REGULARITY OF THE LOCAL FRACTIONAL MAXIMAL FUNCTION

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ABSTRACT. This paper studies smoothing properties of the local fractional maximal operator, which is defined in a proper subdomain of the Euclidean space. We prove new pointwise estimates for the weak gradient of the maximal function, which imply norm estimates in Sobolev spaces. An unexpected feature is that these estimates contain extra terms involving spherical and fractional maximal functions. Moreover, we construct several explicit examples which show that our results are essentially optimal. Extensions to metric measure spaces are also discussed.

1. INTRODUCTION

Fractional maximal operators are standard tools in partial differential equations, potential theory and harmonic analysis. In the Euclidean setting, they have been studied in [3], [4], [5], [28], [30], [32] and [37]. It has been observed in [28] that the global fractional maximal operator $\mathcal{M}_\alpha$, defined by

$$(1.1) \qquad \mathcal{M}_\alpha\, u(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |u(y)|\, dy,$$

has similar smoothing properties as the Riesz potential. More precisely, there is a constant C, depending only on n and α, such that

$$(1.2) \qquad |D\,\mathcal{M}_\alpha\, u(x)| \le C\,\mathcal{M}_{\alpha-1}\, u(x)$$

for almost every $x \in \mathbb{R}^n$. This implies that the fractional maximal operator maps $L^p(\mathbb{R}^n)$ to a certain Sobolev space. If the function itself is a Sobolev function, then the fractional maximal function belongs to a Sobolev space with a higher exponent. This follows quite easily from the Sobolev theorem using the facts that $\mathcal{M}_\alpha$ is sublinear and commutes with translations, see [28, Theorem 2.1]. The regularity properties of the Hardy-Littlewood maximal function, that is (1.1) with $\alpha = 0$, have been studied in [6], [10], [18], [19], [25], [29], [31], [33], [35] and [47].

This paper studies smoothness of the local fractional maximal function

$$\mathcal{M}_{\alpha,\Omega}\, u(x) = \sup r^\alpha \fint_{B(x,r)} |u(y)|\, dy,$$

where the supremum is taken over all radii r satisfying $0 < r < \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$. In this case, the family of balls in the definition of the maximal function depends on the point $x \in \Omega$ and the same arguments as in the global case do not apply. For the Hardy-Littlewood maximal function, the question has been studied in [26] and [19], see also [34]. For the local Hardy-Littlewood maximal operator $\mathcal{M}_\Omega$ with $\alpha = 0$ we have

$$(1.3) \qquad\qquad |D\,\mathcal{M}_\Omega\, u(x)| \leq 2\,\mathcal{M}_\Omega\, |Du|(x)$$

for almost every $x \in \Omega$. In particular, this implies that the maximal function is bounded in Sobolev space $W^{1,p}(\Omega)$ when $1 < p \leq \infty$.

The situation is more delicate for the local fractional maximal operator $\mathcal{M}_{\alpha,\Omega}$ with $\alpha > 0$. One might expect that pointwise estimates (1.2) and (1.3) would also hold in that case. However, this is not true as such. Instead of (1.2), we have

$$|D\,\mathcal{M}_{\alpha,\Omega}\, u(x)| \leq C\big(\,\mathcal{M}_{\alpha-1,\Omega}\, u(x) + \mathcal{S}_{\alpha-1,\Omega} u(x)\big)$$

for almost every $x \in \Omega$, where C depends only on n. The local spherical fractional maximal function is defined as

$$\mathcal{S}_{\alpha-1,\Omega} u(x) = \sup r^{\alpha-1}\!\!\fint_{\partial B(x,r)} |u(y)|\, d\mathcal{H}^{n-1}(y),$$

where the supremum is taken over all radii r for which $0 < r < \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$. Norm estimates for the spherical maximal operator are much more delicate than the corresponding estimates for the standard maximal operator, but they can be obtained along the lines of [40] and [42]. These estimates are of independent interest and they are discussed in Section 2. Consequently, the local fractional maximal function belongs locally to a certain Sobolev space.

We also show that

$$|D\,\mathcal{M}_{\alpha,\Omega}\, u(x)| \leq 2\,\mathcal{M}_{\alpha,\Omega}\, |Du|(x) + \alpha\,\mathcal{M}_{\alpha-1,\Omega}\, u(x)$$

for almost every $x \in \Omega$. This is an extension of (1.3), but again there is and extra term on the right hand side. Because of this the local fractional maximal function of a Sobolev function is not necessarily smoother than the fractional maximal function of an arbitrary function in $L^p(\Omega)$. This is in a strict contrast with the smoothing properties in the global case discussed in [28]. Moreover, we show that $\mathcal{M}_{\alpha,\Omega}\, u$ has zero boundary values in the Sobolev sense and hence it can be potentially used as a test function in the theory of partial differential equations. In Section 4, we construct several explicit examples, which complement our study and show that our results are essentially optimal. Another delicate feature is that the local fractional maximal operator over cubes has worse smoothing properties than $\mathcal{M}_{\alpha,\Omega}$ defined over balls.

In the last section, we extend the regularity results of the local fractional maximal operator in metric measure spaces. As in the non-fractional case [2], we use a discrete version of the maximal operator, because the standard maximal operators do not have the required regularity properties without any additional assumptions on the metric and measure. In the metric setting,

fractional maximal operators have been studied for example in [13], [14], [15], [20], [22], [38], [39] and [48].

2. Notation and preliminaries

Throughout the paper, the characteristic function of a set E is denoted by χ_E. In general, C is a positive constant whose value is not necessarily the same at each occurrence.

Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mathbb{R}^n \setminus \Omega \neq \emptyset$ and let $\alpha \geq 0$. The local fractional maximal function of a locally integrable function u is

$$\mathcal{M}_{\alpha,\Omega}\, u(x) = \sup r^\alpha \fint_{B(x,r)} |u(y)|\, dy,$$

where the supremum is taken over all radii r satisfying $0 < r < \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$. Here

$$\fint_B u(y)\, dy = \frac{1}{|B|} \int_B u(y)\, dy$$

denotes the integral average of u over B. If $\alpha = 0$, we have the local Hardy-Littlewood maximal function

$$\mathcal{M}_\Omega\, u(x) = \sup \fint_{B(x,r)} |u(y)|\, dy.$$

When $\Omega = \mathbb{R}^n$, the supremum is taken over all $r > 0$ and we obtain the fractional maximal function $\mathcal{M}_\alpha\, u$ and the Hardy-Littlewood maximal function $\mathcal{M}\, u$. A Sobolev type theorem for the fractional maximal operator follows easily from the Hardy-Littlewood maximal function theorem.

Theorem 2.1. *Let $p > 1$ and $0 < \alpha < n/p$. There is a constant $C > 0$, independent of u, such that*

$$\| \mathcal{M}_\alpha\, u \|_{L^{p^*}(\mathbb{R}^n)} \leq C \|u\|_{L^p(\mathbb{R}^n)},$$

for every $u \in L^p(\mathbb{R}^n)$ with $p^* = np/(n - \alpha p)$.

Now the corresponding boundedness result for the local fractional maximal function follows easily because for each $u \in L^p(\Omega)$, $p > 1$, we have

$$(2.1) \quad \| \mathcal{M}_{\alpha,\Omega}\, u \|_{L^{p^*}(\Omega)} \leq \| \mathcal{M}_\alpha(u\chi_\Omega) \|_{L^{p^*}(\mathbb{R}^n)} \leq C \|u\chi_\Omega\|_{L^p(\mathbb{R}^n)} = C \|u\|_{L^p(\Omega)}.$$

The local spherical fractional maximal function of u is

$$\mathcal{S}_{\alpha,\Omega} u(x) = \sup r^\alpha \fint_{\partial B(x,r)} |u(y)|\, d\mathcal{H}^{n-1}(y),$$

where the supremum is taken over all radii r for which $0 < r < \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$. Observe that the barred integral denotes the integral average with respect to the Hausdorff measure $\mathcal{H}^{n-1}$. When $\Omega = \mathbb{R}^n$, the supremum is taken over all $r > 0$ and we obtain the global spherical fractional maximal function $\mathcal{S}_\alpha u$.

The following norm estimate for the spherical fractional maximal operator will be useful for us.

Theorem 2.2. *Let $n \geq 2$, $p > n/(n-1)$ and $0 \leq \alpha < \min\{(n-1)/p, \ n - 2n/((n-1)p)\}$. Then*

$$(2.2) \qquad \|\mathcal{S}_\alpha u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)},$$

where $p^* = np/(n - \alpha p)$ and the constant C depends only on n, p and α.

