RELAXATION AND INTEGRAL REPRESENTATION FOR FUNCTIONALS OF LINEAR GROWTH ON METRIC MEASURE SPACES

HEIKKI HAKKARAINEN, JUHA KINNUNEN, PANU LAHTI, PEKKA LEHTELÄ

Abstract. This article studies an integral representation of functionals of linear growth on metric measure spaces with a doubling measure and a Poincaré inequality. Such a functional is defined via relaxation, and it defines a Radon measure on the space. For the singular part of the functional, we get the expected integral representation with respect to the variation measure. A new feature is that in the representation for the absolutely continuous part, a constant appears already in the weighted Euclidean case. As an application we show that in a variational minimization problem involving the functional, boundary values can be presented as a penalty term.

1. Introduction

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex, nondecreasing function that satisfies the linear growth condition

$$mt \le f(t) \le M(1+t)$$

with some constants $0 < m \le M < \infty$. Let Ω be an open set on a metric measure space (X, d, μ). Throughout the work we assume that the measure is doubling and that the space supports a Poincaré inequality. For $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the functional of linear growth via relaxation by

$$\mathcal{F}(u,\Omega) = \inf\left\{ \liminf_{i\to\infty} \int_\Omega f(g_{u_i})\,d\mu : \, u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\, u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\},$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i. For $f(t) = t$, this gives the definition of functions of bounded variation, or BV functions, on metric measure spaces, see [1], [3] and [24]. For $f(t) = \sqrt{1+t^2}$, we get the generalized surface area functional, which has been considered previously in [17] and [18]. Our first result shows that if $\mathcal{F}(u,\Omega) < \infty$,

2010 *Mathematics Subject Classification.* 49Q20, 30L99, 26B30.

The research was supported by the Academy of Finland and the Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation. Part of the work was done during a visit at the Institut Mittag-Leffler (Djursholm, Sweden).

then $\mathcal{F}(u,\cdot)$ is a Borel regular outer measure on Ω. This result is a generalization of [24, Theorem 3.4]. For corresponding results in the Euclidean case with either the Lebesgue measure or more general measures, we refer to [2], [4], [8], [9], [10], [13], [14], and [15].

Our main goal is to study whether the relaxed functional $\mathcal{F}(u,\cdot)$ can be represented as an integral in terms of the variation measure $\|Du\|$, as can be done in the Euclidean setting, see e.g. [2, Section 5.5]. To this end, let $u \in L^1(\Omega)$ with $\mathcal{F}(u,\Omega) < \infty$. Then the growth condition implies that $u \in \mathrm{BV}(\Omega)$. We denote the decomposition of the variation measure $\|Du\|$ into the absolutely continuous and singular parts by $d\|Du\| = a\,d\mu + d\|Du\|^s$, where $a \in L^1(\Omega)$. Similarly, we denote by $\mathcal{F}^a(u,\cdot)$ and $\mathcal{F}^s(u,\cdot)$ the absolutely continuous and singular parts of $\mathcal{F}(u,\cdot)$ with respect to μ. For the singular part, we obtain the integral representation

$$\mathcal{F}^s(u,\Omega) = f_\infty \|Du\|^s(\Omega),$$

where $f_\infty = \lim_{t\to\infty} f(t)/t$. This is analogous to the Euclidean case. However, for the absolutely continuous part we only get an integral representation up to a constant

$$\int_\Omega f(a)\,d\mu \le \mathcal{F}^a(u,\Omega) \le \int_\Omega f(Ca)\,d\mu,$$

where C depends on the doubling constant of the measure and the constants in the Poincaré inequality. Furthermore, we give a counterexample which shows that the constant cannot be dismissed. We observe that working in the general metric context produces significant challenges that are already visible in the Euclidean setting with a weighted Lebesgue measure. In overcoming these challenges, a key technical tool is an equi-integrability result for the discrete convolution of a measure. As a by-product of our analysis, we are able to show that a BV function is actually a Newton-Sobolev function in a set where the variation measure is absolutely continuous.

As an application of the integral representation, we consider a minimization problem related to functionals of linear growth. First we define the concept of boundary values of BV functions, which is a delicate issue already in the Euclidean case. Let $\Omega \Subset \Omega^*$ be bounded open sets in X, and assume that $h \in \mathrm{BV}(\Omega^*)$. We define $\mathrm{BV}_h(\Omega)$ as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = h$ μ-almost everywhere in $\Omega^* \setminus \Omega$. A function $u \in \mathrm{BV}_h(\Omega)$ is a minimizer of the functional of linear growth with boundary values h, if

$$\mathcal{F}(u,\Omega^*) = \inf \mathcal{F}(v,\Omega^*),$$

where the infimum is taken over all $v \in \mathrm{BV}_h(\Omega)$. It was shown in [17] that this problem always has a solution. By using the integral representation, we can express the boundary values as a penalty term. More precisely, under suitable conditions on the space and Ω, we establish

equivalence between the above minimization problem and minimizing the functional

$$\mathcal{F}(u,\Omega) + f_\infty \int_{\partial\Omega} |T_\Omega u - T_{X\setminus\Omega} h|\theta_\Omega \, d\mathcal{H}$$

over all $u \in \mathrm{BV}(\Omega)$. Here $T_\Omega u$ and $T_{X\setminus\Omega} u$ are boundary traces and θ_Ω is a strictly positive density function. This extends the Euclidean results in [14, p. 582] to metric measure spaces. A careful analysis of BV extension domains and boundary traces is needed in the argument.

2. Preliminaries

In this paper, (X, d, μ) is a complete metric measure space with a Borel regular outer measure μ. The measure μ is assumed to be doubling, meaning that there exists a constant $c_d > 0$ such that

$$0 < \mu(B(x,2r)) \le c_d \mu(B(x,r)) < \infty$$

for every ball $B(x,r)$ with center $x \in X$ and radius $r > 0$. For brevity, we will sometimes write λB for $B(x,\lambda r)$. On a metric space, a ball B does not necessarily have a unique center point and radius, but we assume every ball to come with a prescribed center and radius. The doubling condition implies that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge C \left(\frac{r}{R}\right)^Q \tag{2.1}$$

for every $r \le R$ and $y \in B(x,R)$, and some $Q > 1$ and $C \ge 1$ that only depend on c_d. We recall that a complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as the space of functions that are Lipschitz continuous in every $\Omega' \Subset \Omega$ (and other local spaces of functions are defined similarly). Here $\Omega' \Subset \Omega$ means that Ω' is open and that $\overline{\Omega'}$ is a compact subset of Ω.

For any set $A \subset X$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) = \inf\left\{ \sum_{i=1}^\infty \frac{\mu(B(x_i,r_i))}{r_i} : A \subset \bigcup_{i=1}^\infty B(x_i,r_i),\ r_i \le R \right\},$$

where $0 < R < \infty$. The Hausdorff measure of codimension 1 of a set $A \subset X$ is

$$\mathcal{H}(A) = \lim_{R\to 0} \mathcal{H}_R(A).$$

The measure theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ in which both E and its complement have positive density, i.e.

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0.$$

A curve γ is a rectifiable continuous mapping from a compact interval to X. The length of a curve γ is denoted by ℓ_γ. We will assume every

curve to be parametrized by arc-length, which can always be done (see e.g. [16, Theorem 3.2]).

A nonnegative Borel function g on X is an upper gradient of an extended real-valued function u on X if for all curves γ in X, we have

$$|u(x) - u(y)| \le \int_\gamma g\,ds \tag{2.2}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. Here x and y are the end points of γ. If g is a nonnegative μ-measurable function on X and (2.2) holds for 1-almost every curve, then g is a 1-weak upper gradient of u. A property holds for 1-almost every curve if it fails only for a curve family with zero 1-modulus. A family Γ of curves is of zero 1-modulus if there is a nonnegative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho\,ds$ is infinite.

We consider the following norm

$$\|u\|_{N^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all upper gradients g of u. The Newtonian space is defined as

$$N^{1,1}(X) = \{u : \|u\|_{N^{1,1}(X)} < \infty\}/\sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if $\|u - v\|_{N^{1,1}(X)} = 0$. In the definition of upper gradients and Newtonian spaces, the whole space X can be replaced by any μ-measurable (typically open) set $\Omega \subset X$. It is known that for any $u \in N^{1,1}_{\rm loc}(\Omega)$, there exists a minimal 1-weak upper gradient, which we always denote by g_u, satisfying $g_u \le g$ μ-almost everywhere in Ω, for any 1-weak upper gradient $g \in L^1_{\rm loc}(\Omega)$ of u [5, Theorem 2.25]. For more on Newtonian spaces, we refer to [26] and [5].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [1], [3] and [24]. For $u \in L^1_{\rm loc}(X)$, we define the total variation of u as

$$\begin{aligned}&\|Du\|(X)\\ &\quad = \inf\left\{\liminf_{i\to\infty} \int_X g_{u_i}\,d\mu : u_i \in \mathrm{Lip}_{\rm loc}(X),\, u_i \to u \text{ in } L^1_{\rm loc}(X)\right\},\end{aligned}$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i. We say that a function $u \in L^1(X)$ is of bounded variation, and write $u \in \mathrm{BV}(X)$, if $\|Du\|(X) < \infty$. Moreover, a μ-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) = \inf\{\|Du\|(\Omega) : A \subset \Omega,\, \Omega \subset X \text{ is open}\}.$$

If $u \in \mathrm{BV}(\Omega)$, $\|Du\|(\cdot)$ is a finite Radon measure on Ω by [24, Theorem 3.4]. The perimeter of E in Ω is denoted by

$$P(E,\Omega) = \|D\chi_E\|(\Omega).$$

We have the following coarea formula given by Miranda in [24, Proposition 4.2]: if $\Omega \subset X$ is an open set and $u \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega)\, dt. \tag{2.3}$$

For an open set $\Omega \subset X$ and a set of locally finite perimeter $E \subset X$, we know that

$$\|D\chi_E\|(\Omega) = \int_{\partial^* E \cap \Omega} \theta_E \, d\mathcal{H}, \tag{2.4}$$

where $\theta_E : X \to [\alpha, c_d]$, with $\alpha = \alpha(c_d, c_P) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6]. The constant c_P is related to the Poincaré inequality, see below.

The jump set of a function $u \in \mathrm{BV}_{\mathrm{loc}}(X)$ is defined as

$$S_u = \{x \in X : u^\wedge(x) < u^\vee(x)\},$$

where $u^\wedge$ and $u^\vee$ are the lower and upper approximate limits of u defined as

$$u^\wedge(x) = \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0} \frac{\mu(\{u < t\} \cap B(x,r))}{\mu(B(x,r))} = 0\right\}$$

and

$$u^\vee(x) = \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0} \frac{\mu(\{u > t\} \cap B(x,r))}{\mu(B(x,r))} = 0\right\}.$$

Outside the jump set, i.e. in $X \setminus S_u$, $\mathcal{H}$-almost every point is a Lebesgue point of u [20, Theorem 3.5], and we denote the Lebesgue limit at x by $\widetilde{u}(x)$.

We say that X supports a $(1,1)$-Poincaré inequality if there exist constants $c_P > 0$ and $\lambda \geq 1$ such that for all balls $B(x,r)$, all locally integrable functions u, and all 1-weak upper gradients g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq c_P r \fint_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

If the space supports a $(1,1)$-Poincaré inequality, by an approximation argument we get for every $u \in L^1_{\mathrm{loc}}(X)$

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq c_P r \frac{\|Du\|(B(x,\lambda r))}{\mu(B(x,\lambda r))},$$

where the constant c_P and the dilation factor λ are the same as in the $(1,1)$-Poincaré inequality. When $u = \chi_E$ for $E \subset X$, we get the relative isoperimetric inequality

$$\min\{\mu(B(x,r)\cap E), \mu(B(x,r)\setminus E)\} \le 2c_P r \|D\chi_E\|(B(x,\lambda r)). \tag{2.5}$$

Throughout the work we assume, without further notice, that the measure μ is doubling and that the space supports a $(1,1)$-Poincaré inequality.

3. Functional and its measure property

In this section we define the functional that is considered in this paper, and show that it defines a Radon measure. Let f be a convex nondecreasing function that is defined on $[0,\infty)$ and satisfies the linear growth condition

$$mt \le f(t) \le M(1+t) \tag{3.1}$$

for all $t \ge 0$, with some constants $0 < m \le M < \infty$. This implies that f is Lipschitz continuous with constant $L > 0$. Furthermore, we define

$$f_\infty = \sup_{t>0} \frac{f(t)-f(0)}{t} = \lim_{t\to\infty} \frac{f(t)-f(0)}{t} = \lim_{t\to\infty} \frac{f(t)}{t},$$

where the second equality follows from the convexity of f. From the definition of f_∞, we get the simple estimate

$$f(t) \le f(0) + tf_\infty \tag{3.2}$$

for all $t \ge 0$. This will be useful for us later.

Now we give the definition of the functional. For an open set Ω and $u \in N^{1,1}(\Omega)$, we could define it as

$$u \longmapsto \int_\Omega f(g_u)\,d\mu,$$

where g_u is the minimal 1-weak upper gradient of u. For $u \in \mathrm{BV}(\Omega)$, we need to use a relaxation procedure as given in the following definition.

Definition 3.1. Let $\Omega \subset X$ be an open set. For $u \in L^1_{\mathrm{loc}}(\Omega)$, we define

$$\begin{aligned}&\mathcal{F}(u,\Omega)\\ &\quad= \inf\left\{\liminf_{i\to\infty} \int_\Omega f(g_{u_i})\,d\mu :\ u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},\end{aligned}$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i.

