\,d\mu &\leq \int_{\{f>\alpha\}} f_\beta^{r-1} f\,d\mu \\ &= (r-1)\int_\alpha^\infty \lambda^{r-2}\int_{\{f_\beta>\lambda\}} f\,d\mu\,d\lambda + \alpha^{r-1}\int_{\{f>\alpha\}} f\,d\mu \\ &\leq c(r-1)\int_\alpha^\infty \lambda^{r-1}\mu(\{f_\beta>\lambda\})\,d\lambda + c\alpha^r\mu(\{f>\alpha\}).\end{aligned}$$

Next we estimate the first integral on the right side using (4.2) and find

$$\int_\alpha^\infty \lambda^{r-1}\mu(\{f_\beta>\lambda\})\,d\lambda = \frac{1}{r}\Big(\int_{\{f>\alpha\}} f_\beta^r\,d\mu - \alpha^r\mu(\{f>\alpha\})\Big).$$

Hence we obtain

$$\int_{\{f>\alpha\}} f_\beta^r\,d\mu \leq c\frac{r-1}{r}\int_{\{f>\alpha\}} f_\beta^r\,d\mu + \frac{c}{r}\alpha^r\mu(\{f>\alpha\}).$$

Choosing $r > 1$ such that $c(r-1)/r < 1$ and using the fact that all terms in the previous inequality are finite, we conclude that

$$\int_{\{f>\alpha\}} f_\beta^r\,d\mu \leq \frac{c}{c-r(c-1)}\alpha^r\mu(\{f>\alpha\}).$$

Finally, as $\beta \to \infty$, the monotone convergence theorem gives inequality (5.7). This proves the theorem. □

By Chebyshev's inequality $c \geq 1$ in (5.6). If there is $\alpha \geq 0$ such that f satisfies (5.6) with $c = 1$, then $\operatorname{ess\,sup}_X f \leq \alpha$. To see this, we use (5.6) with $\lambda = \alpha$ and get

$$0 \leq \int_{\{f \geq \alpha\}} (f - \alpha)\, d\mu \leq \alpha\mu(\{f \geq \alpha\}) - \alpha\mu(\{f \geq \alpha\}) = 0,$$

from which it follows that

$$\int_{\{f \geq \alpha\}} (f - \alpha)\, d\mu = 0,$$

and, consequently, $f \leq \alpha$ in X. In fact, a function satisfies (5.6) with $c = 1$ if and only if it is essentially bounded. However, Example 5.8 shows that for any $c > 1$ there are unbounded functions satisfying (5.6).

The assumptions of the previous theorem imply that f is integrable in $\{f > \alpha\}$ and, because $\mu(X) < \infty$, it is integrable in X, but the conclusion is that f is integrable in X to any power r, $1 < r < c/(c-1)$. In particular, the degree of integrability increases to infinity as c tends to one, as the borderline case $c = 1$ suggests. For related results, see [BSW, Lemma 2], [Ge, Lemma 1], [I] and [M1, Lemma 4].

Suppose that $f \in \mathcal{A}_1$. Using (5.3) we see that $\mathcal{M}f$ fulfills the assumptions of Theorem 5.5. From (5.7) we conclude that $\mathcal{M}f \in L^r(X)$, and hence $f \in L^r(X)$, for some $r > 1$. Moreover, we obtain estimate

$$\begin{aligned} \int_X f^r\, d\mu &\leq \int_{\{\mathcal{M}f \leq \alpha\}} (\mathcal{M}f)^r\, d\mu + \int_{\{\mathcal{M}f > \alpha\}} (\mathcal{M}f)^r\, d\mu \\ &\leq \alpha^r \mu(\{\mathcal{M}f \leq \alpha\}) + c\alpha^r \mu(\{\mathcal{M}f > \alpha\}) \\ &\leq c\alpha^r \mu(X), \qquad \operatorname*{ess\,inf}_X \mathcal{M}f \leq \alpha < \infty. \end{aligned} \tag{5.9}$$

If $\alpha < \operatorname{ess\,inf}_X \mathcal{M}f$, then in the same way as in (2.4) we see that (5.9) is true for every α such that

$$\frac{1}{c} ⨍_X f\, d\mu \leq \alpha < \infty.$$

In particular, choosing $\alpha = \frac{1}{c} ⨍_X f\, d\mu$, we get

$$⨍_X f^r\, d\mu \leq c\Big(⨍_X f\, d\mu\Big)^r.$$

6. The class $\mathcal{RH}_\infty$. There is a strong analogy between classes $\mathcal{A}_1$ and $\mathcal{RH}_\infty$. In the previous section we used the maximal function to study integrability questions for functions belonging to $\mathcal{A}_1$. In this section we prove analogous results for $\mathcal{RH}_\infty$-functions using the minimal function. It is easy to see that $\mathcal{RH}_\infty$-condition

$$\operatorname*{ess\,sup}_B f \leq c ⨍_B f\, d\mu, \qquad B \subset X \tag{6.1}$$

is equivalent to the requirement that

$$f \le c\, mf \tag{6.2}$$

and hence using (3.4) and (3.5) we get

$$\mu(\{mf < \lambda\}) \le \frac{c}{\lambda} \int_{\{mf<\lambda\}} mf\, d\mu, \qquad 0 < \lambda \le \operatorname*{ess\,sup}_X mf. \tag{6.3}$$

We shall see that this inequality implies that f is locally integrable to a negative power. Observe, that in the Euclidean case with the Lebesgue measure, for the non-centered minimal function $mf = f$ is equivalent to the fact that f is constant, but for the centered minimal function it is equivalent to the fact that f is subharmonic.