For $\alpha = 0$, this was proved by Stein [46] in the case $n \geq 3$ and by Bourgain [9] in the case $n = 2$. For $\alpha > 0$, the result is due to Schlag [40, Theorem 1.3] when $n = 2$ and Schlag and Sogge [42, Theorem 4.1] when $n \geq 3$. In [40] and [42] the result is stated for the operator

$$\widetilde{\mathcal{S}}u(x) = \sup_{1 < r < 2} \fint_{\partial B(x,r)} |u(y)| \, d\mathcal{H}^{n-1}(y),$$

but the corresponding result for $\mathcal{S}_\alpha$ follows by the Littlewood-Paley theory as in [9, p.71–73] , [45, Section 2.4] and [41, Section 3.1]. In particular, Theorem 2.2 implies that the local spherical fractional maximal operator satisfies

$$(2.3) \qquad \|\mathcal{S}_{\alpha,\Omega} u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{L^p(\Omega)}.$$

We recall the definition of the Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : |Du| \in L^p(\Omega)\},$$

where $Du = (D_1 u, \ldots, D_n u)$ is the weak gradient of u. The weak partial derivatives of u, denoted by $D_i u$, $i = 1, \ldots, n$, are defined as such functions $v_i \in L^1_{\text{loc}}(\Omega)$ that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega v_i \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. The Sobolev space with zero boundary values $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \int_\Omega |Du|^p \, dx \right)^{1/p}.$$

3. Derivative of the local fractional maximal function

In this section, we prove pointwise estimates for the weak gradient of the local fractional maximal function. By integrating the pointwise estimates we also get the corresponding norm estimates.

We define the fractional average functions $u_t^\alpha \colon \Omega \to [-\infty, \infty]$, $0 < t < 1$, $0 \leq \alpha < \infty$, of a locally integrable function u as

$$(3.1) \qquad u_t^\alpha(x) = (t\delta(x))^\alpha \fint_{B(x,t\delta(x))} u(y) \, dy,$$

where $\delta(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$. We start by deriving an estimate for the weak gradient of the fractional average function of an L^p-function.

Lemma 3.1. *Let $n \geq 2$, $p > n/(n-1)$, $0 < t < 1$ and $1 \leq \alpha < \min\{(n-1)/p, n - 2n/((n-1)p)\} + 1$. If $u \in L^p(\Omega)$, then $|Du_t^\alpha| \in L^q(\Omega)$ with $q = np/(n - (\alpha - 1)p)$. Moreover,*

$$(3.2) \qquad |Du_t^\alpha(x)| \leq C\big(\mathcal{M}_{\alpha-1,\Omega}\, u(x) + \mathcal{S}_{\alpha-1,\Omega} u(x) \big)$$

for almost every $x \in \Omega$, where the constant C depends only on n.

Proof. Suppose first that $u \in L^p(\Omega) \cap C^\infty(\Omega)$. According to Rademacher's theorem, as a Lipschitz function, δ is differentiable almost everywhere in Ω. Moreover, $|D\delta(x)| = 1$ for almost every $x \in \Omega$. Denoting $\omega_n = |B(0,1)|$, the Leibniz rule gives

$$D_i u_t^\alpha(x) = D_i\Big(\omega_n^{-1}(t\delta(x))^{\alpha-n}\Big) \int_{B(x,t\delta(x))} u(y)\, dy$$
$$+ \omega_n^{-1}(t\delta(x))^{\alpha-n} D_i\Big(\int_{B(x,t\delta(x))} u(y)\, dy\Big), \qquad i = 1, \ldots, n,$$

for almost every $x \in \Omega$, and by the chain rule

$$D_i\Big(\int_{B(x,t\delta(x))} u(y)\, dy\Big) = \int_{B(x,t\delta(x))} D_i u(y)\, dy$$
$$+ t D_i \delta(x) \int_{\partial B(x,t\delta(x))} u(y)\, d\mathcal{H}^{n-1}(y), \qquad i = 1, \ldots, n,$$

for almost every $x \in \Omega$. Here we also used the fact that

$$\frac{\partial}{\partial r} \int_{B(x,r)} u(y)\, dy = \int_{\partial B(x,r)} u(y)\, d\mathcal{H}^{n-1}(y).$$

Collecting the terms in a vector form, we obtain

$$Du_t^\alpha(x) = \omega_n^{-1} t^{\alpha-n}(\alpha - n)\delta(x)^{\alpha-n-1} D\delta(x) \int_{B(x,t\delta(x))} u(y)\, dy$$

$$(3.3) \qquad + \omega_n^{-1}(t\delta(x))^{\alpha-n} \int_{B(x,t\delta(x))} Du(y)\, dy$$

$$+ \omega_n^{-1}(t\delta(x))^{\alpha-n} t D\delta(x) \int_{\partial B(x,t\delta(x))} u(y)\, d\mathcal{H}^{n-1}(y)$$

for almost every $x \in \Omega$. Applying Gauss' theorem to the integral in the second term we have

$$\int_{B(x,t\delta(x))} Du(y)\, dy = \int_{\partial B(x,t\delta(x))} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y),$$

where $\nu(y) = (y - x)/(t\delta(x))$ is the unit outer normal of $B(x, t\delta(x))$.

Modifying the integrals into their average forms, we obtain

$$Du_t^\alpha(x) = (\alpha - n)(t\delta(x))^\alpha \frac{D\delta(x)}{\delta(x)} \fint_{B(x,t\delta(x))} u(y)\, dy$$

(3.4)
$$+ n(t\delta(x))^{\alpha-1} \fint_{\partial B(x,t\delta(x))} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y)$$

$$+ n(t\delta(x))^\alpha \frac{D\delta(x)}{\delta(x)} \fint_{\partial B(x,t\delta(x))} u(y)\, d\mathcal{H}^{n-1}(y)$$

for almost every $x \in \Omega$. For the boundary integral terms, we have used the relation between the Lebesgue measure of a ball and the Hausdorff measure of its boundary $\mathcal{H}^{n-1}(\partial B(x,r)) = n\omega_n r^{n-1}$.

Taking the vector norms in the identity of the derivative and recalling that $0 < t < 1$ and $|D\delta(x)| = 1$ for almost every $x \in \Omega$, we obtain

$$|Du_t^\alpha(x)| \le |\alpha - n|(t\delta(x))^\alpha \frac{|D\delta(x)|}{\delta(x)} \fint_{B(x,t\delta(x))} |u(y)|\, dy$$

$$+ n(t\delta(x))^{\alpha-1} \fint_{\partial B(x,t\delta(x))} |u(y)||\nu(y)|\, d\mathcal{H}^{n-1}(y)$$

$$+ n(t\delta(x))^\alpha \frac{|D\delta(x)|}{\delta(x)} \fint_{\partial B(x,t\delta(x))} |u(y)|\, d\mathcal{H}^{n-1}(y)$$

$$\le n(t\delta(x))^{\alpha-1} \fint_{B(x,t\delta(x))} |u(y)|\, dy$$

$$+ n(t\delta(x))^{\alpha-1} \fint_{\partial B(x,t\delta(x))} |u(y)|\, d\mathcal{H}^{n-1}(y)$$

$$+ n(t\delta(x))^{\alpha-1} \fint_{\partial B(x,t\delta(x))} |u(y)|\, d\mathcal{H}^{n-1}(y)$$

$$\le C\big(\mathcal{M}_{\alpha-1,\Omega}\, u(x) + \mathcal{S}_{\alpha-1,\Omega} u(x)\big)$$

for almost every $x \in \Omega$. Thus, (3.2) holds for smooth functions.

The case $u \in L^p(\Omega)$ follows from an approximation argument. For $u \in L^p(\Omega)$, there is a sequence $\{\varphi_j\}_j$ of functions in $L^p(\Omega) \cap C^\infty(\Omega)$ such that $\varphi_j \to u$ in $L^p(\Omega)$ as $j \to \infty$. Definition (3.1) implies that

$$u_t^\alpha(x) = \lim_{j\to\infty} (\varphi_j)_t^\alpha(x),$$

when $x \in \Omega$. By the proved case for the smooth functions, we have

(3.5) $$\left|D(\varphi_j)_t^\alpha(x)\right| \le C\big(\mathcal{M}_{\alpha-1,\Omega}\, \varphi_j(x) + \mathcal{S}_{\alpha-1,\Omega}\, \varphi_j(x)\big), \qquad j = 1, 2, \ldots,$$

for almost every $x \in \Omega$. This inequality and the boundedness results (2.1) and (2.3) imply that

$$\|D(\varphi_j)_t^\alpha\|_{L^q(\Omega)} \le C\big(\|\mathcal{M}_{\alpha-1,\Omega}\, \varphi_j\|_{L^q(\Omega)} + \|\mathcal{S}_{\alpha-1,\Omega}\, \varphi_j\|_{L^q(\Omega)}\big)$$

$$\le C\|\varphi_j\|_{L^p(\Omega)}, \qquad j = 1, 2, \ldots,$$

where $q = np/(n - (\alpha - 1)p)$ and C depends only on n, p and α. Thus, $\{|D(\varphi_j)_t^\alpha|\}_j$ is a bounded sequence in $L^q(\Omega)$ and has a weakly converging subsequence $\{|D(\varphi_{j_k})_t^\alpha|\}_k$ in $L^q(\Omega)$. Since $(\varphi_j)_t^\alpha$ converges pointwise to u_t^α, we conclude that the weak gradient Du_t^α exists and that $|D(\varphi_{j_k})_t^\alpha|$ converges weakly to $|Du_t^\alpha|$ in $L^q(\Omega)$ as $k \to \infty$. This follows from the definitions of weak convergence and weak derivatives.