Note that we could equally well require that g_{u_i} is *any* 1-weak upper gradient of u_i. We define $\mathcal{F}(u,A)$ for an arbitrary set $A \subset X$ by

$$\mathcal{F}(u,A) = \inf\{\mathcal{F}(u,\Omega) :\ \Omega \text{ is open}, A \subset \Omega\}. \tag{3.3}$$

In this section we show that if $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\mathcal{F}(u,\Omega) < \infty$, then $\mathcal{F}(u,\cdot)$ is a Borel regular outer measure on Ω, extending [24, Theorem 3.4]. The functional clearly satisfies

$$m\|Du\|(A) \le \mathcal{F}(u,A) \le M(\mu(A) + \|Du\|(A)) \tag{3.4}$$

for any $A \subset X$. This estimate follows directly from the definition of the functional, the definition of the variation measure, and (3.1). It is also easy to see that

$$\mathcal{F}(u,B) \le \mathcal{F}(u,A)$$

for any sets $B \subset A \subset X$.

Remark 3.2. In this remainder of this section we do not, in fact, need the convexity of f, or the fact that the space supports a $(1,1)$-Poincaré inequality.

In order to show the measure property, we first prove a few lemmas. The first is the following technical gluing lemma that is similar to [2, Lemma 5.44].

Lemma 3.3. *Let U', U, V', V be open sets in X such that $U' \Subset U$ and $V' \subset V$. Then there exists an open set $H \subset (U \setminus U') \cap V'$, with $H \Subset U$, such that for any $\varepsilon > 0$ and any pair of functions $u \in \mathrm{Lip}_{\mathrm{loc}}(U)$ and $v \in \mathrm{Lip}_{\mathrm{loc}}(V)$, there is a function $\phi \in \mathrm{Lip}_c(U)$ with $0 \le \phi \le 1$ and $\phi = 1$ in a neighborhood of U', such that the function $w = \phi u + (1-\phi)v \in \mathrm{Lip}_{\mathrm{loc}}(U' \cup V')$ satisfies*

$$\int_{U' \cup V'} f(g_w)\,d\mu \le \int_U f(g_u)\,d\mu + \int_V f(g_v)\,d\mu + C\int_H |u-v|\,d\mu + \varepsilon.$$

Here $C = C(U,U',M)$.

Proof. Let $\eta = \operatorname{dist}(U', X \setminus U) > 0$. Define

$$H = \left\{ x \in U \cap V' : \frac{\eta}{3} < \operatorname{dist}(x,U') < \frac{2\eta}{3} \right\}.$$

Now fix $u \in \mathrm{Lip}_{\mathrm{loc}}(U)$, $v \in \mathrm{Lip}_{\mathrm{loc}}(V)$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that

$$M\int_H (1 + g_u + g_v)\,d\mu < \varepsilon k \tag{3.5}$$

if the above integral is finite — otherwise the desired estimate is trivially true. For $i = 1, \dots, k$, define the sets

$$H_i = \left\{ x \in U \cap V' : \frac{(k+i-1)\eta}{3k} < \operatorname{dist}(x,U') < \frac{(k+i)\eta}{3k} \right\},$$

so that $H \supset \bigcup_{i=1}^k H_i$, and define the Lipschitz functions

$$\phi_i(x) = \begin{cases} 0, & \operatorname{dist}(x,U') > \frac{k+i}{3k}\eta, \\ \frac{1}{\eta}((k+i)\eta - 3k\operatorname{dist}(x,U')), & \frac{k+i-1}{3k}\eta \le \operatorname{dist}(x,U') \le \frac{k+i}{3k}\eta, \\ 1, & \operatorname{dist}(x,U') < \frac{k+i-1}{3k}\eta. \end{cases}$$

Now $g_{\phi_i} = 0$ μ-almost everywhere in U' and in $U \cap V' \setminus H_i$ [5, Corollary 2.21]. Let $w_i = \phi_i u + (1-\phi_i)v$ on $U' \cup V'$. We have the estimate

$$g_{w_i} \le \phi_i g_u + (1-\phi_i) g_v + g_{\phi_i}|u-v|,$$

see [5, Lemma 2.18]. By also using the estimate $f(t) \le M(1+t)$, we get

$$\begin{aligned}\int_{U'\cup V'} f(g_{w_i})\,d\mu &\le \int_U f(g_u)\,d\mu + \int_V f(g_v)\,d\mu + \int_{H_i} f(g_{w_i})\,d\mu \\ &\le \int_U f(g_u)\,d\mu + \int_V f(g_v)\,d\mu \\ &\quad + M\int_{H_i}(1+g_u+g_v)\,d\mu + \frac{3Mk}{\eta}\int_{H_i}|u-v|\,d\mu.\end{aligned}$$

Now, since $H \supset \bigcup_{i=1}^k H_i$, we have

$$\begin{aligned}&\frac{1}{k}\sum_{i=1}^k \int_{U'\cup V'} f(g_{w_i})\,d\mu \\ &\quad\le \int_U f(g_u)\,d\mu + \int_V f(g_v)\,d\mu + \frac{M}{k}\int_H (1+g_u+g_v)\,d\mu \\ &\qquad + \frac{3M}{\eta}\int_H |u-v|\,d\mu \\ &\quad\le \int_U f(g_u)\,d\mu + \int_V f(g_v)\,d\mu + C\int_H |u-v|\,d\mu + \varepsilon.\end{aligned}$$

In the last inequality we used (3.5). Thus we can find an index i such that the function $w = w_i$ satisfies the desired estimate. □

In the following lemmas, we assume that $u \in L^1_{\mathrm{loc}}(A \cup B)$.

Lemma 3.4. *Let $A \subset X$ be open with $\mathcal{F}(u, A) < \infty$. Then*

$$\mathcal{F}(u, A) = \sup_{B \Subset A} \mathcal{F}(u, B).$$

Proof. Take open sets $B_1 \Subset B_2 \Subset B_3 \Subset A$ and sequences $u_i \in \mathrm{Lip}_{\mathrm{loc}}(B_3)$, $v_i \in \mathrm{Lip}_{\mathrm{loc}}(A \setminus \overline{B_1})$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(B_3)$, $v_i \to u$ in $L^1_{\mathrm{loc}}(A \setminus \overline{B_1})$,

$$\mathcal{F}(u, B_3) = \lim_{i\to\infty} \int_{B_3} f(g_{u_i})\,d\mu,$$

and

$$\mathcal{F}(u, A \setminus \overline{B_1}) = \lim_{i\to\infty} \int_{A\setminus\overline{B_1}} f(g_{v_i})\,d\mu.$$

By using Lemma 3.3 with $U = B_3$, $U' = B_2$, $V = V' = A \setminus \overline{B_1}$ and $\varepsilon = 1/i$, we find a set $H \subset B_3 \setminus B_2$, $H \Subset B_3$, and a sequence $w_i \in \mathrm{Lip}_{\mathrm{loc}}(A)$ such that $w_i \to u$ in $L^1_{\mathrm{loc}}(A)$, and

$$\int_A f(g_{w_i})\,d\mu \le \int_{B_3} f(g_{u_i})\,d\mu + \int_{A\setminus\overline{B_1}} f(g_{v_i})\,d\mu + C\int_H |u_i - v_i|\,d\mu + \frac{1}{i}$$

for every $i \in \mathbb{N}$. In the above inequality, the last integral converges to zero as $i \to \infty$, since $H \Subset B_3$ and $H \Subset A \setminus \overline{B_1}$. Thus

$$\mathcal{F}(u, A) \leq \liminf_{i\to\infty} \int_A f(g_{w_i})\, d\mu \leq \mathcal{F}(u, B_3) + \mathcal{F}(u, A \setminus \overline{B_1}).$$

Exhausting A with sets B_1 concludes the proof, since then $\mathcal{F}(u, A \setminus \overline{B_1}) \to 0$ by (3.4). □

Lemma 3.5. *Let $A, B \subset X$ be open. Then*

$$\mathcal{F}(u, A \cup B) \leq \mathcal{F}(u, A) + \mathcal{F}(u, B).$$

Proof. First we note that every $C \Subset A \cup B$ can be presented as $C = A' \cup B'$, where $A' \Subset A$ and $B' \Subset B$. Therefore, according to Lemma 3.4, it suffices to show that

$$\mathcal{F}(u, A' \cup B') \leq \mathcal{F}(u, A) + \mathcal{F}(u, B)$$

for every $A' \Subset A$ and $B' \Subset B$. If $\mathcal{F}(u, A) = \infty$ or $\mathcal{F}(u, B) = \infty$, the claim holds. Assume therefore that $\mathcal{F}(u, A) < \infty$ and $\mathcal{F}(u, B) < \infty$. Take sequences $u_i \in \mathrm{Lip}_{\mathrm{loc}}(A)$ and $v_i \in \mathrm{Lip}_{\mathrm{loc}}(B)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(A)$, $v_i \to u$ in $L^1_{\mathrm{loc}}(B)$,

$$\mathcal{F}(u, A) = \lim_{i\to\infty} \int_A f(g_{u_i})\, d\mu,$$

and

$$\mathcal{F}(u, B) = \lim_{i\to\infty} \int_B f(g_{v_i})\, d\mu.$$

By using Lemma 3.3 with $U' = A'$, $U = A$, $V' = B'$, $V = B$ and $\varepsilon = 1/i$, we find a set $H \Subset A$, $H \subset B' \Subset B$, and a sequence $w_i \in \mathrm{Lip}_{\mathrm{loc}}(A' \cup B')$ such that $w_i \to u$ in $L^1_{\mathrm{loc}}(A' \cup B')$, and

$$\int_{A'\cup B'} f(g_{w_i})\, d\mu \leq \int_A f(g_{u_i})\, d\mu + \int_B f(g_{v_i})\, d\mu + C \int_H |u_i - v_i|\, d\mu + \frac{1}{i}$$

for every $i \in \mathbb{N}$. By the properties of H, the last integral in the above inequality converges to zero as $i \to \infty$, and then

$$\mathcal{F}(u, A' \cup B') \leq \mathcal{F}(u, A) + \mathcal{F}(u, B).$$

□

Lemma 3.6. *Let $A, B \subset X$ be open and let $A \cap B = \emptyset$. Then*

$$\mathcal{F}(u, A \cup B) \geq \mathcal{F}(u, A) + \mathcal{F}(u, B).$$

Proof. If $\mathcal{F}(u, A \cup B) = \infty$, the claim holds. Hence we may assume that $\mathcal{F}(u, A \cup B) < \infty$. Take a sequence $u_i \in \mathrm{Lip}_{\mathrm{loc}}(A \cup B)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(A \cup B)$ and

$$\mathcal{F}(u, A \cup B) = \lim_{i\to\infty} \int_{A\cup B} f(g_{u_i})\, d\mu.$$

Then, since A and B are disjoint,

$$\begin{aligned}\mathcal{F}(u, A \cup B) &= \lim_{i\to\infty} \int_{A\cup B} f(g_{u_i})\, d\mu \\ &\geq \liminf_{i\to\infty} \int_A f(g_{u_i})\, d\mu + \liminf_{i\to\infty} \int_B f(g_{u_i})\, d\mu \\ &\geq \mathcal{F}(u, A) + \mathcal{F}(u, B).\end{aligned}$$

□

Now we are ready to prove the measure property of the functional.

Theorem 3.7. *Let $\Omega \subset X$ be an open set, and let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\mathcal{F}(u, \Omega) < \infty$. Then $\mathcal{F}(u, \cdot)$ is a Borel regular outer measure on Ω.*

Proof. First we show that $\mathcal{F}(u, \cdot)$ is an outer measure on Ω. Obviously $\mathcal{F}(u, \emptyset) = 0$. As mentioned earlier, clearly $\mathcal{F}(u, A) \leq \mathcal{F}(u, B)$ for any $A \subset B \subset \Omega$. Take open sets $A_i \subset \Omega$, $i = 1, 2, \ldots$. Let $\varepsilon > 0$. By Lemma 3.4 there exists a set $B \Subset \bigcup_{i=1}^\infty A_i$ such that

$$\mathcal{F}\left(u, \bigcup_{i=1}^\infty A_i\right) < \mathcal{F}(u, B) + \varepsilon.$$

Since $\overline{B} \subset \bigcup_{i=1}^\infty A_i$ is compact, there exists $n \in \mathbb{N}$ such that $B \subset \overline{B} \subset \bigcup_{i=1}^n A_i$. Then by Lemma 3.5,

$$\mathcal{F}(u, B) \leq \mathcal{F}\left(u, \bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathcal{F}(u, A_i),$$

and thus letting $n \to \infty$ and $\varepsilon \to 0$ gives us

$$\mathcal{F}\left(u, \bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mathcal{F}(u, A_i). \tag{3.6}$$

For general sets $A_i \subset \Omega$, we can prove (3.6) by approximation with open sets.