We replace the minimal function by any μ-measurable function $f\colon X \to [0,\infty]$ for which an inequality of type (6.3) is true. Clearly for any such f we have Chebyshev's inequality

$$\int_{\{f<\lambda\}} f\, d\mu \le \lambda\mu(\{f < \lambda\}), \qquad 0 < \lambda < \infty, \tag{6.4}$$

and hence inequality (6.3) is a reverse Chebyshev's inequality. Next we prove an analog of Theorem 5.5, where instead of (5.6) we assume inequality of type (6.3).

6.5. Theorem. *If there are $\alpha > 0$ and $c > 1$ so that*

$$\mu(\{f < \lambda\}) \le \frac{c}{\lambda} \int_{\{f<\lambda\}} f\, d\mu, \qquad 0 < \lambda \le \alpha, \tag{6.6}$$

then for every $0 < r < 1/(c-1)$ we have

$$\int_{\{f<\alpha\}} f^{-r}\, d\mu \le \frac{c}{1 - r(c-1)} \alpha^{-r} \mu(\{f < \alpha\}). \tag{6.7}$$

6.8. Example. The upper bound for the exponent and the constant in (6.7) are sharp. To see this, fix $c > 1$, let $X = (0,1)^n$ and define

$$f\colon X \to [0,\infty], \quad f(x) = x_1^{c-1}.$$

Then (6.6) and (6.7) become equalities and f is not integrable to power $1/(1-c)$.

Proof of Theorem 6.5. Let $0 < \beta < \alpha$ and denote $f_\beta = \max(f,\beta)$. Then

$$\mu(\{f_\beta < \lambda\}) \le \frac{c}{\lambda} \int_{\{f_\beta<\lambda\}} f_\beta\, d\mu, \qquad 0 < \lambda \le \alpha.$$

We multiply both sides by λ^{-r-1} and integrate from 0 to α to get

$$\int_0^\alpha \lambda^{-r-1} \mu(\{f_\beta < \lambda\})\, d\lambda \le c \int_0^\alpha \lambda^{-r-2} \int_{\{f_\beta<\lambda\}} f_\beta\, d\mu\, d\lambda.$$

By (4.5) the integral on the left side equals to

$$\frac{1}{r}\Big(\int_{\{f<\alpha\}} f_\beta^{-r}\,d\mu - \alpha^{-r}\mu(\{f<\alpha\})\Big)$$

and the integral on the right side is

$$\frac{1}{r+1}\Big(\int_{\{f<\alpha\}} f_\beta^{-r}\,d\mu - \alpha^{-r-1}\int_{\{f<\alpha\}} f_\beta\,d\mu\Big).$$

Combining these estimates, we get

$$\int_{\{f<\alpha\}} f_\beta^{-r}\,d\mu \le c\frac{r}{r+1}\int_{\{f<\alpha\}} f_\beta^{-r}\,d\mu + \frac{c}{r+1}\alpha^{-r-1}\int_{\{f<\alpha\}} f_\beta\,d\mu.$$

Because all integrals are finite, we obtain

$$\begin{aligned}\Big(1-c\frac{r}{r+1}\Big)\int_{\{f<\alpha\}} f_\beta^{-r}\,d\mu &\le \frac{c}{r+1}\alpha^{-r-1}\int_{\{f<\alpha\}} f_\beta\,d\mu\\ &\le \frac{c}{r+1}\alpha^{-r}\mu(\{f<\alpha\}).\end{aligned}$$

Finally, choosing $r>0$ so that $r<1/(c-1)$ and letting $\beta\to 0$, we get (6.7). This completes the proof. □

By (6.4) the constant in (6.6) satisfies $c\ge 1$. If $c=1$, then

$$0\le \int_{\{f\le\alpha\}}(\alpha-f)\,d\mu = \alpha\mu(\{f<\alpha\}) - \int_{\{f<\alpha\}} f\,d\mu = 0,$$

and hence

$$\int_{\{f\le\alpha\}}(\alpha-f)\,d\mu = 0.$$

This implies $f\ge\alpha$ in X and therefore $\operatorname{ess\,inf}_X f\ge\alpha$. In this case f is integrable to any negative power in X. Inequality (6.7) implies that f is locally integrable to power $-r$ for every $1<r<1/(c-1)$ and the upper bound increases to infinity as c tends one. For related results, see [N].

If $f\in\mathcal{RH}_\infty$, then Theorem 6.5 implies that

$$\begin{aligned}\int_X f^{-r}\,d\mu &\le \int_{\{mf<\alpha\}}(mf)^{-r}\,d\mu + \int_{\{mf\ge\alpha\}}(mf)^{-r}\,d\mu\\ &\le c\alpha^{-r}\mu(\{mf<\alpha\}) + \alpha^{-r}\mu(\{mf\ge\alpha\})\\ &\le c\alpha^{-r}\mu(X),\qquad 0<\alpha\le c\operatorname*{ess\,sup}_X mf.\end{aligned}\tag{6.9}$$

The same reasoning that gave (3.5) also implies that (6.9) holds for every α with

$$0<\alpha\le c\fint_X f\,d\mu.$$

Hence we may take $\alpha = c\fint_X f\,d\mu$ and we get

$$\fint_X f^{-r}\,d\mu \le c\Big(\fint_X f\,d\mu\Big)^{-r}.$$

This shows that f is integrable to power $-r$.