To establish (3.2), we want to proceed to the limit in (3.5) as $j \to \infty$. By the sublinearity of the maximal operator and (2.1), we obtain

$$\| \mathcal{M}_{\alpha-1,\Omega}\, \varphi_j - \mathcal{M}_{\alpha-1,\Omega}\, u\|_{L^q(\Omega)} \leq \| \mathcal{M}_{\alpha-1,\Omega}(\varphi_j - u)\|_{L^q(\Omega)}$$
$$\leq C\|\varphi_j - u\|_{L^p(\Omega)}, \qquad j = 1, 2, \ldots.$$

Analogously, by (2.3), we get

$$\|\mathcal{S}_{\alpha-1,\Omega}\, \varphi_j - \mathcal{S}_{\alpha-1,\Omega}u\|_{L^q(\Omega)} \leq C\|\varphi_j - u\|_{L^p(\Omega)}, \qquad j = 1, 2, \ldots.$$

Hence $\mathcal{M}_{\alpha-1,\Omega}\, \varphi_j + \mathcal{S}_{\alpha-1,\Omega}\, \varphi_j$ converges to $\mathcal{M}_{\alpha-1,\Omega}\, u + \mathcal{S}_{\alpha-1,\Omega}u$ in $L^q(\Omega)$ as $j \to \infty$.

To complete the proof, we need the following simple property of weak convergence: If $f_k \to f$ and $g_k \to g$ weakly in $L^q(\Omega)$ and $f_k \leq g_k$, $k = 1, 2, \ldots,$ almost everywhere in Ω, then $f \leq g$ almost everywhere in Ω. Applying the property to (3.5) with

$$f_k = \left|D(\varphi_{j_k})_t^\alpha\right| \quad \text{and} \quad g_k = C\left(\mathcal{M}_{\alpha-1,\Omega}\, \varphi_{j_k} + \mathcal{S}_{\alpha-1,\Omega}\, \varphi_{j_k}\right),$$

we obtain (3.2). This completes the proof. $\qquad\square$

The weak gradient of the local fractional maximal function of an L^p-function satisfies a pointwise estimate in terms of a local fractional maximal function and local spherical fractional maximal function of the function itself. The following is the main result of this section.

Theorem 3.2. *Let $n \geq 2$, $p > n/(n-1)$ and let $1 \leq \alpha < \min\{(n-1)/p,\ n - 2n/((n-1)p)\} + 1$. If $u \in L^p(\Omega)$, then $|D\, \mathcal{M}_{\alpha,\Omega}\, u| \in L^q(\Omega)$ with $q = np/(n - (\alpha - 1)p)$. Moreover,*

$$(3.6) \qquad |D\, \mathcal{M}_{\alpha,\Omega}\, u(x)| \leq C\left(\mathcal{M}_{\alpha-1,\Omega}\, u(x) + \mathcal{S}_{\alpha-1,\Omega}u(x)\right)$$

for almost every $x \in \Omega$, where the constant C depends only on n.

Proof. Let t_j, $j = 1, 2, \ldots,$ be an enumeration of the rationals between 0 and 1 and let

$$u_j = |u|_{t_j}^\alpha, \qquad j = 1, 2, \ldots.$$

By Lemma 3.1, we see that $|Du_j| \in L^q(\Omega)$ for every $j = 1, 2, \ldots$ and (3.2) gives us the estimate

$$|Du_j(x)| \leq C\left(\mathcal{M}_{\alpha-1,\Omega}\, u(x) + \mathcal{S}_{\alpha-1,\Omega}u(x)\right), \qquad j = 1, 2, \ldots,$$

for almost every $x \in \Omega$. We define $v_k \colon \Omega \to [-\infty, \infty]$ as the pointwise maximum

$$v_k(x) = \max_{1 \leq j \leq k} u_j(x), \qquad k = 1, 2, \ldots.$$

Then $\{v_k\}_k$ is an increasing sequence of functions converging pointwise to $\mathcal{M}_{\alpha,\Omega}\, u$. Moreover, the weak gradients Dv_k, $k = 1, 2, \dots$, exist since Du_j exists for each $j = 1, 2, \dots$, and we can estimate

$$
(3.7) \qquad |Dv_k(x)| = \big|D \max_{1 \le j \le k} u_j(x)\big| \le \max_{1 \le j \le k} |Du_j(x)|
$$
$$
\le C\big(\mathcal{M}_{\alpha-1,\Omega}\, u(x) + \mathcal{S}_{\alpha-1,\Omega} u(x)\big), \qquad k = 1, 2, \dots,
$$

for almost every $x \in \Omega$.

The rest of the proof goes along the lines of the final part of the proof for Lemma 3.1. By (3.7), (2.1) and (2.3), we obtain

$$
\|Dv_k\|_{L^q(\Omega)} \le C\big(\|\mathcal{M}_{\alpha-1,\Omega}\, u\|_{L^q(\Omega)} + \|\mathcal{S}_{\alpha-1,\Omega} u\|_{L^q(\Omega)}\big)
$$
$$
\le C\|u\|_{L^p(\Omega)}, \qquad k = 1, 2, \dots.
$$

Hence $\{|Dv_k|\}_k$ is a bounded sequence in $L^q(\Omega)$ with $v_k \to \mathcal{M}_{\alpha,\Omega}\, u$ pointwise in Ω as $k \to \infty$. Thus, there is a weakly converging subsequence $\{|Dv_{k_j}|\}_j$ that has to converge weakly to $|D\,\mathcal{M}_{\alpha,\Omega}\, u|$ in $L^q(\Omega)$ as $j \to \infty$. We may proceed to the weak limit in (3.7), using the same argument as in the end of the proof of Lemma 3.1, and claim (3.6) follows. $\qquad \square$

Corollary 3.3. *Let $n \ge 2$, $p > n/(n-1)$ and let $1 \le \alpha < n/p$. If $|\Omega| < \infty$ and $u \in L^p(\Omega)$, then $\mathcal{M}_{\alpha,\Omega}u \in W^{1,q}(\Omega)$ with $q = np/(n - (\alpha - 1)p)$.*

Proof. By (2.1) we have $\mathcal{M}_{\alpha,\Omega}u \in L^{p^*}(\Omega)$ and $|D\mathcal{M}_{\alpha,\Omega}u| \in L^q(\Omega)$ by Theorem 3.2 because

$$
\frac{n}{p} \le \min\left\{\frac{n-1}{p},\ n - \frac{2n}{(n-1)p}\right\} + 1.
$$

Since $q < p^*$, we have

$$
\|\mathcal{M}_{\alpha,\Omega}\, u\|_{L^q(\Omega)} \le |\Omega|^{1/q - 1/p^*} \|\mathcal{M}_{\alpha,\Omega}\, u\|_{L^{p^*}(\Omega)} < \infty
$$

by Hölder's inequality. Hence $\mathcal{M}_{\alpha,\Omega}\, u \in W^{1,q}(\Omega)$. $\qquad \square$

Next we will show that the local fractional maximal operator actually maps $L^p(\Omega)$ to the Sobolev space with zero boundary values. For this we need the following Hardy-type result proved in [27, Theorem 3.13].

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$, be an open set. If $u \in W^{1,p}(\Omega)$ and*

$$
\int_\Omega \left(\frac{u(x)}{\mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)}\right)^p dx < \infty,
$$

then $u \in W_0^{1,p}(\Omega)$.

Corollary 3.5. *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be an open set with $|\Omega| < \infty$. Let $p > n/(n-1)$ and $1 \le \alpha < n/p$. If $u \in L^p(\Omega)$, then $\mathcal{M}_{\alpha,\Omega}u \in W_0^{1,q}(\Omega)$ with $q = np/(n - (\alpha - 1)p)$.*

Proof. By Corollary 3.3, $\mathcal{M}_{\alpha,\Omega}\, u \in W^{1,q}(\Omega)$. It suffices to show that

$$
(3.8) \qquad \int_\Omega \left(\frac{\mathcal{M}_{\alpha,\Omega}\, u(x)}{\mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)}\right)^q dx < \infty.
$$

The claim then follows from Theorem 3.4. Since

$$\mathcal{M}_{\alpha,\Omega}\, u(x) \le \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)\, \mathcal{M}_{\alpha-1,\Omega}\, u(x)$$

for every $x \in \Omega$, inequality (3.8) follows from (2.1). Hence $\mathcal{M}_{\alpha,\Omega}\, u \in W_0^{1,q}(\Omega)$.
$\square$

Next we derive estimates for Sobolev functions. In general, Sobolev functions do satisfy neither any better inequality for weak gradients nor better embedding than L^p-functions, but since no spherical maximal function is needed in the Sobolev setting, the estimate holds also when $1 < p \le n/(n-1)$. The following is a variant of Lemma 3.1.