The next step is to prove that $\mathcal{F}(u, \cdot)$ is a Borel outer measure. Let $A, B \subset \Omega$ satisfy $\operatorname{dist}(A, B) > 0$. Fix $\varepsilon > 0$ and choose an open set $U \supset A \cup B$ such that

$$\mathcal{F}(u, A \cup B) > \mathcal{F}(u, U) - \varepsilon.$$

Define the sets

$$V_A = \left\{x \in \Omega : \operatorname{dist}(x, A) < \frac{\operatorname{dist}(A, B)}{3}\right\} \cap U,$$

$$V_B = \left\{x \in \Omega : \operatorname{dist}(x, B) < \frac{\operatorname{dist}(A, B)}{3}\right\} \cap U.$$

Then V_A, V_B are open and $A \subset V_A$, $B \subset V_B$. Moreover $V_A \cap V_B = \emptyset$. Thus by Lemma 3.6,

$$\begin{aligned}\mathcal{F}(u, A \cup B) &\geq \mathcal{F}(u, V_A \cup V_B) - \varepsilon \\ &\geq \mathcal{F}(u, V_A) + \mathcal{F}(u, V_B) - \varepsilon \\ &\geq \mathcal{F}(u, A) + \mathcal{F}(u, B) - \varepsilon.\end{aligned}$$

Now letting $\varepsilon \to 0$ shows that $\mathcal{F}(u, \cdot)$ is a Borel outer measure by Carathéodory's criterion.

The measure $\mathcal{F}(u, \cdot)$ is Borel regular by construction, since for every $A \subset \Omega$ we may choose open sets V_i such that $A \subset V_i \subset \Omega$ and

$$\mathcal{F}(u, V_i) < \mathcal{F}(u, A) + \frac{1}{i},$$

and by defining $V = \bigcap_{i=1}^{\infty} V_i$, we get $\mathcal{F}(u, V) = \mathcal{F}(u, A)$, where $V \supset A$ is a Borel set. □

As a simple application of the measure property of the functional, we show the following approximation result.

Proposition 3.8. *Let $\Omega \subset X$ be an open set, and let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\mathcal{F}(u, \Omega) < \infty$. Then for any sequence of functions $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ for which $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and*

$$\int_\Omega f(g_{u_i})\, d\mu \to \mathcal{F}(u, \Omega),$$

we also have $f(g_{u_i})\, d\mu \overset{*}{\rightharpoonup} d\mathcal{F}(u, \cdot)$ in Ω.

Proof. For any open set $U \subset \Omega$, we have by the definition of the functional that

$$\mathcal{F}(u, U) \leq \liminf_{i \to \infty} \int_U f(g_{u_i})\, d\mu. \tag{3.7}$$

On the other hand, for any relatively closed set $F \subset \Omega$ we have

$$\begin{aligned}\mathcal{F}(u, \Omega) &= \limsup_{i \to \infty} \int_\Omega f(g_{u_i})\, d\mu \\ &\geq \limsup_{i \to \infty} \int_F f(g_{u_i})\, d\mu + \liminf_{i \to \infty} \int_{\Omega \setminus F} f(g_{u_i})\, d\mu \\ &\geq \limsup_{i \to \infty} \int_F f(g_{u_i})\, d\mu + \mathcal{F}(u, \Omega \setminus F).\end{aligned}$$

The last inequality follows from (3.7), since $\Omega \setminus F$ is open. By the measure property of the functional, we can subtract $\mathcal{F}(u, \Omega \setminus F)$ from both sides to get

$$\limsup_{i \to \infty} \int_F f(g_{u_i})\, d\mu \leq \mathcal{F}(u, F).$$

According to a standard characterization of the weak* convergence of Radon measures, the above inequality and (3.7) together give the result [11, p. 54]. □

4. Integral representation

In this section we study an integral representation for the functional $\mathcal{F}(u,\cdot)$, in terms of the variation measure $\|Du\|$. First we show an estimate from below. Note that due to (3.4), $\mathcal{F}(u,\Omega) < \infty$ always implies $\|Du\|(\Omega) < \infty$.

Theorem 4.1. *Let Ω be an open set, and let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\mathcal{F}(u,\Omega) < \infty$. Let $d\|Du\| = a\,d\mu + d\,\|Du\|^s$ be the decomposition of the variation measure into the absolutely continuous and singular parts, where $a \in L^1(\Omega)$ is a Borel function and $\|Du\|^s$ is the singular part. Then we have*

$$\mathcal{F}(u,\Omega) \geq \int_\Omega f(a)\,d\mu + f_\infty \|Du\|^s(\Omega).$$

Proof. Pick a sequence $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega f(g_{u_i})\,d\mu \to \mathcal{F}(u,\Omega) \quad \text{as} \quad i \to \infty. \tag{4.1}$$

Using the linear growth condition for f, presented in (3.1), we estimate

$$\limsup_{i\to\infty} \int_\Omega g_{u_i}\,d\mu \leq \frac{1}{m} \limsup_{i\to\infty} \int_\Omega f(g_{u_i})\,d\mu < \infty.$$

For a suitable subsequence, which we still denote by g_{u_i}, we have $g_{u_i}\,d\mu \overset{*}{\rightharpoonup} d\nu$ in Ω, where ν is a Radon measure with finite mass in Ω. Furthermore, by the definition of the variation measure, we necessarily have $\nu \geq \|Du\|$, which can be seen as follows. For any open set $U \subset \Omega$ and for any $\varepsilon > 0$, we can pick an open set $U' \Subset U$ such that $\|Du\|(U) < \|Du\|(U') + \varepsilon$; see e.g. Lemma 3.4. We obtain

$$\begin{aligned}\|Du\|(U) < \|Du\|(U') + \varepsilon &\leq \liminf_{i\to\infty} \int_{U'} g_{u_i}\,d\mu + \varepsilon \\ &\leq \limsup_{i\to\infty} \int_{\overline{U'}} g_{u_i}\,d\mu + \varepsilon \leq \nu(\overline{U'}) + \varepsilon \leq \nu(U) + \varepsilon.\end{aligned}$$

On the first line we used the definition of the variation measure, and on the second line we used a property of the weak* convergence of Radon measures, see e.g. [2, Example 1.63]. By approximation we get $\nu(A) \geq \|Du\|(A)$ for any $A \subset \Omega$.

The following lower semicontinuity argument is from [2, p. 64–66]. First we note that as a nonnegative nondecreasing convex function, f can be presented as

$$f(t) = \sup_{j\in\mathbb{N}} (d_j t + e_j), \quad t \geq 0,$$

for some sequences $d_j, e_j \in \mathbb{R}$, with $d_j \geq 0$, $j = 1, 2, \ldots$, and furthermore $\sup_j d_j = f_\infty$ [2, Proposition 2.31, Lemma 2.33]. Given any pairwise disjoint open subsets of Ω, denoted by $A_1, \ldots, A_k$, $k \in \mathbb{N}$, and functions $\phi_j \in C_c(A_j)$ with $0 \leq \phi_j \leq 1$, we have

$$\int_{A_j} (d_j g_{u_i} + e_j)\phi_j \, d\mu \leq \int_{A_j} f(g_{u_i}) \, d\mu$$

for every $j = 1, \ldots, k$ and $i \in \mathbb{N}$. Summing over j and letting $i \to \infty$, we get by the weak* convergence $g_{u_i} \, d\mu \overset{*}{\rightharpoonup} d\nu$

$$\sum_{j=1}^{k} \left(\int_{A_j} d_j \phi_j \, d\nu + \int_{A_j} e_j \phi_j \, d\mu \right) \leq \liminf_{i\to\infty} \int_\Omega f(g_{u_i}) \, d\mu.$$

Since we had $\nu \geq \|Du\|$, this immediately implies

$$\sum_{j=1}^{k} \left(\int_{A_j} d_j \phi_j \, d\|Du\| + \int_{A_j} e_j \phi_j \, d\mu \right) \leq \liminf_{i\to\infty} \int_\Omega f(g_{u_i}) \, d\mu.$$

We recall that $d\|Du\| = a \, d\mu + d\|Du\|^s$. It is known that the singular part $\|Du\|^s$ is concentrated on a Borel set $D \subset \Omega$ that satisfies $\mu(D) = 0$ and $\|Du\|^s(\Omega \setminus D) = 0$, see e.g. [11, p. 42]. Define the Radon measure $\sigma = \mu + \|Du\|^s$, and the Borel functions

$$h_j = \begin{cases} d_j a + e_j, & \text{on } \Omega \setminus D, \\ d_j, & \text{on } D \end{cases}$$

for $j = 1, \ldots, k$, and

$$h = \begin{cases} f(a), & \text{on } \Omega \setminus D, \\ f_\infty, & \text{on } D. \end{cases}$$

As mentioned above, we have $\sup_j h_j = h$, and we can write the previous inequality as

$$\sum_{j=1}^{k} \int_{A_j} h_j \phi_j \, d\sigma \leq \liminf_{i\to\infty} \int_\Omega f(g_{u_i}) \, d\mu.$$

Since the functions $\phi_j \in C_c(A_j)$, $0 \leq \phi_j \leq 1$, were arbitrary, we get

$$\sum_{j=1}^{k} \int_{A_j} h_j \, d\sigma \leq \liminf_{i\to\infty} \int_\Omega f(g_{u_i}) \, d\mu.$$

Since this holds for *any* pairwise disjoint open subsets $A_1, \ldots, A_k \subset \Omega$, by [2, Lemma 2.35] we get

$$\int_\Omega h \, d\sigma \leq \liminf_{i\to\infty} \int_\Omega f(g_{u_i}) \, d\mu.$$

However, by the definitions of h and σ, this is the same as

$$\int_\Omega f(a)\,d\mu + f_\infty\|Du\|^s(\Omega) \le \liminf_{i\to\infty}\int_\Omega f(g_{u_i})\,d\mu.$$

Combining this with (4.1), we get the desired estimate from below. □

It is worth noting that in the above argument, we only needed the weak* convergence of the sequence $g_{u_i}\,d\mu$ to a Radon measure that majorizes $\|Du\|$. Then we could use the fact that the functional for measures

$$\nu \longmapsto \int_\Omega f(\check{a})\,d\mu + f_\infty\nu^s(\Omega), \qquad d\nu = \check{a}\,d\mu + d\nu^s,$$

is lower semicontinuous with respect to weak* convergence of Radon measures. This *lower* semicontinuity is guaranteed by the fact that f is convex, but in order to have *upper* semicontinuity, we should have that f is also concave (and thus linear). Thus there is an important asymmetry in the setting, and for the estimate from above, we will need to use rather different methods where we prove weak or strong L^1-convergence for the sequence of upper gradients, instead of just weak* convergence of measures. To achieve this type of stronger convergence, we need to specifically ensure that the sequence of upper gradients is *equi-integrable*. The price that is paid is that a constant C appears in the final estimate related to the absolutely continuous parts. An example that we provide later shows that this constant cannot be discarded.

We recall that for a μ-measurable set $H \subset X$, the equi-integrability of a sequence of functions $g_i \in L^1(H)$, $i \in \mathbb{N}$, is defined by two conditions. First, for any $\varepsilon > 0$ there must exist a μ-measurable set $A \subset H$ with $\mu(A) < \infty$ such that

$$\int_{H\setminus A} g_i\,d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.$$

Second, for any $\varepsilon > 0$ there must exist $\delta > 0$ such that if $\widetilde{A} \subset H$ is any μ-measurable set with $\mu(\widetilde{A}) < \delta$, then

$$\int_{\widetilde{A}} g_i\,d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.$$

We will need the following equi-integrability result that partially generalizes [12, Lemma 6]. For the construction of Whitney coverings that are needed in the result, see e.g. [6, Theorem 3.1].

Lemma 4.2. *Let $\Omega \subset X$ be open, let $H \subset \Omega$ be μ-measurable, and let ν be a Radon measure with finite mass in Ω. Write the decomposition of ν into the absolutely continuous and singular parts with respect to μ as $d\nu = a\,d\mu + d\nu^s$, and assume that $\nu^s(H) = 0$. Take a sequence of open sets H_i such that $H \subset H_i \subset \Omega$ and $\nu^s(H_i) < 1/i$, $i \in \mathbb{N}$. For a given $\tau \ge 1$ and every $i \in \mathbb{N}$, take a Whitney covering $\{B^i_j = B(x^i_j, r^i_j)\}_{j=1}^\infty$*

of H_i such that $r_j^i \le 1/i$ for every $j \in \mathbb{N}$, $\tau B_j^i \subset H_i$ for every $j \in \mathbb{N}$, every ball τB_k^i meets at most $c_o = c_o(c_d, \tau)$ balls τB_j^i, and if τB_j^i meets τB_k^i, then $r_j^i \le 2r_k^i$. Define the functions

$$g_i = \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\nu(\tau B_j^i)}{\mu(B_j^i)}, \qquad i \in \mathbb{N}.$$

Then the sequence g_i is equi-integrable in H. Moreover, a subsequence of g_i converges weakly in $L^1(H)$ to a function $\check{a}$ that satisfies $\check{a} \le c_o a$ μ-almost everywhere in H.

Remark 4.3. If the measure ν is absolutely continuous in the whole of Ω, then we can choose $H = H_i = \Omega$ for all $i \in \mathbb{N}$.

Proof. To check the first condition of equi-integrability, let $\varepsilon > 0$ and take a ball $B = B(x_0, R)$ with $x_0 \in X$ and $R > 0$ so large that $\nu(\Omega \setminus B(x_0, R)) < \varepsilon/c_o$. Then, by the bounded overlap property of the Whitney balls, we have

$$\int_{H \setminus B(x_0, R+2\tau)} g_i \, d\mu \le c_o \nu(H_i \setminus B(x_0, R)) < \varepsilon$$

for all $i \in \mathbb{N}$.