7. Maximal functions and reverse Hölder inequalities. If $f \in \mathcal{RH}_t$, $t > 1$, a similar argument that lead to (2.3) gives

$$\begin{aligned}\int_{\{\mathcal{M}f>\lambda\}} f^t\,d\mu &\le \sum_{i=1}^{\infty}\int_{\sigma B_i} f^t\,d\mu \le c\sum_{i=1}^{\infty}\Big(\fint_{\sigma B_i} f\,d\mu\Big)^t \mu(\sigma B_i)\\ &\le c\lambda^t\sum_{i=1}^{\infty}\mu(\sigma B_i) \le c\lambda^t\sum_{i=1}^{\infty}\mu(B_i)\\ &\le c\lambda^t\mu(\{\mathcal{M}f>\lambda\}), \qquad \operatorname*{ess\,inf}_X \mathcal{M}f \le \lambda < \infty.\end{aligned} \tag{7.1}$$

The constant c in (7.1) depends only on the constant in the reverse Hölder inequality and the doubling constant. If $\lambda < \operatorname{ess\,inf}_X \mathcal{M}f$, then $\mu(\{\mathcal{M}f > \lambda\}) = \mu(X)$ and (7.1) holds whenever

$$\Big(\frac{1}{c}\fint_X f^t\,d\mu\Big)^{1/t} \le \lambda < \infty$$

and using the hypothesis that f satisfies the reverse Hölder inequality we see that (7.1) holds for any

$$\fint_X f\,d\mu \le \lambda < \infty.$$

We show that (7.1) together with the weak type estimate (2.2) imply that $f \in L^r(X)$ for some $r > t$. We begin with proving a general Hardy type inequality.

7.2. Lemma. *If there is $\alpha > 0$ such that*

$$\mu(\{g>\lambda\}) \le \frac{1}{\lambda}\int_{\{g>\lambda\}} f\,d\mu, \qquad \alpha \le \lambda < \infty, \tag{7.3}$$

and the integral on the right side is finite, then for $1 \le t < \infty$ and $1 < r < \infty$ we have

$$\int_{\{g>\alpha\}} g^r\,d\mu \le \Big(\frac{r}{r-1}\Big)^t \int_{\{g>\alpha\}} g^{r-t}f^t\,d\mu. \tag{7.4}$$

Proof. Let $\beta > \alpha$. Using (7.3) we see that assumption (7.3) holds with g replaced by $g_\beta = \min(g,\beta)$. Using (4.2) we get

$$\begin{aligned}\int_{\{g>\alpha\}} g_\beta^r\,d\mu &= r\int_\alpha^\infty \lambda^{r-1}\mu(\{g_\beta>\lambda\})\,d\lambda + \alpha^r\mu(\{g>\alpha\})\\ &\le r\int_\alpha^\infty \lambda^{r-2}\int_{\{g_\beta>\lambda\}} f\,d\mu\,d\lambda + \alpha^r\mu(\{g>\alpha\})\end{aligned}$$

and

$$\int_\alpha^\infty \lambda^{r-2}\int_{\{g_\beta>\lambda\}} f\,d\mu\,d\lambda = \frac{1}{r-1}\Big(\int_{\{g>\alpha\}} g_\beta^{r-1}f\,d\mu - \alpha^{r-1}\int_{\{g>\alpha\}} f\,d\mu\Big).$$

Hence

$$\int_{\{g>\alpha\}} g_\beta^r \, d\mu \le \frac{r}{r-1} \int_{\{g>\alpha\}} g_\beta^{r-1} f \, d\mu - \frac{1}{r-1} \alpha^r \mu(\{g>\alpha\})$$
$$\le \frac{r}{r-1} \int_{\{g>\alpha\}} g_\beta^{r-1} f \, d\mu.$$

Hölder's inequality gives

$$\int_{\{g>\alpha\}} g_\beta^{r-1} f \, d\mu \le \Big(\int_{\{g>\alpha\}} g_\beta^{r-t} f^t \, d\mu \Big)^{1/t} \Big(\int_{\{g>\alpha\}} g_\beta^r \, d\mu \Big)^{(t-1)/t},$$

and since

$$\int_{\{g>\alpha\}} g_\beta^r \, d\mu \le \beta^r \mu(\{g>\alpha\}) < \infty,$$

we get (7.4) letting $\beta \to \infty$. □

Suppose, that f is integrable to a power $r > 1$ in X and recall the weak type inequality (2.2) for the maximal function. From (7.4) with $t = r > 1$ we obtain

$$\int_X (\mathcal{M} f)^r \, d\mu \le c^r \Big(\frac{r}{r-1} \Big)^r \int_X f^r \, d\mu.$$

Here c is the constant in (2.2). This is the Hardy–Littlewood–Wiener maximal function theorem.

7.5. Theorem. *If there are $\alpha > 0$, $t > 1$, $c_1 \ge 1$ such that (7.3) is true and*

$$\int_{\{g>\lambda\}} f^t \, d\mu \le c_1 \lambda^t \mu(\{g>\lambda\}), \qquad \alpha \le \lambda < \infty, \tag{7.6}$$

then for every $r > t$ for which

$$c_1 \frac{r-t}{r} \Big(\frac{r}{r-1} \Big)^t < 1, \tag{7.7}$$

we have

$$\int_{\{g>\alpha\}} g^r \, d\mu \le c_2 \alpha^r \mu(\{g>\alpha\}). \tag{7.8}$$

Here $c_2 = c_2(r, t, c_1)$.