Lemma 3.6. *Let $n \ge 2$, $1 < p < n$, $1 \le \alpha < n/p$ and let $0 < t < 1$. If $|\Omega| < \infty$ and $u \in W^{1,p}(\Omega)$, then $|Du_t^\alpha| \in L^q(\Omega)$ with $q = np/(n - (\alpha-1)p)$. Moreover,*

$$(3.9) \qquad |Du_t^\alpha(x)| \le 2\,\mathcal{M}_{\alpha,\Omega}\, |Du|(x) + \alpha\, \mathcal{M}_{\alpha-1,\Omega}\, u(x)$$

for almost every $x \in \Omega$.

Proof. Suppose first that $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$. Equation (3.3) in the proof of Lemma 3.1 holds in this case, as well, and modifying the integrals into average forms we obtain

$$Du_t^\alpha(x) = \alpha(t\delta(x))^\alpha \frac{D\delta(x)}{\delta(x)} \fint_{B(x,t\delta(x))} u(y)\, dy$$

$$+ n(t\delta(x))^\alpha \frac{D\delta(x)}{\delta(x)} \left(\fint_{\partial B(x,t\delta(x))} u(y)\, d\mathcal{H}^{n-1}(y) - \fint_{B(x,t\delta(x))} u(y)\, dy \right)$$

$$+ (t\delta(x))^\alpha \fint_{B(x,t\delta(x))} Du(y)\, dy$$

for almost every $x \in \Omega$.

In order to estimate the difference of the two integrals in the parenthesis, we use Green's first identity

$$\int_{\partial B(x,r)} u(y) \frac{\partial v}{\partial \nu}(y)\, d\mathcal{H}^{n-1}(y) = \int_{B(x,r)} \big(u(y)\Delta v(y) + Du(y) \cdot Dv(y) \big)\, dy,$$

where $\nu(y) = (y-x)/r$ is the unit outer normal of $B(x,r)$. We choose $r = t\delta(x)$ and $v(y) = |y - x|^2/2$. With these choices

$$Dv(y) = y - x, \quad \frac{\partial v}{\partial \nu}(y) = r, \quad \Delta v(y) = n$$

and Green's formula reads

$$\fint_{\partial B(x,t\delta(x))} u(y)\, d\mathcal{H}^{n-1}(y) - \fint_{B(x,t\delta(x))} u(y)\, dy = \frac{1}{n} \fint_{B(x,t\delta(x))} Du(y) \cdot (y - x)\, dy.$$

Taking the vector norms in the identity of the derivative and recalling that $|D\delta(x)| = 1$ almost everywhere and $0 < t < 1$, we obtain

$$
\begin{aligned}
|Du_t^\alpha(x)| &\leq \alpha(t\delta(x))^\alpha \frac{|D\delta(x)|}{\delta(x)} \fint_{B(x,t\delta(x))} |u(y)|\,dy \\
&\quad + n(t\delta(x))^\alpha \frac{|D\delta(x)|}{\delta(x)} \frac{1}{n} \fint_{B(x,t\delta(x))} |Du(y)||y-x|\,dy \\
&\quad + (t\delta(x))^\alpha \fint_{B(x,t\delta(x))} |Du(y)|\,dy \\
&\leq \alpha(t\delta(x))^{\alpha-1} \fint_{B(x,t\delta(x))} |u(y)|\,dy \\
&\quad + (t\delta(x))^\alpha \fint_{B(x,t\delta(x))} |Du(y)|\,dy \\
&\quad + (t\delta(x))^\alpha \fint_{B(x,t\delta(x))} |Du(y)|\,dy \\
&\leq \alpha\,\mathcal{M}_{\alpha-1,\Omega}\,u(x) + 2\,\mathcal{M}_{\alpha,\Omega}\,|Du|(x)
\end{aligned}
$$

for almost every $x \in \Omega$. Thus, (3.9) holds for smooth functions.

The case $u \in W^{1,p}(\Omega)$ follows from an approximation argument. For $u \in W^{1,p}(\Omega)$, there is a sequence $\{\varphi_j\}_j$ of functions in $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ such that $\varphi_j \to u$ in $W^{1,p}(\Omega)$ as $j \to \infty$. By definition (3.1) we see that

$$
u_t^\alpha(x) = \lim_{j\to\infty} (\varphi_j)_t^\alpha(x),
$$

when $x \in \Omega$. By the proved case for smooth functions we have

$$
(3.10) \quad \left|D(\varphi_j)_t^\alpha(x)\right| \leq 2\,\mathcal{M}_{\alpha,\Omega}\,|D\varphi_j|(x) + \alpha\,\mathcal{M}_{\alpha-1,\Omega}\,\varphi_j(x), \qquad j = 1, 2, \ldots,
$$

for almost every $x \in \Omega$. Let $p^* = np/(n - \alpha p)$ and $q = np/(n - (\alpha - 1)p)$. Then $\|f\|_{L^q(\Omega)} < C\|f\|_{L^{p^*}(\Omega)}$ for any $f \in L^{p^*}(\Omega)$ since $q < p^*$ and $|\Omega| < \infty$. The estimate (3.10) and the boundedness result (2.1) imply

$$
\begin{aligned}
\left\|D(\varphi_j)_t^\alpha\right\|_{L^q(\Omega)} &\leq 2\left\|\mathcal{M}_{\alpha,\Omega}\,|D\varphi_j|\right\|_{L^q(\Omega)} + \alpha\left\|\mathcal{M}_{\alpha-1,\Omega}\,\varphi_j\right\|_{L^q(\Omega)} \\
&\leq C\left\|\mathcal{M}_{\alpha,\Omega}\,|D\varphi_j|\right\|_{L^{p^*}(\Omega)} + \alpha\left\|\mathcal{M}_{\alpha-1,\Omega}\,\varphi_j\right\|_{L^q(\Omega)} \\
&\leq C\|D\varphi_j\|_{L^p(\Omega)} + C\|\varphi_j\|_{L^p(\Omega)} \\
&\leq C\|\varphi_j\|_{W^{1,p}(\Omega)}, \qquad j = 1, 2, \ldots,
\end{aligned}
$$

where C depends on n, p, α and $|\Omega|$. Thus, $\{D(\varphi_j)_t^\alpha\}_j$ is a bounded sequence in $L^q(\Omega)$ and has a weakly converging subsequence $\{(\varphi_{j_k})_t^\alpha\}_k$. Since $(\varphi_j)_t^\alpha$ converges to u_t^α pointwise, we conclude that the Sobolev derivative Du_t^α exists and that $D(\varphi_{j_k})_t^\alpha \to Du_t^\alpha$ weakly in $L^q(\Omega)$ as $k \to \infty$.

To establish (3.9), we want to proceed to the limit in (3.10) as $j \to \infty$. This goes as in the proof of Lemma 3.1, and we obtain the claim. $\qquad\square$

The following is a variant of Theorem 3.2 for Sobolev functions.

Theorem 3.7. *Let $n \geq 2$, $1 < p < n$ and let $1 \leq \alpha < n/p$. If $|\Omega| < \infty$ and $u \in W^{1,p}(\Omega)$, then $\mathcal{M}_{\alpha,\Omega}\, u \in W^{1,q}(\Omega)$ with $q = np/(n - (\alpha - 1)p)$. Moreover,*

$$|D\,\mathcal{M}_{\alpha,\Omega}\, u(x)| \leq 2\,\mathcal{M}_{\alpha,\Omega}\,|Du|(x) + \alpha\,\mathcal{M}_{\alpha-1,\Omega}\, u(x)$$

for almost every $x \in \Omega$.

The proof is analogous to the proofs of Theorem 3.2 and Corollary 3.3, but using Lemma 3.6 instead of Lemma 3.1.

Remark 3.8. If Ω is bounded with a C^1-boundary, then Theorem 3.7 holds with a better exponent $p^* = np/(n - \alpha p)$ instead of q. Indeed, in this setting we have the Sobolev inequality

$$\|u\|_{L^r(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

where $r = np/(n - p)$ is the Sobolev conjugate of p, and we can estimate

$$\begin{aligned}
\|D\,\mathcal{M}_{\alpha,\Omega}\, u\|_{L^{p^*}(\Omega)} &\leq 2\|\,\mathcal{M}_{\alpha,\Omega}\,|Du|\|_{L^{p^*}(\Omega)} + \alpha\|\,\mathcal{M}_{\alpha-1,\Omega}\, u\|_{L^{p^*}(\Omega)} \\
&\leq C\|Du\|_{L^p(\Omega)} + C\|u\|_{L^r(\Omega)} \\
&\leq C\|Du\|_{L^p(\Omega)} + C\|u\|_{W^{1,p}(\Omega)} \\
&\leq C\|u\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

In the second inequality, we used (2.1) and the fact that p^* can be written as $p^* = nr/(n - (\alpha - 1)r)$.