To check the second condition, assume by contradiction that there is a sequence of μ-measurable sets $A_i \subset H$ with $\mu(A_i) \to 0$, and $\int_{A_i} g_i \, d\mu > \eta > 0$ for all $i \in \mathbb{N}$. Fix $\varepsilon > 0$. We know that there exists $\delta > 0$ such that if $A \subset \Omega$ and $\mu(A) < \delta$, then $\int_A a \, d\mu < \varepsilon$. Note that by the bounded overlap property of the Whitney balls, we have for every $i \in \mathbb{N}$

$$\begin{aligned} \int_{A_i} g_i \, d\mu &= \sum_{j=1}^{\infty} \frac{\mu(A_i \cap B_j^i)}{\mu(B_j^i)} \nu(\tau B_j^i) \\ &\le c_o \nu^s(H_i) + \sum_{j=1}^{\infty} \frac{\mu(A_i \cap B_j^i)}{\mu(B_j^i)} \int_{\tau B_j^i} a \, d\mu. \end{aligned} \tag{4.2}$$

Fix $k \in \mathbb{N}$. We can divide the above sum into two parts: let I_1 consist of those indices $j \in \mathbb{N}$ for which $\mu(A_i \cap B_j^i)/\mu(B_j^i) > 1/k$, and let I_2 consist of the remaining indices. We estimate

$$\mu\left(\bigcup_{j \in I_1} \tau B_j^i\right) \le C \sum_{j \in I_1} \mu(B_j^i) \le Ck \sum_{j \in I_1} \mu(A_i \cap B_j^i) \le Ck\mu(A_i) < \delta,$$

when i is large enough. Now we can further estimate (4.2):

$$\int_{A_i} g_i \, d\mu \le c_o \nu^s(H_i) + \frac{c_o}{k} \int_{H_i} a \, d\mu + c_o \varepsilon$$

for large enough $i \in \mathbb{N}$. By letting first $i \to \infty$, then $k \to \infty$, and finally $\varepsilon \to 0$, we get a contradiction with $\int_{A_i} g_i \, d\mu > \eta > 0$, proving the equi-integrability.

Finally, let us prove the weak convergence in $L^1(H)$. Possibly by taking a subsequence that we still denote by g_i, we have $g_i \to \check{a}$ weakly in $L^1(H)$ for some $\check{a} \in L^1(H)$, by the Dunford-Pettis theorem (see e.g. [2, Theorem 1.38]). By this weak convergence and the bounded overlap property of the Whitney balls, we can estimate for any $x \in H$ and $0 < \widetilde{r} < r$

$$\begin{aligned}\int_{B(x,\widetilde{r})\cap H} \check{a}\,d\mu &= \limsup_{i\to\infty} \int_{B(x,\widetilde{r})\cap H} g_i\,d\mu \\ &= \limsup_{i\to\infty} \sum_{j=1}^{\infty} \frac{\mu(B_j^i \cap B(x,\widetilde{r}) \cap H)}{\mu(B_j^i)} \nu(\tau B_j^i) \\ &\le \limsup_{i\to\infty} \sum_{j\in\mathbb{N}:\, B_j^i\cap B(x,\widetilde{r})\cap H\neq\emptyset} \nu(\tau B_j^i) \\ &\le \limsup_{i\to\infty} c_o \nu(B(x,r)).\end{aligned}$$

By letting $\widetilde{r} \nearrow r$, we get

$$\int_{B(x,r)\cap H} \check{a}\,d\mu \le c_o \nu(B(x,r)).$$

By the Radon-Nikodym theorem, μ-almost every $x \in H$ satisfies

$$\lim_{r\to 0} \fint_{B(x,r)\cap H} \check{a}\,d\mu = \check{a}(x) \quad \text{and} \quad \lim_{r\to 0} \frac{\nu^s(B(x,r))}{\mu(B(x,r))} = 0.$$

By using these estimates as well as the previous one, we get for μ-almost every $x \in H$

$$\begin{aligned}\check{a}(x) &= \lim_{r\to 0} \fint_{B(x,r)\cap H} \check{a}\,d\mu \\ &\le c_o \limsup_{r\to 0} \fint_{B(x,r)} a\,d\mu + c_o \limsup_{r\to 0} \frac{\nu^s(B(x,r))}{\mu(B(x,r))},\end{aligned}$$

where the first term on the right-hand side is $c_o a$ by the Radon-Nikodym theorem, and the second term is zero. Thus we have $\check{a} \le c_o a$ μ-almost everywhere in H. □

Now we are ready to prove the estimate from above.

Theorem 4.4. *Let Ω be an open set, and let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\mathcal{F}(u,\Omega) < \infty$. Let $d\|Du\| = a\,d\mu + d\,\|Du\|^s$ be the decomposition of the variation measure, where $a \in L^1(\Omega)$ and $\|Du\|^s$ is the singular part. Then we have*

$$\mathcal{F}(u,\Omega) \le \int_\Omega f(Ca)\,d\mu + f_\infty \|Du\|^s(\Omega),$$

with $C = C(c_d, c_P, \lambda)$.

Proof. Since the functional $\mathcal{F}(u,\cdot)$ is a Radon measure by Theorem 3.7, we can decompose it into the absolutely continuous and singular parts as $\mathcal{F}(u,\cdot) = \mathcal{F}^a(u,\cdot) + \mathcal{F}^s(u,\cdot)$. The singular parts $\|Du\|^s$ and $\mathcal{F}^s(u,\cdot)$ are concentrated on a Borel set $D \subset \Omega$ that satisfies $\mu(D) = 0$ and

$$\|Du\|^s(\Omega \setminus D) = 0 = \mathcal{F}^s(u, \Omega \setminus D),$$

see e.g. [11, p. 42].

First we prove the estimate for the singular part. Let $\varepsilon > 0$. Choose an open set G with $D \subset G \subset \Omega$, such that $\mu(G) < \varepsilon$ and $\|Du\|(G) < \|Du\|(D) + \varepsilon$. Take a sequence $u_i \in \mathrm{Lip}_{\mathrm{loc}}(G)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(G)$ and

$$\int_G g_{u_i}\, d\mu \to \|Du\|(G) \quad \text{as} \quad i \to \infty.$$

Thus for some $i \in \mathbb{N}$ large enough, we have

$$\int_G g_{u_i}\, d\mu < \|Du\|(G) + \varepsilon$$

and

$$\mathcal{F}(u,G) < \int_G f(g_{u_i})\, d\mu + \varepsilon.$$

The latter inequality necessarily holds for large enough i by the definition of the functional $\mathcal{F}(u,\cdot)$. Now, using the two inequalities above and the estimate for f given in (3.2), we can estimate

$$\begin{aligned}
\mathcal{F}(u,D) \le \mathcal{F}(u,G) &\le \int_G f(g_{u_i})\, d\mu + \varepsilon \\
&\le \int_G f(0)\, d\mu + f_\infty \int_G g_{u_i}\, d\mu + \varepsilon \\
&\le f(0)\mu(G) + f_\infty \|Du\|(G) + f_\infty \varepsilon + \varepsilon \\
&\le f(0)\varepsilon + f_\infty(\|Du\|(D) + \varepsilon) + f_\infty \varepsilon + \varepsilon.
\end{aligned}$$

In the last inequality we used the properties of the set G given earlier. Letting $\varepsilon \to 0$, we get the estimate from above for the singular part, i.e.

$$\mathcal{F}^s(u,\Omega) = \mathcal{F}(u,D) \le f_\infty \|Du\|(D) = f_\infty \|Du\|^s(\Omega). \tag{4.3}$$

Next let us consider the absolutely continuous part. Let D be defined as above, and let $H = \Omega \setminus D$. Let $\varepsilon > 0$. Take an open set G such that $H \subset G \subset \Omega$, and $\|Du\|(G) < \|Du\|(H) + \varepsilon$.

For every $i \in \mathbb{N}$, take a Whitney covering $\{B^i_j = B(x^i_j, r^i_j)\}_{j=1}^\infty$ of G such that $r^i_j \le 1/i$ for every $j \in \mathbb{N}$, $5\lambda B^i_j \subset G$ for every $j \in \mathbb{N}$, every ball $5\lambda B^i_k$ meets at most $C = C(c_d, \lambda)$ balls $5\lambda B^i_j$, and if $5\lambda B^i_j$ meets $5\lambda B^i_k$, then $r^i_j \le 2r^i_k$. Then take a partition of unity $\{\phi^i_j\}_{j=1}^\infty$ subordinate to this cover, such that $0 \le \phi^i_j \le 1$, each ϕ^i_j is a $C(c_d)/r^i_j$-Lipschitz function, and $\mathrm{supp}(\phi^i_j) \subset 2B^i_j$ for every $j \in \mathbb{N}$ (see

e.g. [6, Theorem 3.4]). Define discrete convolutions with respect to the Whitney coverings by

$$u_i = \sum_{j=1}^{\infty} u_{B_j^i} \phi_j^i, \qquad i \in \mathbb{N}.$$

We know that $u_i \to u$ in $L^1(G)$ as $i \to \infty$, and that each u_i has an upper gradient

$$g_i = C \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\|Du\|(5\lambda B_j^i)}{\mu(B_j^i)}$$

with $C = C(c_d, c_P)$, see e.g. the proof of [20, Proposition 4.1]. We can of course write the decomposition $g_i = g_i^a + g_i^s$, where

$$g_i^a = C \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\int_{5\lambda B_j^i} a \, d\mu}{\mu(B_j^i)}$$

and

$$g_i^s = C \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\|Du\|^s(5\lambda B_j^i)}{\mu(B_j^i)}.$$

By the bounded overlap property of the coverings, we can easily estimate

$$\int_G g_i^s \, d\mu \le \widetilde{C} \|Du\|^s(G) < \widetilde{C}\varepsilon \tag{4.4}$$

for every $i \in \mathbb{N}$, with $\widetilde{C} = \widetilde{C}(c_d, c_P, \lambda)$. Furthermore, by Lemma 4.2 we know that the sequence g_i^a is equi-integrable and that a subsequence, which we still denote g_i^a, converges weakly in $L^1(G)$ to a function $\check{a} \le Ca$, with $C = C(c_d, \lambda)$. By Mazur's lemma we have for certain convex combinations, denoted by a hat,

$$\widehat{g_i^a} = \sum_{j=i}^{N_i} d_{i,j} g_j^a \to \check{a} \quad \text{in } L^1(G) \text{ as } i \to \infty,$$

where $d_{i,j} \ge 0$ and $\sum_{j=i}^{N_i} d_{i,j} = 1$ for every $i \in \mathbb{N}$ [25, Theorem 3.12]. We note that $\widehat{u_i} \in \operatorname{Lip}_{\mathrm{loc}}(G)$ for every $i \in \mathbb{N}$ (the hat always means that we take the same convex combinations), $\widehat{u_i} \to u$ in $L^1_{\mathrm{loc}}(G)$, and $g_{\widehat{u_i}} \le \widehat{g_i}$ μ-almost everywhere for every $i \in \mathbb{N}$ (recall that g_u always means the minimal 1-weak upper gradient of u). Using the definition

of $\mathcal{F}(u,\cdot)$, the fact that f is L-Lipschitz, and (4.4), we get

$$\begin{aligned}\mathcal{F}(u,H) \le \mathcal{F}(u,G) &\le \liminf_{i\to\infty} \int_G f(g_{\widehat{u}_i})\,d\mu \\ &\le \liminf_{i\to\infty} \int_G f(\widehat{g}_i)\,d\mu \le \liminf_{i\to\infty} \left(\int_G f(\widehat{g}_i^a)\,d\mu + \int_G L\widehat{g}_i^s\,d\mu \right) \\ &\le \liminf_{i\to\infty} \left(\int_G f(\widehat{g}_i^a)\,d\mu + L\widetilde{C}\varepsilon \right) = \int_G f(\check{a})\,d\mu + L\widetilde{C}\varepsilon \\ &\le \int_G f(Ca)\,d\mu + L\widetilde{C}\varepsilon \le \int_\Omega f(Ca)\,d\mu + L\widetilde{C}\varepsilon.\end{aligned}$$

By letting $\varepsilon \to 0$ we get the estimate from above for the absolutely continuous part, i.e.

$$\mathcal{F}^a(u,\Omega) = \mathcal{F}(u,H) \le \int_\Omega f(Ca)\,d\mu.$$

By combining this with (4.3), we get the desired estimate from above. □

Remark 4.5. By using Theorems 4.1 and 4.4, as well as the definition of the functional for general sets given in (3.3), we can conclude that for any μ-measurable set $A \subset \Omega \subset X$ with $\mathcal{F}(u,\Omega) < \infty$, we have

$$\mathcal{F}^s(u,A) = f_\infty \|Du\|^s(A)$$

and

$$\int_A f(a)\,d\mu \le \mathcal{F}^a(u,A) \le \int_A f(Ca)\,d\mu,$$

where $\mathcal{F}^a(u,\cdot)$ and $\mathcal{F}^s(u,\cdot)$ are again the absolutely continuous and singular parts of the measure given by the functional.

Since locally Lipschitz functions are dense in the Newtonian space $N^{1,1}(\Omega)$ with Ω open [5, Theorem 5.47], from the definition of total variation we know that if $u \in N^{1,1}(\Omega)$, then $u \in \mathrm{BV}(\Omega)$ with $\|Du\|$ absolutely continuous, and more precisely

$$\|Du\|(\Omega) \le \int_\Omega g_u\,d\mu.$$

We obtain, to some extent as a by-product of the latter part of the proof of the previous theorem, the following converse, which also answers a question posed in [20]. A later example will show that the constant C is necessary here as well.