The next example shows that the upper bound given by (7.7) is the best possible.

7.9. Example. Let $X=(0,1)^n$ and fix $c_1>1$. Suppose that $t>1$ and $r>t$ are such that

$$c_1\frac{r-t}{r}\Big(\frac{r}{r-1}\Big)^t=1.$$

Note that by continuity, we can always pick such a number r for any given c_1 and t. We define

$$f\colon X\to\mathbf{R},\quad f(x)=x_1^{-1/r}\qquad\text{and}\qquad g\colon X\to\mathbf{R},\quad g(x)=\frac{r}{r-1}f(x).$$

It is easy to see that the hypotheses of Theorem 7.5 are fulfilled. However, g is not integrable to the power r in $\{g>\alpha\}$. This example also shows that the constant in (7.4) is sharp.

Proof of Theorem 7.5. Let $\beta>\alpha$. The truncated function $g_\beta=\min(g,\beta)$ satisfies the assumptions of the theorem. Using (4.2) we get

$$\begin{aligned}\int_\alpha^\infty\lambda^{r-t-1}\int_{\{g_\beta>\lambda\}}f^t\,d\mu\,d\lambda&\le c_1\int_\alpha^\infty\lambda^{r-1}\mu(\{g_\beta>\lambda\})\,d\lambda\\&=\frac{c_1}{r}\Big(\int_{\{g>\alpha\}}g_\beta^r\,d\mu-\alpha^r\mu(\{g>\alpha\})\Big)\end{aligned}$$

and

$$\int_{\{g>\alpha\}}g_\beta^{r-t}f^t\,d\mu=(r-t)\int_\alpha^\infty\lambda^{r-t-1}\int_{\{g_\beta>\lambda\}}f^t\,d\mu\,d\lambda+\alpha^{r-t}\int_{\{g>\alpha\}}f^t\,d\mu.$$

Therefore we have

$$\int_{\{g>\alpha\}}g_\beta^{r-t}f^t\,d\mu\le c_1\frac{r-t}{r}\int_{\{g>\alpha\}}g_\beta^r\,d\mu+c_1\frac{t}{r}\alpha^r\mu(\{g>\alpha\})\tag{7.10}$$

and inequality (7.4) yields

$$\int_{\{g>\alpha\}}g_\beta^r\,d\mu\le c_1\frac{r-t}{r}\Big(\frac{r}{r-1}\Big)^t\int_{\{g>\alpha\}}g_\beta^r\,d\mu+c_1\frac{t}{r}\Big(\frac{r}{r-1}\Big)^t\alpha^r\mu(\{g>\alpha\}).$$

Since

$$\int_{\{g>\alpha\}}g_\beta^r\,d\mu\le\beta^r\mu(\{g>\alpha\})<\infty,$$

we may choose $r>t$ so that (7.7) holds and we get

$$\int_{\{g>\alpha\}}g_\beta^r\,d\mu\le c_2\alpha^r\mu(\{g>\alpha\}),$$

where $c_2=c_2(r,t,c_1)$. Letting $\beta\to\infty$, we see that (7.8) holds and the proof is complete. $\square$

By Hölder's inequality $c_1 \geq 1$. If (7.6) holds with $c_1 = 1$ and all the other assumptions of Theorem 7.5 are satisfied, then by Hölder's inequality we have

$$\begin{aligned}\lambda\mu(\{g > \lambda\}) &\leq \int_{\{g>\lambda\}} f\,d\mu \leq \Big(\int_{\{g>\lambda\}} f^t\,d\mu\Big)^{1/t} \mu(\{g > \lambda\})^{1-1/t} \\ &\leq \lambda\mu(\{g > \lambda\}), \qquad \alpha \leq \lambda < \infty,\end{aligned}$$

and hence all the inequalities are equalities. This is possible if and only if $\mu(\{g > \alpha\}) = 0$ or $f = \alpha$ in $\{g > \alpha\}$, since an equality occurs in Hölder's inequality only in that case. ¿From this it follows that $\operatorname{ess\,sup} g \leq \alpha$.

Assumption (7.6) implies that $f \in L^t(X)$ and Lemma 7.2 implies that also $g \in L^t(X)$. ¿From (7.8) we conclude that $g \in L^r(X)$ for any power $r > t$ for which (7.7) holds. In particular, if c_1 tends to one, the degree of integrability increases to infinity corresponding to the borderline case $c_1 = 1$. On the other hand, if c_1 goes to infinity, the degree of local integrability decreases to t.

If $f \in \mathcal{RH}_t$, $t > 0$, then the assumptions of Theorem 7.5 are fulfilled by cf and $\mathcal{M}f$, where c is the constant in (2.2), see inequalities (7.1) and (2.2). Using Theorem 7.5 we see that

$$\begin{aligned}\int_X f^r\,d\mu &\leq \int_{\{\mathcal{M}f\leq\alpha\}} (\mathcal{M}f)^r\,d\mu + \int_{\{\mathcal{M}f>\alpha\}} (\mathcal{M}f)^r\,d\mu \\ &\leq c\alpha^r\mu(\{\mathcal{M}f \leq \alpha\}) + \alpha^r\mu(\{\mathcal{M}f > \alpha\}) \\ &\leq c\alpha^r\mu(X), \qquad c\fint_X f\,d\mu \leq \alpha < \infty.\end{aligned}$$

Hence we get

$$\fint_X f^r\,d\mu \leq c\Big(\fint_X f\,d\mu\Big)^r$$

and consequently $f \in L^r(X)$ for some $r > t$.