4. EXAMPLES

Our first example shows that the inequality

$$(4.1) \qquad |D\mathcal{M}_{\alpha,\Omega}u(x)| \leq C\mathcal{M}_{\alpha-1,\Omega}u(x),$$

for almost every $x \in \Omega$, cannot hold in general. Hence, the term containing the spherical maximal function in (3.6) cannot be dismissed.

Example 4.1. Let $n \geq 2$ and $\Omega = B(0,1) \subset \mathbb{R}^n$. Let $1 < p < \infty$, $\alpha \geq 1$ and let $0 < \beta < 1$. Then the function u,

$$u(x) = (1 - |x|)^{-\beta/p},$$

belongs to $L^p(\Omega) \cap L^1(\Omega)$. When $0 < |x| < \rho$, ρ small enough, the maximizing radius for the maximal functions $\mathcal{M}_{\alpha,\Omega}u(x)$ and $\mathcal{M}_{\alpha-1,\Omega}u(x)$ is the largest possible, i.e. $1 - |x|$. To see this, it suffices to consider $\mathcal{M}_\Omega\, u$ and averages without the fractional coefficient. Denote $f \colon \{(x,r) : r \geq 0, |x| + r < 1\} \to \mathbb{R}$,

$$f(x,r) = \fint_{\partial B(x,r)} u(y)\, d\mathcal{H}^{n-1}(y) - \fint_{B(x,r)} u(y)\, dy,$$

which is continuous because u is continuous. Since $f(x,r) \to \infty$ as $x \to 0$ and $r \to 1$, there exists $\rho_1 > 0$ such that $f(x,r) > 1$ whenever $|x| < \rho_1$ and $1 - 2\rho_1 < r < 1 - |x|$. Then denote $g \colon B(0, \rho_1) \to \mathbb{R}$,

$$g(x) = \mathcal{M}_\Omega\, u(x) - \max_{0 \leq r \leq 1-2\rho_1} \fint_{B(x,r)} u(y)\, dy,$$

which is continuous because u is continuous. Since $g(0) > 0$, there exists $\rho_2 > 0$ such that $g(x) > 0$ when $|x| < \rho_2$. This implies that

$$\mathcal{M}_\Omega u(x) > \fint_{B(x,r)} u(y)\, dy$$

for every $0 \le r < 1 - |x|$ whenever $|x| < \min\{\rho_1, \rho_2\} = \rho$.

Thus, by (3.4) in the proof of Lemma 3.1,

$$D\mathcal{M}_{\alpha,\Omega} u(x) = (n - \alpha)\frac{x}{|x|}(1 - |x|)^{\alpha-1}\fint_{B(x,1-|x|)} u(y)\, dy$$

$$+ n(1 - |x|)^{\alpha-1}\fint_{\partial B(x,1-|x|)} u(y)\nu(y)\, d\mathcal{H}^{n-1}(y)$$

$$- n\frac{x}{|x|}(1 - |x|)^{\alpha-1}\fint_{\partial B(x,1-|x|)} u(y)\, d\mathcal{H}^{n-1}(y)$$

for almost every x with $|x| < \rho$. By symmetry, the contribution from the integral in the second term has the same direction $\frac{x}{|x|}$ as the first term, whereas the direction of the last term is the opposite. Thus, all the terms lie in the same line of $\mathbb{R}^n$ and it is sufficient to compare the vector norm of the first term and the sum of the latter terms. For the first term,

$$\left|(n - \alpha)\frac{x}{|x|}(1 - |x|)^{\alpha-1}\fint_{B(x,1-|x|)} u(y)\, dy\right| = |n - \alpha|\mathcal{M}_{\alpha-1,\Omega} u(x) \le M,$$

where M depends only on n, p, α, β and ρ. For the latter terms,

$$\left|\fint_{\partial B(x,1-|x|)} u(y)\left(\nu(y) - \frac{x}{|x|}\right) d\mathcal{H}^{n-1}(y)\right| \ge \frac{1}{2}\fint_{S(x)} u(y)\, d\mathcal{H}^{n-1}(y),$$

where $S(x)$ is the half sphere $S(x) = \{y \in \partial B(x, 1 - |x|) : (y - x) \cdot x < 0\}$. Further, when $|x| < \varepsilon$,

$$n(1 - |x|)^{\alpha-1}\frac{1}{2}\fint_{S(x)} u(y)\, d\mathcal{H}^{n-1}(y) \ge \frac{n(1 - \varepsilon)^{\alpha-1}}{2(2\varepsilon)^{\beta/p}},$$

which goes to ∞ as $\varepsilon \to 0$. We conclude that for small values of $|x|$, the boundary integral terms dominate, and thus (4.1) cannot hold.

The next example shows that Theorem 3.7 is sharp. There are domains $\Omega \subset \mathbb{R}^n$, $n \ge 2$, for which $\mathcal{M}_{\alpha,\Omega}(W^{1,p}(\Omega)) \not\subset W^{1,\hat{q}}(\Omega)$ when $\hat{q} > q = np/(n - (\alpha - 1)p)$. This is in strict contrast with the global case, where $\mathcal{M}_\alpha \colon W^{1,p}(\mathbb{R}^n) \hookrightarrow W^{1,p^*}(\mathbb{R}^n)$ with $p^* = np/(n - \alpha p)$, see [28, Theorem 2.1].

Example 4.2. Let $n \ge 2$, $\alpha \ge 1$ and $(\alpha - 1)p < n$. Let

$$\Omega = \operatorname{int}\left(\bigcup_{k=1}^{\infty} B_k \cup C_k\right),$$

where

$$B_k = [k, k + 2^{-k}] \times [0, 2^{-k}]^{n-1} \quad \text{and} \quad C_k = [k + 2^{-k}, k + 1] \times [0, 2^{-3k}]^{n-1}$$

is a corridor connecting B_k to B_{k+1}. We will show that for every $\hat{q} > q = np/(n - (\alpha - 1)p)$, there exists $u \in W^{1,p}(\Omega)$ such that

$$|D\,\mathcal{M}_{\alpha,\Omega}\,u| \notin L^{\hat{q}}(\Omega).$$

Let $\hat{q} > q$ and let $\hat{p} = n\hat{q}/(n + (\alpha - 1)\hat{q})$. Then $\hat{p} > p$. Define u such that $u = 2^{kn/\hat{p}}$ on B_k and u increases linearly from $2^{kn/\hat{p}}$ to $2^{(k+1)n/\hat{p}}$ on C_k. Then it is easy to see that $u \in W^{1,p}(\Omega)$.

If $x \in \frac{1}{2}B_k$, where $\frac{1}{2}B_k$ is a cube with the same center as B_k and with side length half side length of B_k, we have that

$$\mathcal{M}_{\alpha,\Omega}\,u(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus B_k)^\alpha 2^{kn/\hat{p}}.$$

Hence, for almost every $x \in \frac{1}{2}B_k$,

$$|D\,\mathcal{M}_{\alpha,\Omega}\,u(x)| = \alpha \operatorname{dist}(x, \mathbb{R}^n \setminus B_k)^{\alpha-1} 2^{kn/\hat{p}} \geq C2^{k(n/\hat{p}-\alpha+1)} = C2^{kn/\hat{q}},$$

which implies that

$$\int_\Omega |D\,\mathcal{M}_{\alpha,\Omega}\,u(x)|^{\hat{q}}\,dx \geq C \sum_{k=1}^\infty \int_{\frac{1}{2}B_k} 2^{nk}\,dx = \infty.$$

Define the local fractional maximal function over cubes by setting

$$\widetilde{\mathcal{M}}_{\alpha,\Omega}u(x) = \sup_{Q(x,r) \subset \Omega} r^\alpha \fint_{Q(x,r)} |u(y)|\,dy,$$

where $Q(x,r) = (x_1 - r, x_1 + r) \times \cdots \times (x_n - r, x_n + r)$ is a cube with center $x = (x_1, \ldots, x_n)$ and of side length $2r$. As noted in [28], in the global case the maximal operator over cubes behaves similarly as the maximal operator over balls. Somewhat surprisingly, in the local case, the smoothing properties of the maximal operator over cubes are much worse. Indeed, we show that there are domains $\Omega \subset \mathbb{R}^n$ such that $\widetilde{\mathcal{M}}_{\alpha,\Omega}(L^p(\Omega)) \not\subset W^{1,\hat{p}}(\Omega)$ when $\hat{p} > p$.