Theorem 4.6. *Let $\Omega \subset X$ be an open set, let $u \in \mathrm{BV}(\Omega)$, and let $d\|Du\| = a\,d\mu + d\|Du\|^s$ be the decomposition of the variation measure, where $a \in L^1(\Omega)$ and $\|Du\|^s$ is the singular part. Let $H \subset \Omega$ be a μ-measurable set for which $\|Du\|^s(H) = 0$. Then, by modifying u on a set of μ-measure zero if necessary, we have $u|_H \in N^{1,1}(H)$ and $g_u \le Ca$ μ-almost everywhere in H, with $C = C(c_d, c_P, \lambda)$.*

Proof. We pick a sequence of open sets H_i such that $H \subset H_i \subset \Omega$ and $\|Du\|^s(H_i) < 1/i$, $i = 1, 2, \ldots$. Then, as described in Lemma 4.2, we pick Whitney coverings $\{B_j^i\}_{j=1}^\infty$ of the sets H_i, with the constant $\tau = 5\lambda$.

Furthermore, as we did in the latter part of the proof of Theorem 4.4 with the open set G, we define for every $i \in \mathbb{N}$ a discrete convolution u_i of the function u with respect to the Whitney covering $\{B_j^i\}_{j=1}^\infty$. Every u_i has an upper gradient

$$g_i = C \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\|Du\|(5\lambda B_j^i)}{\mu(B_j^i)}$$

in H_i, with $C = C(c_d, c_P)$, and naturally g_i is then also an upper gradient of u_i in H. We have $u_i \to u$ in $L^1(H)$ (see e.g. the proof of [20, Proposition 4.1]) and, according to Lemma 4.2 and up to a subsequence, $g_i \to \check{a}$ weakly in $L^1(H)$, where $\check{a} \le Ca$ μ-almost everywhere in H. We now know by [16, Lemma 7.8] that by modifying u on a set of μ-measure zero, if necessary, we have that $\check{a}$ is a 1-weak upper gradient of u in H. Thus we have the result. □

Remark 4.7. As in Lemma 4.2, if $\|Du\|$ is absolutely continuous on the whole of Ω, we can choose simply $H = \Omega$, and then we also have the inequality

$$\int_\Omega g_u \, d\mu \le C\|Du\|(\Omega)$$

with $C = C(c_d, c_P, \lambda)$. Note also that the proof of [16, Lemma 7.8], which we used above, is also based on Mazur's lemma, so the techniques used above are very similar to those used in the proof of Theorem 4.4.

Finally we give the counterexample that shows that in general, we can have

$$\mathcal{F}^a(u, \Omega) > \int_\Omega f(a) \, d\mu \quad \text{and} \quad \|Du\|(\Omega) < \int_\Omega g_u \, d\mu.$$

The latter inequality answers a question raised in [24] and later in [3].

Example 4.8. Take the space $X = [0, 1]$, equipped with the Euclidean distance and a measure μ, which we will next define. First we construct a fat Cantor set A as follows. Take $A_0 = [0, 1]$, whose measure we denote by $\alpha_0 = \mathcal{L}^1(A_0) = 1$, where $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure. Then in each step $i \in \mathbb{N}$ we define A_i by removing from A_{i-1} the set B_i, which consists of 2^{i-1} open intervals of length 2^{-2i}, centered at the middle points of the intervals that make up A_{i-1}. We denote $\alpha_i = \mathcal{L}^1(A_i)$, and define $A = \bigcap_{i=1}^\infty A_i$. Then we have

$$\alpha = \mathcal{L}^1(A) = \lim_{i \to \infty} \alpha_i = 1/2.$$

Now, equip the space X with the weighted Lebesgue measure $d\mu = w\,d\mathcal{L}^1$, where $w = 2$ in A and $w = 1$ in $X \setminus A$. Define

$$g = \frac{1}{\alpha}\chi_A = 2\chi_A \quad \text{and} \quad g_i = \frac{1}{\alpha_{i-1} - \alpha_i}\chi_{B_i}, \quad i \in \mathbb{N}.$$

The unweighted integral of g and each g_i over X is 1. Next define the function

$$u(x) = \int_0^x g\,d\mathcal{L}^1.$$

Now u is in $N^{1,1}(X)$ and even in $\mathrm{Lip}(X)$, since g is bounded. In this 1-dimensional setting, it can be seen that every 1-weak upper gradient of u is in fact an upper gradient, and then it is easy to see that the minimal 1-weak upper gradient of u is g. Approximate u with the functions

$$u_i(x) = \int_0^x g_i\,d\mathcal{L}^1, \quad i \in \mathbb{N}.$$

The functions u_i are Lipschitz, and they converge to u in $L^1(X)$ and even uniformly, which can be seen as follows. Given $i \in \mathbb{N}$, the set A_i consists of 2^i intervals of length $\alpha_i/2^i$. If I is one of these intervals, we have

$$2^{-i} = \int_I g\,d\mathcal{L}^1 = \int_I g_{i+1}\,d\mathcal{L}^1, \tag{4.5}$$

and also

$$\int_{X\setminus A_i} g\,d\mathcal{L}^1 = 0 = \int_{X\setminus A_i} g_{i+1}\,d\mathcal{L}^1.$$

Hence $u_{i+1} = u$ at the end points of the intervals that make up A_i, and elsewhere $|u_{i+1} - u|$ is at most 2^{-i} by (4.5).

Clearly the minimal 1-weak upper gradient of u_i is g_i. However, we have

$$\int_0^1 g\,d\mu = 2 > 1 = \lim_{i\to\infty}\int_0^1 g_i\,d\mu \geq \|Du\|([0,1]).$$

Thus the total variation is strictly smaller than the integral of the minimal 1-weak upper gradient, demonstrating the necessity of the constant C in Theorem 4.6. On the other hand, any approximating sequence $v_i \in \mathrm{Lip}(X)$ satisfying $v_i \to u$ in $L^1(X)$ converges, up to a subsequence, to u also pointwise μ- and thus $\mathcal{L}^1$-almost everywhere, and then we necessarily have for some such sequence

$$\|Du\|([0,1]) = \lim_{i\to\infty}\int_0^1 g_{v_i}\,d\mu \geq \limsup_{i\to\infty}\int_0^1 g_{v_i}\,d\mathcal{L}^1 \geq 1. \tag{4.6}$$

Hence we have $\|Du\|([0,1]) = 1$. Let us show that more precisely, $d\|Du\| = a\,d\mu$ with $a = \chi_A$. The fact that u is Lipschitz implies that

$\|Du\|$ is absolutely continuous with respect to μ. Since u_i converges to u uniformly, for any interval $(d,e) \subset (0,1)$ we must have

$$\lim_{i\to\infty} \int_{(d,e)} g_i \, d\mathcal{L}^1 = \int_{(d,e)} g \, d\mathcal{L}^1,$$

and since for the weight we had $w = 1$ where $g_i > 0$, and $w = 2$ where $g > 0$, we now get

$$\lim_{i\to\infty} \int_{(d,e)} g_i \, d\mu = \frac{1}{2} \int_{(d,e)} g \, d\mu.$$

By the definition of the variation measure, we have at any point $x \in X$ for small enough $r > 0$

$$\|Du\|((x-r,x+r)) \le \liminf_{i\to\infty} \int_{(x-r,x+r)} g_i \, d\mu = \frac{1}{2} \int_{(x-r,x+r)} g \, d\mu.$$

Now, if $x \in A$, we can estimate the Radon-Nikodym derivative

$$\limsup_{r\to 0} \frac{\|Du\|(B(x,r))}{\mu(B(x,r))} \le 1,$$

and if $x \in X \setminus A$, we clearly have that the derivative is 0. On the other hand, if the derivative were strictly smaller than 1 in a subset of A of positive μ-measure, we would get $\|Du\|(X) < 1$, which is a contradiction with the fact that $\|Du\|(X) = 1$. Thus $d\|Du\| = a \, d\mu$ with $a = \chi_A$. [1]

To show that we can have $\mathcal{F}^a(u,X) > \int_X f(a) \, d\mu$ — note that $\mathcal{F}^a(u,X) = \mathcal{F}(u,X)$ — assume that f is given by

$$f(t) = \begin{cases} t, & t \in [0,1], \\ 2t-1, & t > 1. \end{cases}$$

(We could equally well consider other nonlinear f that satisfy the earlier assumptions.) Since $a = \chi_A$, we have

$$\int_X f(a) \, d\mu = \int_X a \, d\mu = 2 \int_X \chi_A \, d\mathcal{L}^1 = 1.$$

On the other hand, for some sequence of Lipschitz functions $v_i \to u$ in $L^1(X)$, we have

$$\begin{aligned} \mathcal{F}(u,X) &= \lim_{i\to\infty} \int_X f(g_{v_i}) \, d\mu \\ &= \lim_{i\to\infty} \left(2 \int_A f(g_{v_i}) \, d\mathcal{L}^1 + \int_{X\setminus A} f(g_{v_i}) \, d\mathcal{L}^1 \right). \end{aligned} \tag{4.7}$$

[1] We can further show that $g_i \, d\mu \overset{*}{\rightharpoonup} a \, d\mu$ in X, but we do not have $g_i \to a$ weakly in $L^1(X)$, demonstrating the subtle difference between the two types of weak convergence.

By considering a subsequence, if necessary, we may assume that $v_i \to u$ pointwise μ- and thus $\mathcal{L}^1$-almost everywhere. By Proposition 3.8, we have for any closed set $F \subset X \setminus A$

$$\limsup_{i\to\infty} \int_F f(g_{v_i})\,d\mu \le \mathcal{F}(u,F) \le \mathcal{F}(u, X\setminus A) \le \int_{X\setminus A} f(g_u)\,d\mu = 0,$$

which implies that

$$\lim_{i\to\infty} \int_F f(g_{v_i})\,d\mathcal{L}^1 = 0 = \lim_{i\to\infty} \int_F g_{v_i}\,d\mathcal{L}^1.$$

Applying these two equalities together with the inequality $f(t) \ge 2t-1$, we obtain

$$\begin{aligned}
\limsup_{i\to\infty} \int_{X\setminus A} f(g_{v_i})\,d\mathcal{L}^1 &= \limsup_{i\to\infty} \int_{X\setminus(A\cup F)} f(g_{v_i})\,d\mathcal{L}^1 \\
&\ge \limsup_{i\to\infty} \int_{X\setminus(A\cup F)} (2g_{v_i}-1)\,d\mathcal{L}^1 \\
&\ge \limsup_{i\to\infty} \int_{X\setminus(A\cup F)} 2g_{v_i}\,d\mathcal{L}^1 - \mathcal{L}^1(X\setminus(A\cup F)) \\
&= \limsup_{i\to\infty} \int_{X\setminus A} 2g_{v_i}\,d\mathcal{L}^1 - \mathcal{L}^1(X\setminus(A\cup F)).
\end{aligned}$$

The last term on the last line can be made arbitrarily small. Inserting this into (4.7), we get

$$\begin{aligned}
\mathcal{F}(u,X) &= \limsup_{i\to\infty} \left(2\int_A f(g_{v_i})\,d\mathcal{L}^1 + \int_{X\setminus A} f(g_{v_i})\,d\mathcal{L}^1 \right) \\
&\ge 2\liminf_{i\to\infty} \int_A f(g_{v_i})\,d\mathcal{L}^1 + 2\limsup_{i\to\infty} \int_{X\setminus A} g_{v_i}\,d\mathcal{L}^1 \\
&\ge 2\liminf_{i\to\infty} \int_0^1 g_{v_i}\,d\mathcal{L}^1 \ge 2.
\end{aligned}$$

The last inequality follows from the pointwise convergence of v_i to u $\mathcal{L}^1$-almost everywhere.

Roughly speaking, we note that the total variation $\|Du\|(X)$ is found to be unexpectedly small because the growth of the approximating functions u_i is concentrated outside the Cantor set A, where it is "cheaper" due to the smaller value of the weight function. However, when we calculate $\mathcal{F}(u,X)$, the same does not work, because now the nonlinear function f places "extra weight" on upper gradients that take values larger than 1.

5. Minimization problem

Let us consider a minimization problem related to the functional of linear growth. First we specify what we mean by boundary values of BV functions.

Definition 5.1. Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $h \in \mathrm{BV}(\Omega^*)$. We define $\mathrm{BV}_h(\Omega)$ as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = h$ μ-almost everywhere in $\Omega^* \setminus \Omega$.

Now we give the definition of our minimization problem.

Definition 5.2. A function $u \in \mathrm{BV}_h(\Omega)$ is a minimizer of the functional of linear growth with the boundary values $h \in \mathrm{BV}(\Omega^*)$, if

$$\mathcal{F}(u, \Omega^*) = \inf \mathcal{F}(v, \Omega^*),$$

where the infimum is taken over all $v \in \mathrm{BV}_h(\Omega)$.

Note that if $u \in L^1_{\mathrm{loc}}(\Omega^*)$ and $u = h$ in $\Omega^* \setminus \Omega$, then $u \in L^1(\Omega^*)$. Furthermore, if $\mathcal{F}(u, \Omega^*) < \infty$, then $\|Du\|(\Omega^*) < \infty$ by (3.4). Thus it makes sense to restrict u to the class $\mathrm{BV}(\Omega^*)$ in the above definition. Observe that the minimizers do not depend on Ω^*, but the value of the functional does. Note also that the minimization problem always has a solution and that the solution is not necessarily continuous, see [17].