If $f \in \mathcal{RH}_{-t}$, $t > 0$, exactly the same way as in (2.2) we get

$$\begin{aligned}\int_{\{\mathcal{M}f>\lambda\}} f^{-t}\,d\mu &\leq \sum_{i=1}^{\infty}\fint_{\sigma B_i} f^{-t}\,d\mu \leq c\sum_{i=1}^{\infty}\Big(\fint_{\sigma B_i} f\,d\mu\Big)^{-t}\mu(\sigma B_i) \\ &\leq c\sum_{i=1}^{\infty}\Big(\fint_{B_i} f\,d\mu\Big)^{-t}\mu(B_i) \leq c\lambda^{-t}\sum_{i=1}^{\infty}\mu(B_i) \\ &\leq c\lambda^{-t}\mu(\{\mathcal{M}f > \lambda\}), \qquad 0 < \lambda < \infty.\end{aligned} \tag{7.11}$$

The constant c in (7.11) depends only on the constant in the reverse Hölder inequality and the doubling constant.

Next we show that estimate (7.11) and the reverse weak type inequality (2.3) imply that f is locally integrable to a slightly greater power than one. We emphasize that qualitatively this result follows from Theorem 7.5, because by Hölder's inequality we have

$$\fint_B f\,d\mu \leq c\Big(\fint_B f^{-t}\,d\mu\Big)^{-1/t} \leq c\Big(\fint_B f^s\,d\mu\Big)^{1/s}, \qquad B \subset X,$$

for any $0 < s < 1$. But in order to obtain sharp results, we need another argument. First we prove an analog of Lemma 7.2 where assumption (7.3) is replaced by an inequality of type (7.11).

7.12. Lemma. *Suppose that $-\infty < -t < 0 < r < \infty$. If there are $\alpha > 0$ and $c_1 \ge 1$ such that*

$$\int_{\{g>\lambda\}} f^{-t}\,d\mu \le c_1\lambda^{-t}\mu(\{g>\lambda\}), \qquad \alpha \le \lambda < \infty, \tag{7.13}$$

and the integral on the right side is finite, then

$$\int_{\{g>\alpha\}} g^r\,d\mu \le \Big(c_1\frac{r+t}{r}\Big)^{1/t}\int_{\{g>\alpha\}} g^{r-1}f\,d\mu \tag{7.14}$$

Proof. We observe that the truncated function $g_\beta = \min(g,\beta)$, $\beta > \alpha$, satisfies (7.13). Then we proceed exactly as in the proof of Theorem 7.5 and get

$$\begin{aligned}\int_{\{g>\alpha\}} g_\beta^{r+t}f^{-t}\,d\mu &\le c_1\frac{r+t}{r}\int_{\{g>\alpha\}} g_\beta^r\,d\mu - c_1\frac{t}{r}\alpha^r\mu(\{g>\alpha\})\\ &\le c_1\frac{r+t}{r}\int_{\{g>\alpha\}} g_\beta^r\,d\mu.\end{aligned}$$

Hölder's inequality implies

$$\int_{\{g>\alpha\}} g_\beta^{r+t}f^{-t}\,d\mu \ge \Big(\int_{\{g>\alpha\}} g_\beta^{r-1}f\,d\mu\Big)^{-t}\Big(\int_{\{g>\alpha\}} g_\beta^r\,d\mu\Big)^{t+1}$$

and hence

$$\Big(\int_{\{g>\alpha\}} g_\beta^r\,d\mu\Big)^t \le c_1\frac{r+t}{r}\Big(\int_{\{g>\alpha\}} g_\beta^{r-1}f\,d\mu\Big)^t.$$

We get the claim letting $\beta\to\infty$. □

7.15. Theorem. *If there are $\alpha > 0$, $t > 0$, $c_1 \ge 1$ such that (7.13) holds and*

$$\int_{\{g>\lambda\}} f\,d\mu \le \lambda\mu(\{g>\lambda\}), \qquad \alpha \le \lambda < \infty, \tag{7.16}$$

then for every $r > 1$ for which

$$c_1\frac{r+t}{r}\Big(\frac{r-1}{r}\Big)^t < 1, \tag{7.17}$$

we have

$$\int_{\{g>\alpha\}} g^r\,d\mu \le c_2\alpha^r\mu(\{g>\alpha\}) \tag{7.18}$$

with $c_2 = c_2(t, r, c_1)$.