Example 4.3. Let $\Omega = (0,2) \times (-1,2)^{n-1}$ and let $u \colon \Omega \to \mathbb{R}$ be of the form $u(x) = v(x_1)$, where v is non-negative and continuous. If $Q(x,r) \subset \Omega$, then

$$r^\alpha \fint_{Q(x,r)} |u(y)|\,dy = \frac{1}{2} r^{\alpha-1} \int_{x_1-r}^{x_1+r} v(t)\,dt.$$

Hence, for $\alpha > 1$ and $x \in (0,1)^n$, we have

$$\widetilde{\mathcal{M}}_{\alpha,\Omega}u(x) = \frac{1}{2} x_1^{\alpha-1} \int_0^{2x_1} v(t)\,dt$$

and

$$D_1\widetilde{\mathcal{M}}_{\alpha,\Omega}u(x) = \frac{1}{2}(\alpha-1)x_1^{\alpha-2} \int_0^{2x_1} v(t)dt \;+\; x_1^{\alpha-1}v(2x_1).$$

It follows that

$$D_1\widetilde{\mathcal{M}}_{\alpha,\Omega}u(x) \geq Cv(2x_1),$$

for $x \in (1/2, 1) \times (0,1)^{n-1}$, which shows that $D_1\widetilde{\mathcal{M}}_{\alpha,\Omega}u$ cannot belong to a higher L^p space than u.

In all our results in Section 3, we assumed that $\alpha \geq 1$. Our final example shows that, in the case $0 < \alpha < 1$, $\mathcal{M}_{\alpha,\Omega}\, u$ can be very irregular, even when u is a constant function. Indeed, we show that for any $r > 0$, there exists a domain Ω such that the weak gradient of the fractional maximal function of a constant function does not belong to $L^r(\Omega)$.

Example 4.4. Let $n \geq 1$, $0 < \alpha < 1$ and $r > 0$. We will construct a bounded open set $\Omega \subset \mathbb{R}^n$ such that, for $u \equiv 1$, we have

$$\mathcal{M}_{\alpha,\Omega}\, u = \mathrm{dist}(\cdot, \mathbb{R}^n \setminus \Omega)^\alpha$$

and the weak gradient of $\mathcal{M}_{\alpha,\Omega}\, u$ does not belong to $L^r(\Omega)$. Let β be an integer satisfying $\beta \geq n/((1-\alpha)r)$, and let

$$\Omega = B(0,2) \setminus \overline{\bigcup_{k \geq 1} S_k},$$

where

$$S_k = \{2^{-k} + j2^{-(1+\beta)k} : j = 1, \ldots, 2^{\beta k}\}^n.$$

If $x \in S_k$ and $y \in S_l$ with $x \neq y$, then the balls $B(x, 2^{-(1+\beta)k-1})$ and $B(y, 2^{-(1+\beta)l-1})$ are disjoint. For each $y \in B(x, 2^{-(1+\beta)k-1}) \setminus \{x\}$, we have $\mathcal{M}_{\alpha,\Omega}\, u(y) = |y - x|^\alpha$, which implies that

$$|D\,\mathcal{M}_{\alpha,\Omega}\, u(y)| = \alpha|y - x|^{\alpha-1} \geq C2^{-(1+\beta)(\alpha-1)k}.$$

It follows that

$$\int_\Omega |D\,\mathcal{M}_{\alpha,\Omega}\, u(y)|^r\, dy \geq \sum_{k \geq 1} \sum_{x \in S_k} \int_{B(x, 2^{-(1+\beta)k-1})} |D\,\mathcal{M}_{\alpha,\Omega}\, u(y)|^r\, dy$$

$$\geq C \sum_{k \geq 1} 2^{\beta kn} 2^{-(1+\beta)kn} 2^{-(1+\beta)(\alpha-1)rk}$$

$$= C \sum_{k \geq 1} 2^{((1+\beta)(1-\alpha)r-n)k} = \infty,$$

and hence the weak gradient of $\mathcal{M}_{\alpha,\Omega}\, u$ does not belong to $L^r(\Omega)$.

5. THE LOCAL DISCRETE FRACTIONAL MAXIMAL FUNCTION IN METRIC SPACE

In this section, we study the smoothing properties of the local discrete fractional maximal function in a metric space which is equipped with a doubling measure. We begin by recalling some definitions.

5.1. **Sobolev spaces on metric spaces.** Let $X = (X, d, \mu)$ be a locally compact metric measure space equipped with a metric d and a Borel regular, doubling outer measure μ. The doubling property means that there is a fixed constant $c_d > 0$, called a doubling constant of μ, such that

$$\mu(B(x, 2r)) \leq c_d \mu(B(x, r))$$

for each ball $B(x,r) = \{y \in X : d(y,x) < r\}$. We also assume that nonempty open sets have positive measure and bounded sets have finite measure. We say that the measure μ satisfies a measure lower bound condition if there exist constants $Q \geq 1$ and $c_l > 0$ such that

$$(5.1) \qquad\qquad \mu(B(x,r)) \geq c_l r^Q$$

for all $x \in X$ and $r > 0$. This assumption is needed for the boundedness of the fractional maximal operator in L^p.

General metric spaces lack the notion of smooth functions, but there exists a natural counterpart of Sobolev spaces, defined by Shanmugalingam in [43] and based on upper gradients. A Borel function $g \geq 0$ is an upper gradient of a function u on an open set $\Omega \subset X$, if for all curves γ joining points x and y in Ω,

$$(5.2) \qquad\qquad |u(x) - u(y)| \leq \int_\gamma g \, ds,$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g \, ds = \infty$ otherwise. By a curve, we mean a nonconstant, rectifiable, continuous mapping from a compact interval to X.

If $g \geq 0$ is a measurable function and (5.2) only fails for a curve family with zero p-modulus, then g is a p-weak upper gradient of u on Ω. For the p-modulus on metric measure spaces and the properties of upper gradients, see for example [7], [16], [23], [43], and [44]. If $1 \leq p < \infty$ and $u \in L^p(\Omega)$, let

$$\|u\|_{N^{1,p}(\Omega)} = \left(\int_\Omega |u|^p \, d\mu + \inf_g \int_\Omega g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The Sobolev space on Ω is the quotient space

$$N^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$.

For a measurable set $E \subset X$, the Sobolev space with zero boundary values is

$$N_0^{1,p}(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ in } X \setminus E\}.$$

By [44, Theorem 4.4], also the space $N_0^{1,p}(E)$, equipped with the norm inherited from $N^{1,p}(X)$, is a Banach space. Note that we obtain the same class of functions as above if we require u to vanish p-quasi everywhere in $X \setminus E$ in the sense of p-capacity, since Sobolev functions are defined pointwise outside sets of zero capacity, see [43] and [8].

In Theorems 5.1 and 5.8, we assume, in addition to the doubling condition, that X supports a (weak) $(1,p)$-Poincaré inequality, which means that there exist constants $c_P > 0$ and $\lambda \geq 1$ such that for all balls B, all locally integrable

functions u and for all p-weak upper gradients g_u of u, we have

$$\fint_B |u - u_B|\,d\mu \le c_P r \left(\fint_{\lambda B} g_u^p\,d\mu\right)^{1/p},$$

where

$$u_B = \fint_B u\,d\mu = \mu(B)^{-1}\int_B u\,d\mu$$

is the integral average of u over B.

In the Euclidean space with the Lebesgue measure, $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ for all domains $\Omega \subset \mathbb{R}^n$ and $g_u = |Du|$ is a minimal upper gradient of u, see [43] and [44]. Standard examples of doubling metric spaces supporting Poincaré inequalities include (weighted) Euclidean spaces, compact Riemannian manifolds, metric graphs, and Carnot-Carathéodory spaces. See for instance [17] and [16], and the references therein, for more extensive lists of examples and applications.

The following Hardy-type condition for functions in Sobolev spaces with zero boundary values has been proved in [1] and in [24].

Theorem 5.1. *Assume that X supports a $(1,p)$-Poincaré inequality with $1 < p < \infty$. Let $\Omega \subset X$ be an open set. If $u \in N^{1,p}(\Omega)$ and*

$$\int_\Omega \left(\frac{u(x)}{\operatorname{dist}(x, X \setminus \Omega)}\right)^p d\mu(x) < \infty,$$

then $u \in N_0^{1,p}(\Omega)$.

5.2. **The fractional maximal function.** Let $\Omega \subset X$ be an open set such that $X \setminus \Omega \ne \emptyset$ and let $\alpha \ge 0$. The local fractional maximal function of a locally integrable function u is

$$\mathcal{M}_{\alpha,\Omega}\, u(x) = \sup r^\alpha \fint_{B(x,r)} |u|\,d\mu,$$

where the supremum is taken over all radii r satisfying $0 < r < \operatorname{dist}(x, X \setminus \Omega)$. If $\alpha = 0$, we have the local Hardy-Littlewood maximal function

$$\mathcal{M}_\Omega\, u(x) = \sup \fint_{B(x,r)} |u|\,d\mu.$$

When $\Omega = X$, the supremum is taken over all $r > 0$ and we obtain the fractional maximal function $\mathcal{M}_\alpha u$ and the Hardy-Littlewood maximal function $\mathcal{M} u$.