Remark 5.3. We point out that any minimizer is also a local minimizer in the following sense. A minimizer $u \in \mathrm{BV}_h(\Omega)$ of $\mathcal{F}(\cdot, \Omega^*)$ with the boundary values $h \in \mathrm{BV}(\Omega^*)$ is a minimizer of $\mathcal{F}(\cdot, \Omega'')$ with the boundary values $u \in \mathrm{BV}_u(\Omega')$ for every $\Omega' \Subset \Omega'' \subset \Omega^*$, with $\Omega' \subset \Omega$. This can be seen as follows. Every $v \in \mathrm{BV}_u(\Omega')$ can be extended to a BV function in Ω^* by defining $v = u$ in $\Omega^* \setminus \Omega''$. The minimality of u and the measure property of the functional (Theorem 3.7) then imply that

$$\mathcal{F}(u, \Omega^* \setminus \Omega'') + \mathcal{F}(u, \Omega'') \leq \mathcal{F}(v, \Omega^* \setminus \Omega'') + \mathcal{F}(v, \Omega'').$$

Since $u = v$ μ-almost everywhere in $\Omega^* \setminus \Omega'$, the first terms on both sides of the inequality cancel out, and we have

$$\mathcal{F}(u, \Omega'') \leq \mathcal{F}(v, \Omega'').$$

Now we wish to express the boundary values of the minimization problem as a penalty term involving an integral over the boundary. To this end, we need to discuss boundary traces and extensions of BV functions.

Definition 5.4. An open set Ω is a strong BV extension domain, if for every $u \in \mathrm{BV}(\Omega)$ there is an extension $Eu \in \mathrm{BV}(X)$ such that $Eu|_\Omega = u$, there is a constant $1 \leq c_\Omega < \infty$ such that $\|Eu\|_{\mathrm{BV}(X)} \leq c_\Omega \|u\|_{\mathrm{BV}(\Omega)}$, and $\|D(Eu)\|(\partial\Omega) = 0$.

The word "strong" refers to the condition $\|D(Eu)\|(\partial\Omega) = 0$, which is not (necessarily) part of the conventional definition of a BV extension domain. It can be understood as an additional regularity condition for the domain. As an example of a BV extension domain that fails to

satisfy this additional condition, consider $X = \mathbb{C} = \mathbb{R}^2$ and the slit disk

$$\Omega = B(0,1) \setminus \{z = (x_1, x_2) : \, x_1 > 0, x_2 = 0\}.$$

This is a BV extension domain according to [21, Theorem 1.1]. However, the function $u(z) = \operatorname{Arg}(z) \in \mathrm{BV}(\Omega)$ clearly cannot be extended such that the condition $\|D(Eu)\|(\partial\Omega) = 0$ would be satisfied.

Definition 5.5. We say that a μ-measurable set Ω satisfies the *weak measure density condition* if for $\mathcal{H}$-almost every $x \in \partial\Omega$ we have

$$\liminf_{r\to 0} \frac{\mu(B(x,r)\cap\Omega)}{\mu(B(x,r))} > 0.$$

These are the two conditions we will impose in order to have satisfactory results on the boundary traces of BV functions. Based on results found in [7], we proved in [22] that every bounded *uniform* domain is a strong BV extension domain and satisfies the weak measure density condition. An open set Ω is A-uniform, with constant $A \geq 1$, if for every $x, y \in \Omega$ there is a curve γ in Ω connecting x and y such that $\ell_\gamma \leq Ad(x,y)$, and for all $t \in [0, \ell_\gamma]$, we have

$$\operatorname{dist}(\gamma(t), X \setminus \Omega) \geq A^{-1} \min\{t, \ell_\gamma - t\}.$$

The standard assumption in the classical Euclidean theory of boundary traces is a bounded domain with a Lipschitz boundary, see e.g. [2, Theorem 3.87]. It can be checked that such a domain is always a uniform domain, and so the theory we develop here is a natural generalization of the classical theory to the metric setting.

Now we give the definition of boundary traces.

Definition 5.6. For a μ-measurable set Ω and a μ-measurable function u on Ω, a real-valued function $T_\Omega u$ defined on $\partial\Omega$ is a boundary trace of u if for $\mathcal{H}$-almost every $x \in \partial\Omega$, we have

$$\lim_{r\to 0} \fint_{\Omega\cap B(x,r)} |u - T_\Omega u(x)| \, d\mu = 0.$$

Often we will also call $T_\Omega u(x)$ a boundary trace if the above condition is satisfied at the point x. If the trace exists at a point $x \in \partial\Omega$, we clearly have

$$T_\Omega u(x) = \lim_{r\to 0} \fint_{B(x,r)\cap\Omega} u \, d\mu = \operatorname*{ap\,lim}_{y\in\Omega,\, y\to x} u(y),$$

where ap lim denotes the approximate limit. Furthermore, we can show that the trace is always a Borel function.

Let us recall the following decomposition result for the variation measure of a BV function from [3, Theorem 5.3]. For any open set

$\Omega \subset X$, any $u \in \mathrm{BV}(\Omega)$, and any Borel set $A \subset \Omega$ that is σ-finite with respect to $\mathcal{H}$, we have

$$\|Du\|(\Omega) = \|Du\|(\Omega \setminus A) + \int_A \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x). \tag{5.1}$$

The function θ and the lower and upper approximate limits $u^\wedge$ and $u^\vee$ were defined in Section 2. In particular, by [3, Theorem 5.3] the jump set S_u is known to be σ-finite with respect to $\mathcal{H}$.

The following is our main result on boundary traces.

Theorem 5.7. *Assume that Ω is a strong* BV *extension domain that satisfies the weak measure density condition, and let $u \in \mathrm{BV}(\Omega)$. Then the boundary trace $T_\Omega u$ exists, that is, $T_\Omega u(x)$ is defined for $\mathcal{H}$-almost every $x \in \partial\Omega$.*

Proof. Extend u to a function $Eu \in \mathrm{BV}(X)$. By the fact that

$$\|D(Eu)\|(\partial\Omega) = 0$$

and the decomposition (5.1), we have $\mathcal{H}(S_{Eu} \cap \partial\Omega) = 0$ — recall that the function θ is bounded away from zero. Here

$$S_{Eu} = \{x \in X : (Eu)^\wedge(x) < (Eu)^\vee(x)\},$$

as usual. On the other hand, by [20, Theorem 3.5] we know that $\mathcal{H}$-almost every point $x \in \partial\Omega \setminus S_{Eu}$ is a Lebesgue point of Eu. In these points we define $T_\Omega u(x)$ simply as the Lebesgue limit $\widetilde{Eu}(x)$. For $\mathcal{H}$-almost every $x \in \partial\Omega$ the weak measure density condition implies

$$\liminf_{r\to 0} \frac{\mu(B(x,r)\cap\Omega)}{\mu(B(x,r))} = c(x) > 0.$$

Thus for $\mathcal{H}$-almost every $x \in \partial\Omega$ we can estimate

$$\begin{aligned}\limsup_{r\to 0} \fint_{B(x,r)\cap\Omega} |u - T_\Omega u(x)|\, d\mu \\ \le \limsup_{r\to 0} \frac{1}{c(x)\mu(B(x,r))} \int_{B(x,r)} |Eu - \widetilde{Eu}(x)|\, d\mu = 0.\end{aligned}$$

□

Due to the Lebesgue point theorem [20, Theorem 3.5], we have in fact

$$\limsup_{r\to 0} \fint_{B(x,r)\cap\Omega} |u - T_\Omega u(x)|^{Q/(Q-1)}\, d\mu = 0$$

for $\mathcal{H}$-almost every $x \in \partial\Omega$, where $Q > 1$ was given in (2.1). However, we will not need this stronger result.

Let us list some general properties of boundary traces.

Proposition 5.8. *Assume that Ω is a μ-measurable set and that u and v are μ-measurable functions on Ω. The boundary trace operator enjoys the following properties for any $x \in \partial\Omega$ for which both $T_\Omega u(x)$ and $T_\Omega v(x)$ exist:*

(i) $T_\Omega(\alpha u + \beta v)(x) = \alpha\, T_\Omega u(x) + \beta\, T_\Omega v(x)$ *for any* $\alpha, \beta \in \mathbb{R}$.

(ii) If $u \geq v$ μ-almost everywhere in Ω, then $T_\Omega u(x) \geq T_\Omega v(x)$. In particular, if $u = v$ μ-almost everywhere in Ω, then $T_\Omega u(x) = T_\Omega v(x)$.

(iii) $T_\Omega(\max\{u, v\})(x) = \max\{T_\Omega u(x), T_\Omega v(x)\}$ *and* $T_\Omega(\min\{u, v\})(x) = \min\{T_\Omega u(x), T_\Omega v(x)\}$.

(iv) Let $h > 0$ and define the truncation $u_h = \min\{h, \max\{u, -h\}\}$. Then $T_\Omega u_h(x) = (T_\Omega u(x))_h$.

(v) If Ω is a μ-measurable set such that both Ω and its complement satisfy the weak measure density condition, and w is a μ-measurable function on X, then for $\mathcal{H}$-almost every $x \in \partial\Omega$ for which both traces $T_\Omega w(x)$ and $T_{X\setminus\Omega} w(x)$ exist, we have

$$\{T_\Omega w(x), T_{X\setminus\Omega} w(x)\} = \{w^\wedge(x), w^\vee(x)\}.$$

Proof. Assertions (i) and (ii) are clear. Since minimum and maximum can be written as sums by using absolute values, property (iii) follows from (i) and the easily verified fact that $T_\Omega|u|(x) = |T_\Omega u(x)|$. Assertion (iv) follows from (iii). In proving assertion (v), due to the symmetry of the situation we can assume that $T_\Omega w(x) \geq T_{X\setminus\Omega} w(x)$. By using the definition of traces and Chebyshev's inequality, we deduce that for every $\varepsilon > 0$,

$$\lim_{r\to 0} \frac{\mu(\{|w - T_\Omega w(x)| > \varepsilon\} \cap B(x, r) \cap \Omega)}{\mu(B(x, r) \cap \Omega)} = 0$$

and

$$\lim_{r\to 0} \frac{\mu(\{|w - T_{X\setminus\Omega} w(x)| > \varepsilon\} \cap B(x, r) \setminus \Omega)}{\mu(B(x, r) \setminus \Omega)} = 0.$$

To determine the lower and upper approximate limits, we use these results to compute

$$\limsup_{r\to 0} \frac{\mu(\{w>t\}\cap B(x,r))}{\mu(B(x,r))}$$
$$= \limsup_{r\to 0} \left[\frac{\mu(\{w>t\}\cap B(x,r)\cap\Omega)}{\mu(B(x,r))} + \frac{\mu(\{w>t\}\cap B(x,r)\setminus\Omega)}{\mu(B(x,r))}\right]$$
$$\begin{cases} = 0+0 & \text{if } t > T_\Omega w(x), \\ = \limsup_{r\to 0} \frac{\mu(B(x,r)\cap\Omega)}{\mu(B(x,r))} + 0 & \text{if } T_{X\setminus\Omega} w(x) < t < T_\Omega w(x), \\ = \limsup_{r\to 0} \left[\frac{\mu(B(x,r)\cap\Omega)}{\mu(B(x,r))} + \frac{\mu(B(x,r)\setminus\Omega)}{\mu(B(x,r))}\right] & \text{if } t < T_{X\setminus\Omega} w(x), \end{cases}$$
$$\begin{cases} = 0 & \text{if } t > T_\Omega w(x), \\ \in (0,1) & \text{if } T_{X\setminus\Omega} w(x) < t < T_\Omega w(x), \\ = 1 & \text{if } t < T_{X\setminus\Omega} w(x). \end{cases}$$

To obtain the result "$\in (0,1)$" above, we used the weak measure density conditions. We conclude that $w^\vee(x) = T_\Omega w(x)$, and since "lim sup" can be replaced by "lim inf" in the above calculation, we also get $w^\wedge(x) = T_{X\setminus\Omega} w(x)$. □

A minor point to be noted is that any function that is in the class $\mathrm{BV}(X)$, such as an extension Eu of $u \in \mathrm{BV}(\Omega)$, is also in the class $\mathrm{BV}(\Omega)$, and thus $T_\Omega Eu = T_\Omega u$.

Eventually we will also need to make an additional assumption on the space, as described in the following definition that is from [3, Definition 6.1]. The function θ_E was introduced earlier in (2.4).

Definition 5.9. We say that X is a *local space* if, given any two sets of locally finite perimeter $E_1 \subset E_2 \subset X$, we have $\theta_{E_1}(x) = \theta_{E_2}(x)$ for $\mathcal{H}$-almost every $x \in \partial^* E_1 \cap \partial^* E_2$.