Proof. The proof goes along the lines of the proof of Theorem 7.5 and we use the same notation as there. Using assumption (7.16) and the case $c_1 = 1$ $t = 1$ of inequality (7.10) we get

$$\int_{\{g>\alpha\}} g_\beta^{r-1} f \, d\mu \le \frac{r-1}{r} \int_{\{g>\alpha\}} g_\beta^r \, d\mu + \frac{\alpha^r}{r} \mu(\{g > \alpha\}).$$

Observe, that in the proof of (7.10) we assumed that $t > 1$, but it is also valid when $t = 1$. Then (7.14) implies

$$\int_{\{g>\alpha\}} g_\beta^r \, d\mu \le \frac{r-1}{r} \Big(c_1 \frac{r+t}{r} \Big)^{1/t} \int_{\{g>\alpha\}} g_\beta^r \, d\mu + \frac{1}{r} \Big(c_1 \frac{r+t}{r} \Big)^{1/t} \alpha^r \mu(\{g > \alpha\}).$$

Since

$$\int_{\{g>\alpha\}} g_\beta^r \, d\mu \le \beta^r \mu(\{g > \alpha\}) < \infty,$$

we may choose $r > 1$ such that (7.17) holds and we get

$$\int_{\{g>\alpha\}} g_\beta^r \, d\mu \le c_2 \alpha^r \mu(\{g > \alpha\}),$$

where $c_2 = c_2(t, r, c_1)$. Letting $\beta \to \infty$, we see that (7.18) holds and the proof is complete. □

Modifying the functions in Example 7.9, we see that the upper bound given by (7.17) is the best possible. Combining (7.13), (7.16) and Hölder's inequality, we see that $c_1 \ge 1$. If $c_1 = 1$, then g is essentially bounded.

If $f \in \mathcal{RH}_{-t}$, $t > 0$, then $c^{-1} f$, here c is the constant in (2.3), and $\mathcal{M} f$ fulfill the hypotheses of Theorem 7.15. Thus (7.18) implies

$$\begin{aligned} \int_X f^r \, d\mu &\le \int_{\{\mathcal{M}f \le \alpha\}} (\mathcal{M}f)^r \, d\mu + \int_{\{\mathcal{M}f > \alpha\}} (\mathcal{M}f)^r \, d\mu \\ &\le \alpha^r \mu(\{\mathcal{M}f \le \alpha\}) + c\alpha^r \mu(\{\mathcal{M}f > \alpha\}) \\ &\le c\alpha^r \mu(X), \end{aligned}$$

for any $r > 1$ such that (7.17) holds. By (2.4) we may take $\alpha = c^{-1} \fint_X f \, d\mu$ and hence we have

$$\fint_X f^r \, d\mu \le c \Big(\fint_X f \, d\mu \Big)^r$$

This shows that $f \in L^r(X)$.

8. Minimal functions and reverse Hölder inequalities. If $f \in \mathcal{RH}_{-t}$, $t > 0$, then in exactly the same way as deriving (3.4) we get

$$\begin{aligned}\int_{\{mf<\lambda\}} f^{-t}\,d\mu &\le \sum_{i=1}^{\infty}\int_{\sigma B_i} f^{-t}\,d\mu \le c\sum_{i=1}^{\infty}\Big(\fint_{\sigma B_i} f\,d\mu\Big)^{-t}\mu(\sigma B_i)\\ &\le c\lambda^{-t}\sum_{i=1}^{\infty}\mu(\sigma B_i) \le c\lambda^{-t}\sum_{i=1}^{\infty}\mu(B_i)\\ &\le c\lambda^{-t}\mu(\{mf<\lambda\}), \qquad 0<\lambda\le \operatorname*{ess\,sup}_X mf.\end{aligned} \tag{8.1}$$

In fact, if $\operatorname{ess\,sup}_X f < \lambda < \infty$, then $\mu(\{mf < \lambda\}) = \mu(X)$ and (8.1) is true whenever

$$0 < \lambda \le \Big(\frac{1}{c}\fint_X f^{-t}\,d\mu\Big)^{-1/t}.$$

Since $f \in \mathcal{RH}_{-t}$, we see that (8.1) holds if

$$0 < \lambda \le \fint_X f\,d\mu.$$

We begin with proving that weak type inequality (3.3) implies a strong type inequality for the negative powers.

8.2. Lemma. *Suppose that $0 < r < \infty$ and $0 < t < \infty$. If there is $\alpha > 0$ so that*

$$\int_{\{h<\lambda\}} f\,d\mu \le \lambda\mu(\{h<\lambda\}), \qquad 0<\lambda\le\alpha, \tag{8.3}$$

then

$$\int_{\{h<\alpha\}} h^{-r}\,d\mu \le \Big(\frac{r+1}{r}\Big)^t \int_{\{h<\alpha\}} h^{t-r} f^{-t}\,d\mu, \tag{8.4}$$

whenever the right side of (8.4) is finite.

Proof. Let $0 < \beta < \alpha$ and denote $h_\beta = \max(h,\beta)$. Then h_β satisfies the assumptions of the theorem. We multiply both sides by $\lambda^{-(r+2)}$ and integrate from 0 to α to get

$$\int_0^\alpha \lambda^{-(r+2)}\int_{\{h_\beta<\lambda\}} f\,d\mu\,d\lambda \le \int_0^\alpha \lambda^{-(r+1)}\mu(\{h_\beta<\lambda\})\,d\lambda.$$

By (4.5) the left side equals to

$$\frac{1}{r+1}\Big(\int_{\{h<\alpha\}} h_\beta^{-r-1} f\,d\mu - \alpha^{-(r+1)}\int_{\{h<\alpha\}} f\,d\mu\Big)$$

and the right side is

$$\frac{1}{r}\Big(\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu - \alpha^{-r}\mu(\{h<\alpha\})\Big).$$

Combining these and using Hölder's inequality, we find

$$\begin{aligned}\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu &\ge \frac{r}{r+1}\int_{\{h<\alpha\}} h_\beta^{-r-1} f\,d\mu + \frac{1}{r+1}\alpha^{-r-1}\int_{\{h<\alpha\}} f\,d\mu\\ &\ge \frac{r}{r+1}\int_{\{h<\alpha\}} h_\beta^{-r-1} f\,d\mu\\ &\ge \frac{r}{r+1}\Big(\int_{\{h<\alpha\}} h_\beta^{t-r} f^{-t}\,d\mu\Big)^{-1/t}\Big(\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu\Big)^{1+1/t}.\end{aligned}$$