Sobolev type theorem for the fractional maximal operator follows easily from the Hardy-Littlewood maximal function theorem. For the proof, see [12], [13] or [20].

Theorem 5.2. *Assume that measure lower bound condition* (5.1) *holds. If $p > 1$ and $0 < \alpha < Q/p$, then there is a constant $C > 0$, independent of u, such that*

$$\|\mathcal{M}_\alpha u\|_{L^{p^*}(X)} \le C\|u\|_{L^p(X)},$$

for every $u \in L^p(X)$ with $p^* = Qp/(Q - \alpha p)$. If $p = 1$ and $0 < \alpha < Q$, then

$$\mu(\{\mathcal{M}_\alpha u > \lambda\}) \leq C\big(\lambda^{-1}\|u\|_{L^1(X)}\big)^{Q/(Q-\alpha)}$$

for every $u \in L^1(X)$. The constant $C > 0$ depends only on the doubling constant, the constant in the measure lower bound and α.

Now the corresponding boundedness results for the local fractional maximal function follow easily because for each open set $\Omega \subset X$ and for each $u \in L^p(\Omega)$, $p > 1$, we have

$$(5.3) \qquad \|\mathcal{M}_{\alpha,\Omega}\, u\|_{L^{p^*}(\Omega)} \leq \|\mathcal{M}_\alpha(u\chi_\Omega)\|_{L^{p^*}(X)} \leq C\|u\chi_\Omega\|_{L^p(X)} = C\|u\|_{L^p(\Omega)}.$$

Similarly, we obtain a weak type estimate when $p = 1$,

$$(5.4) \qquad \mu(\{x \in \Omega : \mathcal{M}_{\alpha,\Omega}\, u(x) > \lambda\}) \leq C\big(\lambda^{-1}\|u\|_{L^1(\Omega)}\big)^{Q/(Q-\alpha)}.$$

The weak type estimate implies that the fractional maximal operator maps L^1 locally to L^s whenever $1 < s < Q/(Q - \alpha)$.

Corollary 5.3. *Assume that measure lower bound condition (5.1) holds. Let $0 < \alpha < Q$ and $1 \leq s < Q/(Q - \alpha)$. If $\Omega \subset X$, $\mu(\Omega) < \infty$ and $u \in L^1(\Omega)$, then $\mathcal{M}_{\alpha,\Omega}\, u \in L^s(\Omega)$ and*

$$(5.5) \qquad \|\mathcal{M}_{\alpha,\Omega}\, u\|_{L^s(\Omega)} \leq C\|u\|_{L^1(\Omega)},$$

where the constant C depends on the doubling constant, the constant in the measure lower bound, s, α and $\mu(\Omega)$.

Proof. Let $a > 0$. Now

$$\int_\Omega (\mathcal{M}_{\alpha,\Omega}\, u)^s\, d\mu = s \int_0^\infty t^{s-1} \mu(\{x \in \Omega : \mathcal{M}_{\alpha,\Omega}\, u(x) > t\})\, dt$$

$$= s\left(\int_0^a + \int_a^\infty\right),$$

where

$$\int_0^a t^{s-1} \mu(\{x \in \Omega : \mathcal{M}_{\alpha,\Omega}\, u(x) > t\})\, dt \leq a^s \mu(\Omega).$$

For the second term, (5.4) together with the assumption $1 \leq s < Q/(Q - \alpha)$ implies that

$$\int_a^\infty t^{s-1} \mu(\{x \in \Omega : \mathcal{M}_{\alpha,\Omega}\, u(x) > t\})\, dt \leq C\|u\|_{L^1(\Omega)}^{Q/(Q-\alpha)} \int_a^\infty t^{s-1-Q/(Q-\alpha)}\, dt$$

$$= C\|u\|_{L^1(\Omega)}^{Q/(Q-\alpha)} a^{s-Q/(Q-\alpha)}.$$

Now norm estimate (5.5) follows by choosing $a = \|u\|_{L^1(\Omega)}$. $\qquad \square$

5.3. **The discrete fractional maximal function.** We begin the construction of the local discrete fractional maximal function in the metric setting with a Whitney covering as in [2, Lemma 4.1], see also the classical references [11] and [36]. Let $\Omega \subset X$ be an open set such that $X \setminus \Omega \neq \emptyset$, let $0 \leq \alpha \leq Q$ and let $0 < t < 1$ be a scaling parameter. There exist balls $B_i = B(x_i, r_i)$, $i = 1, 2, \ldots$, with $r_i = \frac{1}{18} t \operatorname{dist}(x_i, X \setminus \Omega)$, for which

$$\Omega = \bigcup_{i=1}^{\infty} B_i \quad \text{and} \quad \sum_{i=1}^{\infty} \chi_{6B_i}(x) \leq N < \infty$$

for all $x \in \Omega$. The constant N depends only on the doubling constant. Moreover, for all $x \in 6B_i$,

$$(5.6) \qquad\qquad 12r_i \leq t \operatorname{dist}(x, X \setminus \Omega) \leq 24r_i.$$

Using the definition of r_i, it is easy to show that if $x \in B_i$ and $B_i \cap 6B_j \neq \emptyset$, then

$$(5.7) \qquad\qquad r_i \leq \frac{24}{17} r_j \leq \frac{3}{2} r_j \quad \text{and} \quad r_j \leq \frac{19}{12} r_i \leq \frac{5}{3} r_i.$$

Related to the Whitney covering $\{B_i\}_i$, there is a sequence of Lipschitz functions $\{\varphi_i\}_i$, called partition of unity, for which

$$\sum_{i=1}^{\infty} \varphi_i(x) = 1$$

for all $x \in \Omega$. Moreover, for each i, the functions φ_i satisfy the following properties: $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ in $X \setminus 6B_i$, $\varphi_i \geq \nu$ in $3B_i$, φ_i is Lipschitz with constant L/r_i where $\nu > 0$ and $L > 0$ depend only on the doubling constant.

Now the discrete fractional convolution of a locally integrable function u at the scale t is u_t^α,

$$u_t^\alpha(x) = \sum_{i=1}^{\infty} \varphi_i(x) r_i^\alpha u_{3B_i}, \quad x \in X.$$

Let t_j, $j = 1, 2, \ldots$ be an enumeration of the positive rationals of the interval $(0, 1)$. For every scale t_j, choose a covering of Ω and a partition of unity as above. The local discrete fractional maximal function of u in Ω is $\mathcal{M}_{\alpha,\Omega}^* u$,

$$\mathcal{M}_{\alpha,\Omega}^* u(x) = \sup_j |u|_{t_j}^\alpha(x), \quad x \in X.$$

For $\alpha = 0$, we obtain the local discrete maximal function studied in [2]. The construction depends on the choice of the coverings, but the estimates below are independent of them.

The local discrete fractional maximal function is comparable to the standard local fractional maximal function. The proof of the following lemma is similar as for local discrete maximal function and local Hardy-Littlewood maximal function in [2, Lemma 4.2].

Lemma 5.4. *There is a constant $C \geq 1$, depending only on the doubling constant of μ, such that*

$$C^{-1} \mathcal{M}^{24}_{\alpha,\Omega} u(x) \leq \mathcal{M}^*_{\alpha,\Omega} u(x) \leq C \mathcal{M}_{\alpha,\Omega} u(x)$$

for every $x \in X$ and for each locally integrable function u.

Above, 24 is the constant from (5.6) and

$$\mathcal{M}^{\beta}_{\alpha,\Omega} u(x) = \sup r^{\alpha} \fint_{B(x,r)} |u| \, d\mu,$$

where the supremum is taken over all radii r for which $0 < \beta r < \operatorname{dist}(x, X \setminus \Omega)$, is the restricted local fractional maximal function.

Since the discrete and the standard fractional maximal functions are comparable, the integrability estimates hold for the local discrete fractional maximal function as well, see Theorem 5.2 and (5.3).

5.4. **Sobolev boundary values.** In the metric setting, smoothing properties of the discrete fractional maximal operator in the global case have been studied in [20] and of the standard fractional maximal operator $\mathcal{M}_{\alpha}$ in [21]. In the local case, by [2, Theorem 5.6], the local discrete maximal operator preserves the boundary values in the Newtonian sense, that is, $|u| - \mathcal{M}^*_{\Omega} u \in N^{1,p}_0(\Omega)$ whenever $u \in N^{1,p}(\Omega)$. Intuitively, the definition of the fractional maximal function says that it has to be small near the boundary. In Theorem 5.8, we will show that if Ω has finite measure, then the local discrete fractional maximal operator maps $L^p(\Omega)$-functions to Sobolev functions with zero boundary values.