See [3] and [22] for some examples of local spaces, and the upcoming paper [23] for an example of a space that is not local, despite having a doubling measure and a Poincaré inequality. The assumption $E_1 \subset E_2$ can, in fact, be removed as follows. Note that for a set of locally finite perimeter E, we have $\|D\chi_E\| = \|D\chi_{X\setminus E}\|$, i.e. the two measures are equal [24, Proposition 4.7]. From this it follows that $\theta_E(x) = \theta_{X\setminus E}(x)$ for $\mathcal{H}$-almost every $x \in \partial^* E$. Now, if E_1 and E_2 are arbitrary sets of locally finite perimeter, we know that $E_1 \cap E_2$ and $E_1 \setminus E_2$ are also sets of locally finite perimeter [24, Proposition 4.7]. For every $x \in \partial^* E_1 \cap \partial^* E_2$ we have either $x \in \partial^*(E_1 \cap E_2)$ or $x \in \partial^*(E_1 \setminus E_2)$. Thus by the locality condition, we have for $\mathcal{H}$-almost every $x \in \partial^* E_1 \cap \partial^* E_2$ either

$$\theta_{E_1}(x) = \theta_{E_1\cap E_2}(x) = \theta_{E_2}(x)$$

or

$$\theta_{E_1}(x) = \theta_{E_1\setminus E_2}(x) = \theta_{X\setminus E_2}(x) = \theta_{E_2}(x).$$

Thus we have $\theta_{E_1}(x) = \theta_{E_2}(x)$ for $\mathcal{H}$-almost every $x \in \partial^* E_1 \cap \partial^* E_2$.

In a local space the decomposition (5.1) takes a simpler form, as proved in the following lemma.

Lemma 5.10. *If X is a local space, Ω is a set of locally finite perimeter, $u \in \mathrm{BV}(X)$, and $A \subset \partial^*\Omega$ is a Borel set, then we have*

$$\int_A \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x) = \int_A (u^\vee(x) - u^\wedge(x))\theta_\Omega\, d\mathcal{H}(x).$$

Note that since Ω is a set of locally finite perimeter, $A \subset \partial^*\Omega$ is σ-finite with respect to $\mathcal{H}$.

Proof. We have

$$\begin{aligned}
&\int_A \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x) \\
&= \int_A \int_{-\infty}^{\infty} \chi_{\{(u^\wedge(x),u^\vee(x))\}}(t)\theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x) \\
&= \int_{-\infty}^{\infty} \int_A \chi_{\{(-\infty,t)\}}(u^\wedge(x))\chi_{\{(t,\infty)\}}(u^\vee(x))\theta_{\{u>t\}}(x)\, d\mathcal{H}(x)\, dt \\
&= \int_{-\infty}^{\infty} \int_{A\cap\partial^*\{u>t\}} \chi_{\{(-\infty,t)\}}(u^\wedge(x))\chi_{\{(t,\infty)\}}(u^\vee(x))\theta_{\{u>t\}}(x)\, d\mathcal{H}(x)\, dt.
\end{aligned}$$

On the third line we used Fubini's theorem. On the fourth line we used the fact that if $u^\wedge(x) < t < u^\vee(x)$, then $x \in \partial^*\{u > t\}$. This follows from the definitions of the lower and upper approximate limits. By the locality condition we see that the right-hand side above equals

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{A\cap\partial^*\{u>t\}} \chi_{\{(-\infty,t)\}}(u^\wedge(x))\chi_{\{(t,\infty)\}}(u^\vee(x))\theta_\Omega(x)\, d\mathcal{H}(x)\, dt \\
&= \int_{-\infty}^{\infty} \int_A \chi_{\{(-\infty,t)\}}(u^\wedge(x))\chi_{\{(t,\infty)\}}(u^\vee(x))\theta_\Omega(x)\, d\mathcal{H}(x)\, dt \\
&= \int_A \int_{-\infty}^{\infty} \chi_{\{(u^\wedge(x),u^\vee(x))\}}(t)\, dt\, \theta_\Omega(x)\, d\mathcal{H}(x) \\
&= \int_A (u^\vee(x) - u^\wedge(x))\theta_\Omega(x)\, d\mathcal{H}(x).
\end{aligned}$$

□

Now we prove two propositions concerning boundary traces that are based on [2, Theorem 3.84] and [2, Theorem 3.86].

Proposition 5.11. *Let Ω and Ω^* be open sets such that Ω and $\Omega^* \setminus \Omega$ satisfy the weak measure density condition, $\overline{\Omega} \subset \Omega^*$, and Ω is of finite perimeter. Let $u, v \in \mathrm{BV}(\Omega^*)$, and let $w = u\chi_\Omega + v\chi_{\Omega^*\setminus\Omega}$. Then $w \in \mathrm{BV}(\Omega^*)$ if and only if*

$$\int_{\partial\Omega} |T_\Omega u - T_{\Omega^*\setminus\overline{\Omega}}\, v|\, d\mathcal{H} < \infty. \tag{5.2}$$

In the above characterization, we implicitly assume that the integral is well-defined — in particular, this is the case if Ω and $\Omega^* \setminus \overline{\Omega}$ are also strong BV *extension domains, due to Theorem 5.7. Furthermore, if X is a local space, we then have*

$$\|Dw\|(\Omega^*) = \|Du\|(\Omega) + \|Dv\|(\Omega^* \setminus \overline{\Omega}) + \int_{\partial\Omega} |T_\Omega u - T_{\Omega^*\setminus\overline{\Omega}} v|\theta_\Omega \, d\mathcal{H}.$$

Proof. First note that by the weak measure density conditions, we have $\mathcal{H}(\partial\Omega \setminus \partial^*\Omega) = 0$, and thus $\mathcal{H}(\partial\Omega) < \infty$. This further implies that $\mu(\partial\Omega) = 0$ [19, Lemma 6.1], and by this and the weak measure density conditions again,

$$\mathcal{H}(\partial\Omega \setminus \partial\overline{\Omega}) = 0 \quad \text{and} \quad T_{\Omega^*\setminus\overline{\Omega}} = T_{\Omega^*\setminus\Omega}.$$

To prove one direction of the proposition, let us assume (5.2). In particular, we assume that $T_\Omega u(x)$ and $T_{\Omega^*\setminus\overline{\Omega}} v(x)$ exist for $\mathcal{H}$-almost every $x \in \partial\Omega$. For $h > 0$, define the truncated functions

$$u_h = \min\{h, \max\{u, -h\}\} \qquad \text{and} \qquad v_h = \min\{h, \max\{v, -h\}\}.$$

Clearly $u_h, v_h, \chi_\Omega, \chi_{\Omega^*\setminus\Omega} \in \mathrm{BV}(\Omega^*) \cap L^\infty(\Omega^*)$. Then

$$w_h = u_h\chi_\Omega + v_h\chi_{\Omega^*\setminus\Omega} \in \mathrm{BV}(\Omega^*) \cap L^\infty(\Omega^*),$$

see e.g. [20, Proposition 4.2]. Based on the decomposition of the variation measure given in (5.1),

(5.3)
$$\begin{aligned}
&\|Dw_h\|(\Omega^*) \\
&= \|Du_h\|(\Omega) + \|Dv_h\|(\Omega^* \setminus \overline{\Omega}) + \int_{\partial\Omega} \int_{w_h^\wedge(x)}^{w_h^\vee(x)} \theta_{\{w_h > t\}}(x) \, dt \, d\mathcal{H}(x) \\
&\le \|Du\|(\Omega) + \|Dv\|(\Omega^* \setminus \overline{\Omega}) + \int_{\partial\Omega} c_d |w_h^\vee(x) - w_h^\wedge(x)| \, d\mathcal{H}(x).
\end{aligned}$$

By Proposition 5.8 (iv), the boundary traces T_Ω of u, u_h, w_h, and $T_{\Omega^*\setminus\overline{\Omega}}$ of v, v_h, w_h, exist $\mathcal{H}$-almost everywhere on the boundary $\partial\Omega$. For w_h this fact follows from the definition of boundary traces, by which we have that $T_\Omega w_h = T_\Omega u_h$, and similarly $T_{\Omega^*\setminus\overline{\Omega}} w_h = T_{\Omega^*\setminus\overline{\Omega}} v_h$. Proposition 5.8 (v) now gives

(5.4)
$$\{w_h^\wedge(x), w_h^\vee(x)\} = \{T_\Omega w_h(x), T_{\Omega^*\setminus\overline{\Omega}} w_h(x)\} = \{T_\Omega u_h(x), T_{\Omega^*\setminus\overline{\Omega}} v_h(x)\}$$

for $\mathcal{H}$-almost every $x \in \partial\Omega$. Using Proposition 5.8 (iv) again, for $\mathcal{H}$-almost every $x \in \partial\Omega$ we have

(5.5)
$$\begin{aligned}
&T_\Omega u_h(x) = \min\{h, \max\{T_\Omega u(x), -h\}\}, \\
&T_{\Omega^*\setminus\overline{\Omega}} v_h(x) = \min\{h, \max\{T_{\Omega^*\setminus\overline{\Omega}} v(x), -h\}\}.
\end{aligned}$$

By the lower semicontinuity of the total variation as well as (5.3), (5.4) and (5.5), we now get

$$\begin{aligned}\|Dw\|(\Omega^*) &\le \liminf_{h\to\infty} \|Dw_h\|(\Omega^*)\\ &\le \|Du\|(\Omega) + \|Dv\|(\Omega^*\setminus\overline{\Omega}) + \liminf_{h\to\infty} c_d \int_{\partial\Omega} |T_\Omega u_h - T_{\Omega^*\setminus\overline{\Omega}}\, v_h|\, d\mathcal{H}\\ &= \|Du\|(\Omega) + \|Dv\|(\Omega^*\setminus\overline{\Omega}) + c_d \int_{\partial\Omega} |T_\Omega u - T_{\Omega^*\setminus\overline{\Omega}}\, v|\, d\mathcal{H} < \infty.\end{aligned}$$

Thus $w \in \mathrm{BV}(\Omega^*)$.

To prove the converse, assume that $w \in \mathrm{BV}(\Omega^*)$. Here we can simply again write the decomposition of the variation measure

$$\infty > \|Dw\|(\Omega^*) \ge \|Du\|(\Omega) + \|Dv\|(\Omega^*\setminus\overline{\Omega}) + \alpha \int_{\partial\Omega} |w^\vee - w^\wedge|\, d\mathcal{H},$$

where $\alpha = \alpha(c_d, c_P) > 0$, and just as earlier, note that

$$|w^\vee(x) - w^\wedge(x)| = |T_\Omega w(x) - T_{\Omega^*\setminus\overline{\Omega}}\, w(x)| = |T_\Omega u(x) - T_{\Omega^*\setminus\overline{\Omega}}\, v(x)| \tag{5.6}$$

for $\mathcal{H}$-almost every $x \in \partial\Omega$. This combined with the previous estimate gives the desired result. If X is a local space, we combine the decomposition of the variation measure (5.1), Lemma 5.10, and (5.6) to obtain the last claim. □

Next we show that if a set A (which could be e.g. the boundary $\partial\Omega$) is in a suitable sense of codimension one, traces of BV functions are indeed integrable on A. Let us first recall the following fact from the theory of sets of finite perimeter. Given any set of finite perimeter $E \subset X$, for $\mathcal{H}$-almost every $x \in \partial^* E$ we have

$$\gamma \le \liminf_{r\to 0} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} \le \limsup_{r\to 0} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} \le 1-\gamma, \tag{5.7}$$

where $\gamma \in (0, 1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality [1, Theorem 5.4].

Proposition 5.12. *Let $\Omega^* \subset X$ be open, let $u \in \mathrm{BV}(\Omega^*)$, and let $A \subset \Omega^*$ be a bounded Borel set that satisfies $\operatorname{dist}(A, X\setminus\Omega^*) > 0$ and*

$$\mathcal{H}(A\cap B(x,r)) \le c_A \frac{\mu(B(x,r))}{r} \tag{5.8}$$

for every $x \in A$ and $r \in (0, R]$, where $R \in (0, \operatorname{dist}(A, X\setminus\Omega^*))$ and $c_A > 0$ are constants. Then

$$\int_A (|u^\wedge| + |u^\vee|)\, d\mathcal{H} \le C\|u\|_{\mathrm{BV}(\Omega^*)}, \tag{5.9}$$

where $C = C(c_d, c_P, \lambda, A, R, c_A)$.