Since all integrals are finite, we get

$$\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu \le \Big(\frac{r+1}{r}\Big)^t \int_{\{h<\alpha\}} h^{t-r} f^{-t}\,d\mu.$$

The claim follows from the monotone convergence theorem as $\beta \to 0$. □

Lemma 8.2 shows that if $f\colon X \to [0,\infty]$ is integrable to power $-r$, then

$$\int_X (mf)^{-r}\,d\mu \le c^r\Big(\frac{r+1}{r}\Big)^r \int_X f^{-r}\,d\mu.$$

This corresponds the Hardy–Littlewood–Wiener theorem for the minimal function.

8.5. Theorem. *If there are $\alpha > 0$, $t > 0$ and $c_1 \ge 1$ such that (8.3) holds and*

$$\int_{\{h<\lambda\}} f^{-t}\,d\mu \le c_1\lambda^{-t}\mu(\{h<\lambda\}), \qquad 0<\lambda\le\alpha, \tag{8.6}$$

then for every $r > t$ for which

$$c_1\frac{r-t}{r}\Big(\frac{r+1}{r}\Big)^t < 1, \tag{8.7}$$

there is $c_2 = c_2(t,r,c_1)$ so that

$$\int_{\{h<\alpha\}} h^{-r}\,d\mu \le c_2\alpha^{-r}\mu(\{h<\alpha\}). \tag{8.8}$$

Proof. Let $0 < \beta < \alpha$ and denote $h_\beta = \max(h,\beta)$. Then h_β fulfills the assumptions of the theorem. By (4.5) we have

$$\begin{aligned}\int_{\{h<\alpha\}} h_\beta^{t-r} f^{-t}\,d\mu &= (r-t)\int_0^\alpha \lambda^{-(r-t+1)}\int_{\{h_\beta<\lambda\}} f^{-t}\,d\mu\,d\lambda + \alpha^{t-r}\int_{\{h<\alpha\}} f^{-t}\,d\mu\\ &\le c_1(r-t)\int_0^\alpha \lambda^{-(r+1)}\mu(\{h_\beta<\lambda\})\,d\lambda + c_1\alpha^{-r}\mu(\{h<\alpha\})\\ &= c_1\frac{r-t}{r}\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu + c_1\frac{t}{r}\alpha^{-r}\mu(\{h<\alpha\}).\end{aligned}$$

Then Lemma 8.2 yields

$$\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu \le c_1\frac{r-t}{r}\Big(\frac{r+1}{r}\Big)^t\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu + c_1\frac{t}{r}\Big(\frac{r+1}{r}\Big)^t\alpha^{-r}\mu(\{h<\alpha\}).$$

Since all integrals are finite, we conclude that there is $c_2 = c_2(t,r,c_1)$ so that

$$\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu \le c_2\alpha^{-r}\mu(\{h<\alpha\})$$

whenever $r > t$ such that (8.7) holds. The claim follows letting $\beta \to 0$. □

Again, the upper bound given by (8.7) is sharp. If $c_1 = 1$, then $\operatorname{ess\,inf} h \geq \alpha$ and h is integrable to any negative power. If c_1 tends to one, then the upper bound given by (8.7) goes to infinity as we may expect.

Suppose that $f \in \mathcal{RH}_{-t}$, $t > 0$. Inequalities (8.1) and (3.3) ensure that cf, c is the constant in (3.3), and mf satisfy the assumptions in Theorem 8.5. Hence, using (8.8), we find that

$$\begin{aligned}\int_X f^{-r}\,d\mu &= \int_{\{mf<\alpha\}} (mf)^{-r}\,d\mu + \int_{\{mf\geq\alpha\}} (mf)^{-r}\,d\mu \\ &\leq c\alpha^{-r}\mu(\{mf<\alpha\}) + \alpha^{-r}\mu(\{mf\geq\alpha\}) \\ &\leq c\alpha^{-r}\mu(X).\end{aligned}$$

The discussion after (8.1) shows that we may take $\alpha = c\fint_X f\,d\mu$ and a substitution gives

$$\fint_X f^{-r}\,d\mu \leq c\Big(\fint_X f\,d\mu\Big)^{-r}.$$

Hence there is $r > t$ so that f is integrable to power $-r$.

If $f \in \mathcal{RH}_t$, $t > 1$, then exactly the same way as in (3.2) we have

$$\begin{aligned}\int_{\{mf<\lambda\}} f^t\,d\mu &\leq \sum_{i=1}^{\infty}\int_{\sigma B_i} f^t\,d\mu \leq c\sum_{i=1}^{\infty}\Big(\fint_{\sigma B_i} f\,d\mu\Big)^t\mu(\sigma B_i) \\ &\leq c\sum_{i=1}^{\infty}\Big(\fint_{B_i} f\,d\mu\Big)^t\mu(B_i) \leq c\lambda^t\sum_{i=1}^{\infty}\mu(B_i) \\ &\leq c\lambda^t\mu(\{mf<\lambda\}), \qquad 0<\lambda<\infty.\end{aligned} \tag{8.9}$$

Observe that we used the additional hypothesis (3.2) here.