The next theorem, a local version of [20, Theorem 6.1], shows that the local discrete fractional maximal function of an L^p-function has a weak upper gradient and both $\mathcal{M}^*_{\alpha,\Omega} u$ and the weak upper gradient belong to a higher Lebesgue space than u.

We use the following simple fact in the proof: Assume that u_i, $i = 1, 2, \ldots$, are functions and g_i, $i = 1, 2, \ldots$, are p-weak upper gradients of u_i, respectively. Let $u = \sup_i u_i$ and $g = \sup_i g_i$. If u is finite almost everywhere, then g is a p-weak upper gradient of u. For the proof, we refer to [7].

Theorem 5.5. *Assume that measure lower bound condition* (5.1) *holds. Let $\Omega \subset X$ be an open set and let $u \in L^p(\Omega)$ with $1 < p < Q$. Let $1 \leq \alpha < Q/p$, $p^* = Qp/(Q - \alpha p)$ and $q = Qp/(Q - (\alpha - 1)p)$. Then $C\mathcal{M}_{\alpha-1,\Omega} u$ is a weak upper gradient of $\mathcal{M}^*_{\alpha,\Omega} u$. Moreover,*

$$\| \mathcal{M}^*_{\alpha,\Omega} u \|_{L^{p^*}(\Omega)} \leq C \|u\|_{L^p(\Omega)} \quad and \quad \| \mathcal{M}_{\alpha-1,\Omega} u \|_{L^q(\Omega)} \leq C \|u\|_{L^p(\Omega)}.$$

The constants $C > 0$ depend only on the doubling constant, the constant in the measure lower bound, p and α.

Proof. We begin by showing that $C\,\mathcal{M}_{\alpha-1,\Omega}\,u$ is a weak upper gradient of $|u|_t^\alpha$. Let $t \in (0,1) \cap \mathbb{Q}$ be a scale and let $\{B_i\}_i$ be a Whitney covering of Ω. Since

$$|u|_t^\alpha(x) = \sum_{j=1}^\infty \varphi_j(x) r_j^\alpha |u|_{3B_j},$$

each φ_j is L/r_j-Lipschitz continuous and has a support in $6B_j$, the function

$$g_t(x) = L \sum_{j=1}^\infty r_j^{\alpha-1} |u|_{3B_j} \chi_{6B_j}(x)$$

is a weak upper gradient of $|u|_t^\alpha$. We want to find an upper bound for g_t. Let $x \in \Omega$ and let i be such that $x \in B_i$. Then, by (5.7), $3B_j \subset B(x, 4r_i) \subset 15B_j$ whenever $B_i \cap 6B_j \neq \emptyset$ and hence

$$|u|_{3B_j} \leq C \fint_{B(x,4r_i)} |u|\,d\mu.$$

The bounded overlap property of the balls $6B_j$ together with estimate (5.7) implies that

$$g_t(x) \leq C r_i^{\alpha-1} \fint_{B(x,4r_i)} |u|\,d\mu \leq C\,\mathcal{M}_{\alpha-1,\Omega}\,u(x).$$

Consequently, $C\,\mathcal{M}_{\alpha-1,\Omega}\,u$ is a weak upper gradient of $|u|_t^\alpha$.

By (5.3), the function $\mathcal{M}_{\alpha,\Omega}^*\,u$ belongs to $L^{p^*}(\Omega)$ and hence it is finite almost everywhere. As

$$\mathcal{M}_{\alpha,\Omega}^*\,u(x) = \sup_j |u|_{t_j}^\alpha(x),$$

and because $C\,\mathcal{M}_{\alpha-1,\Omega}\,u$ is an upper gradient of $|u|_{t_j}^\alpha$ for every $t = 1, 2, \ldots,$ we conclude that it is an upper gradient of $\mathcal{M}_{\alpha,\Omega}^*\,u$ as well. The norm bounds follow from Lemma 5.4 and (5.3). $\qquad\square$

Remark 5.6. With the assumptions of Theorem 5.5, $\mathcal{M}_{\alpha,\Omega}^*\,u \in N_{\mathrm{loc}}^{1,q}(\Omega)$ and

$$\|\mathcal{M}_{\alpha,\Omega}^*\,u\|_{N^{1,q}(A)} \leq C\mu(A)^{1/q - 1/p^*} \|u\|_{L^p(A)}$$

for all open sets $A \subset \Omega$ with $\mu(A) < \infty$.

Remark 5.7. Similar arguments as in the proof of Theorem 5.5 together with Corollary 5.3 show that if the measure lower bound condition holds, $\Omega \subset X$ is an open set, $u \in L^1(\Omega)$, $\mu(\Omega) < \infty$, and $1 \leq s' \leq s < Q/(Q - (\alpha - 1))$, then $C\,\mathcal{M}_{\alpha-1,\Omega}\,u$ is a weak upper gradient of $\mathcal{M}_{\alpha,\Omega}^*\,u$ and

$$\|\mathcal{M}_{\alpha,\Omega}^*\,u\|_{L^s(\Omega)} \leq C\|u\|_{L^1(\Omega)} \quad \text{and} \quad \|\mathcal{M}_{\alpha-1,\Omega}\,u\|_{L^{s'}(\Omega)} \leq C\|u\|_{L^1(\Omega)}.$$

In particular, we have that $\mathcal{M}_{\alpha,\Omega}^*\,u \in N^{1,s'}(\Omega)$ and

$$\|\mathcal{M}_{\alpha,\Omega}^*\,u\|_{N^{1,s'}(\Omega)} \leq C\mu(\Omega)^{1/s' - 1/s} \|u\|_{L^1(\Omega)}.$$

The next result shows that the local discrete fractional maximal operator actually maps $L^p(\Omega)$ to the Sobolev space with zero boundary values.

Theorem 5.8. *Assume that measure lower bound condition (5.1) holds and that X supports a $(1,p)$-Poincaré inequality with $1 < p < Q$. Let $\Omega \subset X$ be an open set with $\mu(\Omega) < \infty$ and let $u \in L^p(\Omega)$. Let $1 \le \alpha < Q/p$ and $q = Qp/(Q - (\alpha - 1)p)$. Then $\mathcal{M}^*_{\alpha,\Omega} u \in N^{1,q}_0(\Omega)$.*

Proof. Let $u \in L^p(\Omega)$. By Remark 5.6, $\mathcal{M}^*_{\alpha,\Omega} u \in N^{1,q}(\Omega)$ and hence, by Theorem 5.1, it suffices to show that

$$(5.8) \qquad \int_\Omega \left(\frac{\mathcal{M}^*_{\alpha,\Omega} u(x)}{\operatorname{dist}(x, X \setminus \Omega)} \right)^q d\mu(x) < \infty.$$

We begin by considering $|u|^\alpha_t$. Let $t \in (0,1) \cap \mathbb{Q}$ be a scale and let $\{B_i\}_i$ be a Whitney covering of Ω. Let $x \in \Omega$ and let i be such that $x \in B_i$. Now

$$|u|^\alpha_t(x) = \sum_j \varphi_j(x) r^\alpha_j |u|_{3B_j},$$

where the sum is over such indices j for which $B_i \cap 6B_j \ne \emptyset$. As in the proof of Theorem 5.5, we use (5.7), the doubling property, the bounded overlap of the balls B_j and (5.6) to obtain that

$$|u|_{3B_j} \le C \fint_{B(x,4r_i)} |u| \, d\mu$$

for all such j, and that

$$|u|^\alpha_t(x) \le C r^\alpha_i \fint_{B(x,4r_i)} |u| \, d\mu \le C \operatorname{dist}(x, X \setminus \Omega) \mathcal{M}_{\alpha-1,\Omega} u(x).$$

By taking the supremum on the left side we have

$$\mathcal{M}^*_{\alpha,\Omega} u(x) \le C \operatorname{dist}(x, X \setminus \Omega) \mathcal{M}_{\alpha-1,\Omega} u(x).$$

This together with (5.3) implies that

$$\int_\Omega \left(\frac{\mathcal{M}^*_{\alpha,\Omega} u(x)}{\operatorname{dist}(x, X \setminus \Omega)} \right)^q d\mu(x) \le C \int_\Omega \left(\mathcal{M}_{\alpha-1,\Omega} u \right)^q d\mu \le C \|u\|^q_{L^p(\Omega)}.$$

Hence (5.8) holds and the claim follows. $\qquad\square$

Remark 5.9. The same proof using Remark 5.7 and norm estimate (5.5) gives a corresponding result for $p = 1$. Namely, if $u \in L^1(\Omega)$, $\mu(\Omega) < \infty$, $1 < \alpha < Q$, and $1 < s' < Q/(Q - (\alpha - 1))$, then $\mathcal{M}^*_{\alpha,\Omega} u \in N^{1,s'}_0(\Omega)$.

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