Proof. We may assume that $u \ge 0$. Let

$$c = \inf_{x\in A} \mu(B(x, R/(5\lambda)));$$

by the doubling property of μ we have $c = c(A, R, c_d, \lambda) > 0$. First consider a set $E \subset X$ that is of finite perimeter in Ω^* and satisfies $\mu(E) < \delta$, where $\delta > 0$ is a constant that will be determined below. Define

$$E^\gamma = \left\{ x \in \Omega^* : \liminf_{r \to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \geq \gamma \right\},$$

where $\gamma = \gamma(c_d, c_P, \lambda) > 0$ is the constant from (5.7). Pick any $x \in E^\gamma \cap A$. We note that

$$\frac{\mu(E \cap B(x, R/(5\lambda)))}{\mu(B(x, R/(5\lambda)))} \leq \frac{\mu(E)}{\mu(B(x, R/(5\lambda)))} < \frac{\delta}{c}.$$

By choosing $\delta > 0$ small enough, we have

$$\frac{\mu(E \cap B(x, R/(5\lambda)))}{\mu(B(x, R/(5\lambda)))} \leq \frac{\gamma}{2}.$$

Thus we have $\delta = \delta(c, \gamma)$, and consequently $\delta = \delta(c_d, c_P, \lambda, A, R)$. By the definition of E^γ, we can find a number $r \in (0, R/5]$ that satisfies

$$\frac{\gamma}{2c_d} < \frac{\mu(E \cap B(x, r/\lambda))}{\mu(B(x, r/\lambda))} \leq \frac{\gamma}{2}.$$

This can be done by repeatedly halving the radius $R/5$ until the right-hand side of the above inequality does not hold, and picking the last radius for which it did hold. From the relative isoperimetric inequality (2.5) we conclude that

$$\frac{\mu(B(x, r/\lambda))}{r/\lambda} \leq \frac{2c_d}{\gamma} \frac{\mu(E \cap B(x, r/\lambda))}{r/\lambda} \leq \frac{C}{\gamma} P(E, B(x, r)). \tag{5.10}$$

Using the radii chosen this way, we get a covering $\{B(x, r(x))\}_{x \in A \cap E^\gamma}$ of the set $A \cap E^\gamma$. By the 5-covering lemma, we can select a countable family of disjoint balls $\{B(x_i, r_i)\}_{i=1}^\infty$ such that the balls $B(x_i, 5r_i)$ cover $A \cap E^\gamma$. By using (5.8) and (5.10), we get

$$\begin{aligned} \mathcal{H}(E^\gamma \cap A) &\leq \sum_{i=1}^\infty \mathcal{H}(E^\gamma \cap A \cap B(x_i, 5r_i)) \\ &\leq c_A \sum_{i=1}^\infty \frac{\mu(B(x_i, 5r_i))}{5r_i} \leq C \sum_{i=1}^\infty \frac{\mu(B(x_i, r_i/\lambda))}{r_i/\lambda} \\ &\leq C \sum_{i=1}^\infty P(E, B(x_i, r_i)) \leq C P(E, \Omega^*), \end{aligned} \tag{5.11}$$

where $C = (c_d, c_P, \lambda, c_A)$.

Then we consider the function u. Assume that $x \in A \cap S_u$ and $u^\wedge(x) + u^\vee(x) > t$, with $t > 0$. By the definitions of the lower and upper approximate limits, we know that $x \in \partial^*\{u > s\}$ for all $s \in (u^\wedge(x), u^\vee(x))$. By the coarea formula (2.3), the sets $\{u > s\}$ are of finite perimeter in Ω^* for every $s \in T$, where T is a countable dense subset of $\mathbb{R}$. Thus, outside a $\mathcal{H}$-negligible set, (5.7) holds for every

$x \in \partial^*\{u > s\}$ and $s \in T$. Assuming that x is outside this $\mathcal{H}$-negligible set, we can find $s \in ((u^\wedge(x) + u^\vee(x))/2, u^\vee(x)) \cap T$ and estimate

$$\liminf_{r\to 0} \frac{\mu(\{u > t/2\} \cap B(x,r))}{\mu(B(x,r))} \geq \liminf_{r\to 0} \frac{\mu(\{u > s\} \cap B(x,r))}{\mu(B(x,r))} \geq \gamma,$$

which means that $x \in \{u > t/2\}^\gamma$. By Chebyshev's inequality we get

$$\mu(\{u > t/2\}) \leq \frac{\|u\|_{L^1(\Omega^*)}}{t/2} < \delta$$

if $t > t_0$, where $t_0 = C(c_d, c_P, \lambda, A, R)\|u\|_{L^1(\Omega^*)}$ due to the dependencies of δ given earlier. By the coarea formula, $\{u > t/2\}$ is of finite perimeter in Ω^* for almost every $t \in \mathbb{R}$, and Cavalieri's principle and (5.11) then imply that

$$\begin{aligned}
\int_{A\cap S_u} (u^\wedge + u^\vee)\, d\mathcal{H} &= \int_0^\infty \mathcal{H}(\{x \in A \cap S_u : u^\wedge(x) + u^\vee(x) > t\})\, dt \\
&\leq \int_0^\infty \mathcal{H}(\{u > t/2\}^\gamma \cap A)\, dt \\
&\leq t_0\mathcal{H}(A) + \int_{t_0}^\infty C(c_d, c_P, \lambda, c_A) P(\{u > t/2\}, \Omega^*)\, dt \\
&\leq C(c_d, c_P, \lambda, A, R)\|u\|_{L^1(\Omega^*)}\mathcal{H}(A) + C(c_d, c_P, \lambda, c_A)\|Du\|(\Omega^*).
\end{aligned}$$

This gives the estimate for $A \cap S_u$. For $A \setminus S_u$, we simply note that if $x \in A \setminus S_u$ and $u^\wedge(x) = u^\vee(x) > t$, then the approximate limit of u at x is larger than t, which easily gives $x \in \{u > t\}^\gamma$, and then we can use Cavalieri's principle as above. □

Finally we get the desired representation for the minimization problem.

Theorem 5.13. *Assume that X is a local space, and let $\Omega \Subset \Omega^*$ be bounded open sets such that Ω and $\Omega^* \setminus \Omega$ satisfy the weak measure density condition, Ω is a strong* BV *extension domain, and $\partial\Omega$ satisfies the assumptions of Proposition 5.12. Assume also that $h \in \mathrm{BV}(\Omega^*)$ and that the trace $T_{X\setminus\overline{\Omega}}h(x)$ exists for $\mathcal{H}$-almost every $x \in \partial\Omega$, which in particular is true if $\Omega^* \setminus \overline{\Omega}$ is also a strong* BV *extension domain. Then the minimization problem given in Definition 5.2, with boundary values h, can be reformulated as the minimization of the functional*

$$\mathcal{F}(u, \Omega) + f_\infty \int_{\partial\Omega} |T_\Omega u - T_{X\setminus\overline{\Omega}}h|\theta_\Omega\, d\mathcal{H} \tag{5.12}$$

over all $u \in \mathrm{BV}(\Omega)$.

Note that this formulation contains no reference to Ω^*.

Proof. First note that due to the conditions of Proposition 5.12, we have $\mathcal{H}(\partial\Omega) < \infty$, and thus $\mu(\partial\Omega) = 0$ and Ω is a set of finite perimeter,

see e.g. [19, Lemma 6.1, Proposition 6.3]. By the weak measure density conditions,

$$\mathcal{H}(\partial\Omega \setminus \partial\overline{\Omega}) = 0 \quad \text{and} \quad T_{\Omega^*\setminus\overline{\Omega}} = T_{\Omega^*\setminus\Omega}.$$

Now, for any $u \in \mathrm{BV}_h(\Omega)$, we have $u \in \mathrm{BV}(\Omega^*)$ by definition, and $\mathcal{F}(u,\Omega^*) < \infty$ by (3.4). Then

$$\begin{aligned}
&\mathcal{F}(u,\Omega^*)\\
&\quad = \mathcal{F}(u,\Omega) + \mathcal{F}^s(u,\partial\Omega) + \mathcal{F}(h,\Omega^*\setminus\overline{\Omega})\\
&\quad = \mathcal{F}(u,\Omega) + f_\infty\|Du\|^s(\partial\Omega) + \mathcal{F}(h,\Omega^*\setminus\overline{\Omega})\\
&\quad = \mathcal{F}(u,\Omega) + f_\infty\int_{\partial\Omega} |u^\vee - u^\wedge|\theta_\Omega\, d\mathcal{H} + \mathcal{F}(h,\Omega^*\setminus\overline{\Omega})\\
&\quad = \mathcal{F}(u,\Omega) + f_\infty\int_{\partial\Omega} |T_\Omega u - T_{X\setminus\overline{\Omega}}h|\theta_\Omega\, d\mathcal{H} + \mathcal{F}(h,\Omega^*\setminus\overline{\Omega}),
\end{aligned} \tag{5.13}$$

where the first equality follows from the measure property of $\mathcal{F}(u,\cdot)$ as well as the fact that $\mu(\partial\Omega) = 0$, the second equality follows from the integral representation of the functional (see Remark 4.5), the third equality follows from the decomposition (5.1) and Lemma 5.10, and the fourth equality follows from Proposition 5.8 (v). Now, the term $\mathcal{F}(h,\Omega^*\setminus\overline{\Omega})$ does not depend on u, so in fact we need to minimize (5.12).

Conversely, assume that $u \in \mathrm{BV}(\Omega)$. Then we can extend u to $Eu \in \mathrm{BV}(\Omega^*)$. By Proposition 5.8 (v) we have

$$\{T_\Omega h(x), T_{X\setminus\overline{\Omega}}h(x)\} = \{h^\wedge(x), h^\vee(x)\}$$

for $\mathcal{H}$-almost every $x \in \partial\Omega$. By the proof of Theorem 5.7 we have that $T_\Omega Eu(x)$ is the Lebesgue limit of Eu for $\mathcal{H}$-almost every $x \in \partial\Omega$. By Proposition 5.12, we now get

$$\int_{\partial\Omega} |T_\Omega Eu - T_{X\setminus\overline{\Omega}}h|\, d\mathcal{H} \le C(\|Eu\|_{\mathrm{BV}(\Omega^*)} + \|h\|_{\mathrm{BV}(\Omega^*)}) < \infty.$$

By Proposition 5.11 we deduce that $w = (Eu)\chi_\Omega + h\chi_{\Omega^*\setminus\Omega} \in \mathrm{BV}(\Omega^*)$, and in fact we have $w = u\chi_\Omega + h\chi_{\Omega^*\setminus\Omega} \in \mathrm{BV}_h(\Omega)$. This completes the proof. □

Remark 5.14. Note that in the latter part of the above proof we showed that, under the assumptions on the space and on Ω, the spaces $\mathrm{BV}(\Omega)$ and $\mathrm{BV}_h(\Omega) \subset \mathrm{BV}(\Omega^*)$ can be identified.

References

[1] L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Calculus of variations, nonsmooth analysis and related topics. Set-Valued Anal. **10** (2002), no. 2-3, 111–128.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] L. Ambrosio, M. Miranda, Jr., and D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., **14**, Dept. Math., Seconda Univ. Napoli, Caserta, 2004.

[4] G. Bellettini, G. Bouchitté and I. Fragalà, *BV functions with respect to a measure and relaxation of metric integral functionals*, J. Convex Anal. **6** (1999), no. 2, 349–366.

[5] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011.

[6] A. Björn, J. Björn, and N. Shanmugalingam, *Sobolev extensions of Hölder continuous and characteristic functions on metric spaces*, Canad. J. Math. **59** (2007), no. 6, 1135–1153.

[7] J. Björn and N. Shanmugalingam, *Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces*, J. Math. Anal. Appl. **332** (2007), no. 1, 190–208.

[8] G. Bouchitté and E. Dal Maso, *Integral representation and relaxation of convex local functionals on* $\mathrm{BV}(\Omega)$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993) no. 4, 483–533.

[9] G. Bouchitté and M. Valadier, *Integral representation of convex functionals on a space of measures*, J. Funct. Anal. **80** (1988) no. 2, 398–420.

[10] G. Buttazzo, M. Giaquinta, and S. Hildebrandt, *One-dimensional variational problems. An introduction*, Oxford Lecture Series in Mathematics and its Applications, 15. The Clarendon Press, Oxford University Press, New York, 1998.

[11] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[12] B. Franchi, P. Hajłasz, and P. Koskela, *Definitions of Sobolev classes on metric spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 6, 1903–1924.

[13] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents in the calculus of variations I. Cartesian currents*, Springer, 1998.

[14] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents in the calculus of variations II. Variational integrals*, Springer, 1998.

[15] M. Giaquinta, G. Modica, and J. Souček, *Functionals with linear growth in the calculus of variations. I, II.*, Comment. Math. Univ. Carolin. **20** (1979), no. 1, 143–156, 157–172.

[16] P. Hajłasz, *Sobolev spaces on metric-measure spaces*, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173–218. Contemp. Math., **338**, Amer. Math. Soc., Providence, RI, 2003,

[17] H. Hakkarainen, J. Kinnunen, and P. Lahti, *Regularity of minimizers of the area functional in metric spaces*, Adv. Calc. Var. **8** (2015), no. 1, 55–68.

[18] H. Hakkarainen, R. Korte, P. Lahti, and N. Shanmugalingam, *Stability and continuity of functions of least gradient*, Anal. Geom. Metr. Spaces **3** (2015), Art. 9.

[19] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *A characterization of Newtonian functions with zero boundary values*, Calc. Var. Partial Differential Equations **43** (2012), no. 3-4, 507–528.

[20] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *Pointwise properties of functions of bounded variation on metric spaces*, Rev. Mat. Complut. **27** (2014), no. 1, 41–67.

[21] P. Koskela, M. Miranda Jr., and N. Shanmugalingam, *Geometric properties of planar BV extension domains*, Function Spaces; Topics Around the Research of Prof. Maz'ya I, International Mathematical Series (2010) (Springer collection), 255–272.
[22] P. Lahti, *Extensions and traces of functions of bounded variation on metric spaces*, J. Math. Anal. Appl. **423** (2015), no. 1, 521–537.
[23] P. Lahti and N. Shanmugalingam, Fine properties and a notion of quasicontinuity for BV functions on metric spaces, preprint 2015.
[24] M. Miranda, Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004.
[25] W. Rudin, *Functional analysis*, Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991. xviii+424 pp.
[26] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279.

Addresses:

H.H.: Department of Mathematical Sciences, P.O. Box 3000, FI-90014 University of Oulu, Finland.
E-mail: heikki.hakkarainen@oulu.fi

J.K.: Aalto University, School of Science, Department of Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland.
E-mail: juha.k.kinnunen@aalto.fi

P.L.: Aalto University, School of Science, Department of Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland.
E-mail: panu.lahti@aalto.fi

P.L.: Aalto University, School of Science, Department of Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland.
E-mail: pekka.lehtela@aalto.fi