8.10. Lemma. *Suppose that $0 < r < \infty$, $1 \leq t < \infty$. If there are $\alpha > 0$ and $c_1 \geq 1$ such that*

$$\int_{\{h<\lambda\}} f^t\,d\mu \leq c_1\lambda^t\mu(\{h<\lambda\}), \qquad 0<\lambda\leq\alpha, \tag{8.11}$$

then

$$\int_{\{h<\alpha\}} h^{-r-1}f\,d\mu \leq \Big(c_1\frac{r+t}{r}\Big)^{1/t}\int_{\{h<\alpha\}} h^{-r}\,d\mu, \tag{8.12}$$

whenever the right side of (8.12) is finite.

Proof. Using the same argument as in the proof of Theorem 8.5 we obtain

$$\begin{aligned}\int_{\{h<\alpha\}} h_\beta^{-t-r}f^t\,d\mu &\leq c_1\frac{r+t}{r}\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu - c_1\frac{t}{r}\alpha^{-r}\mu(\{h<\alpha\}) \\ &\leq c_1\frac{r+t}{r}\int_{\{h<\alpha\}} h_\beta^{-r}\,d\mu.\end{aligned}$$

Hölder's inequality implies

$$\int_{\{h<\alpha\}} h_\beta^{-1-r} f \, d\mu \le \Big(\int_{\{h<\alpha\}} h_\beta^{-t-r} f^t \, d\mu \Big)^{1/t} \Big(\int_{\{h<\alpha\}} h_\beta^{-r} \, d\mu \Big)^{1-1/t}$$

and therefore

$$\int_{\{h<\alpha\}} h_\beta^{-1-r} f \, d\mu \le \Big(c_1 \frac{r+t}{r} \Big)^{1/t} \int_{\{h<\alpha\}} h_\beta^{-r} \, d\mu.$$

We get the claim making $\beta \to \infty$. □

8.13. Theorem. *If there are $\alpha > 0$, $t > 1$ and $c_1 \ge 1$ such that (8.11) is true and*

$$\mu(\{h < \lambda\}) \le \frac{1}{\lambda} \int_{\{h<\lambda\}} f \, d\mu \qquad 0 < \lambda \le \alpha, \tag{8.14}$$

then for every $r > 0$ for which

$$c_1 \frac{r+t}{r} \Big(\frac{r}{r+1} \Big)^t < 1 \tag{8.15}$$

we have

$$\int_{\{h<\alpha\}} h^{-r} \, d\mu \le c_2 \alpha^{-r} \mu(\{h < \alpha\}). \tag{8.16}$$

Here $c_2 = c_2(t, r, c_1)$.

Proof. Let $0 < \beta < \alpha$ and denote $h_\beta = \max(h, \beta)$. Then h_β satisfies the assumptions of the theorem. By (4.5) we have

$$\begin{aligned}\int_{\{h<\alpha\}} h_\beta^{-r} \, d\mu &= r \int_0^\alpha \lambda^{-(r+1)} \mu(\{h_\beta < \lambda\}) \, d\lambda + \alpha^{-r} \mu(\{h < \alpha\}) \\ &\le r \int_0^\alpha \lambda^{-(r+2)} \int_{\{h_\beta<\lambda\}} f \, d\mu \, d\lambda + \alpha^{-r} \mu(\{h < \alpha\}) \\ &\le \frac{r}{r+1} \int_{\{h<\alpha\}} h_\beta^{-r-1} f \, d\mu + \frac{\alpha^{-r}}{r} \mu(\{h < \alpha\}).\end{aligned}$$

Then we apply (8.12) and get

$$\int_{\{h<\alpha\}} h_\beta^{-r} \, d\mu \le \frac{r}{r+1} \Big(c_1 \frac{r+t}{r} \Big)^{1/t} \int_{\{h<\alpha\}} h_\beta^{-r} \, d\mu + \frac{\alpha^{-r}}{r} \mu(\{h < \alpha\}).$$

Since all terms are finite, we conclude that

$$\int_{\{h<\alpha\}} h_\beta^{-r} \, d\mu \le c_2 \alpha^{-r} \mu(\{h < \alpha\})$$

with $c_2 = c_2(t, r, c_1)$ if $r > t$ such that (8.15) holds. The claim follows letting $\beta \to 0$. □

If $c_1 = 1$, then $\operatorname{ess\,inf} h \geq \alpha$ and h is integrable to any negative power. If c_1 tends to one then the upper bound given by (8.15) increases to infinity.

Suppose that $f \in \mathcal{RH}_t$, $t > 1$. Using (8.9) and (3.4) we see that cf and mf satisfy the hypotheses of Theorem 8.13. Hence

$$\begin{aligned}\int_X f^{-r}\,d\mu &\leq \int_{\{mf<\alpha\}} (mf)^{-r}\,d\mu + \int_{\{mf\geq\alpha\}} (mf)^{-r}\,d\mu \\ &\leq c\alpha^{-r}\mu(\{mf<\alpha\}) + \alpha^{-r}\mu(\{mf\geq\alpha\}) \\ &\leq c\alpha^{-r}\mu(X), \qquad 0<\alpha\leq c\fint_X f\,d\mu.\end{aligned}$$

Substitution $\alpha = c\fint_X f\,d\mu$ yields

$$\fint_X f^{-r}\,d\mu \leq c\Big(\fint_X f\,d\mu\Big)^{-r}.$$

Therefore f is integrable to power $-r$.

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