STABILITY FOR PARABOLIC QUASIMINIMIZERS

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Abstract. This paper studies parabolic quasiminimizers which are solutions to parabolic variational inequalities. We show that, under a suitable regularity condition on the boundary, parabolic Q-quasiminimizers related to the parabolic p-Laplace equations with given boundary values are stable with respect to parameters Q and p. The argument is based on variational techniques, higher integrability results and regularity estimates in time. This shows that stability does not only hold for parabolic partial differential equations but it also holds for variational inequalities.

1. Introduction

This paper investigates a stability question for parabolic quasiminimizers related to the parabolic p-Laplace equations

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad \frac{2N}{N+2} < p < \infty,$$

on a space time cylinder $\Omega_T = \Omega \times (0,T)$ in $\mathbb{R}^{N+1}$. A function u is called a parabolic Q-quasiminimizer, if for some $Q \geq 1$ we have

$$-\int_{\Omega_T} u\partial_t\phi\,dx\,dt + \frac{1}{p}\int_{\operatorname{supp}\phi}|\nabla u|^p\,dx\,dt \leq \frac{Q}{p}\int_{\operatorname{supp}\phi}|\nabla(u-\phi)|^p\,dx\,dt,$$

for all $\phi \in C_o^\infty(\Omega_T)$. Here, the data on the parabolic boundary $\partial_{\mathrm{par}}\Omega_T = \Omega \times \{0\} \cup \partial\Omega \times (0,T)$ are taken in an appropriate sense. Parabolic quasiminimizers with $Q = 1$ are called minimizers and in that case there is a one to one correspondence between minimizers and solutions: every minimizer is a solution of the corresponding partial differential equation. However, when $Q > 1$ being a quasiminimizer is not only a local property (see [8]) and, consequently, there is no connection to the partial differential equation and only variational methods are available. Parabolic quasiminimizers were introduced in [22], and later they have been studied, for example, in [2, 5, 7, 15, 16, 17, 18, 25, 26].

We consider stability of parabolic quasiminimizers with respect to parameters p and Q. More precisely, assume that we have sequences $Q_i \to Q$ and $p_i \to p$ as $i \to \infty$ and let u_i be a parabolic Q_i-quasiminimizer of the

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variational inequality with exponent p_i with the same initial and boundary conditions. It is well known that the quasiminimizers with $Q_i > 1$ are not unique. To ensure the existence of a limit function, we assume that the sequence parabolic quasiminimizers converges pointwise. This extra condition is redundant in the case that the sequence converges to a minimizer, since minimizers are known to be unique.

According to our main results (Theorems 2.1 and 2.2) the limit function u is a parabolic Q-quasiminimizer with the same boundary and initial values as all terms of the sequence. Furthermore, if $Q = 1$, then the limit function is a minimizer and u_i converges to u in the parabolic Sobolev space. Stability of elliptic quasiminimizers has been studied in [6, 7] and [14]. For elliptic equations, see [11, 12, 13, 23, 24]. We investigate the stability question from a purely variational point of view, by using the assumption that the complement of Ω satisfies a uniform capacity density condition. This paper extends the results of [9] for parabolic quasiminimizers and the argument is based on local and global higher integrability results for the gradients of parabolic quasiminimizers, see [19] and [2]. In particular, the paper [9] only covers the degenerate case $p \geq 2$ and hence our results are new even for the parabolic p-Laplace equation in the singular case $2N/(N+2) < p < 2$. New arguments are needed to compensate the lack of the partial differential equation and the results show that the class of parabolic quasiminimizers is stable under parturbations of the parameters. A careful analysis of the regularity in time plays a decisive role in the argument.

2. Notation and basic definitions

2.1. **Notation.** Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, and $T > 0$. The usual first order Sobolev space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The Sobolev space with zero boundary values $W_o^{1,p}(\Omega)$ is defined as a completion of $C_o^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. The parabolic Sobolev spaces $L^p(0,T;W^{1,p}(\Omega))$ and $L^p(0,T;W_o^{1,p}(\Omega))$ consist of all measurable functions $u : \Omega \times (0,T) \to \mathbb{R}$ such that $u(\cdot,t) \in W^{1,p}(\Omega)$ and $u(\cdot,t) \in W_o^{1,p}(\Omega)$ for almost every $t \in (0,T)$, respectively, and

$$\|u\|_{L^p(0,T;W^{1,p}(\Omega))} := \left(\int_0^T \|u(\cdot,t)\|^p_{W^{1,p}(\Omega)} dt \right)^{1/p} < \infty.$$

Functions in the parabolic space $L^p(0,T;W^{1,p}(\Omega))$, for which there exist t_1, $t_2 \in (0,T)$ with $t_1 < t_2$ such that $u(x,t) = 0$ for almost every $x \in \Omega$ when $t \notin [t_1,t_2]$, are denoted by $L^p_c(0,T;W^{1,p}(\Omega))$. Moreover, we define the space $\mathrm{Lip}_c(0,T;W_o^{1,p}(\Omega))$ as the space of all functions $u \in L^p_c(0,T;W^{1,p}(\Omega))$ for which

$$\|u(\cdot,t)\|_{W^{1,p}(\Omega)} \in \mathrm{Lip}(0,T).$$

For $x \in \mathbb{R}^N$ and $r > 0$, we define the p-capacity of a closed set $E \subset B_r(x)$ with respect to $B_r(x)$ by

$$\begin{aligned}&\operatorname{cap}_p(E, B_r(x))\\&\quad := \inf\left\{\int_{B_r(x)} |\nabla u(y)|^p dy : u \in C_o^\infty(B_r(x)) \text{ with } u \geq 1 \text{ in } E\right\}.\end{aligned}$$

Here $B_r(x)$ denotes the open ball with the radius $r > 0$ and the center at x. The set $\mathbb{R}^N \setminus \Omega$ is called *uniformly p-thick* if there exist positive constants μ and r_0 such that

$$\operatorname{cap}_p((\mathbb{R}^N \setminus \Omega) \cap \overline{B_r(x)}, B_{2r}(x)) \geq \mu \operatorname{cap}_p(\overline{B_r(x)}, B_{2r}(x)) \tag{2.1}$$

for all $x \in \mathbb{R}^N \setminus \Omega$ and $r \in (0, r_0)$.

2.2. **Parabolic quasiminimizers.** Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, $T > 0$, and $Q \geq 1$. A function $u \in L^p(0,T;W^{1,p}(\Omega))$ is called a (local) *parabolic Q-quasiminimizer* if it satisfies

$$-\int_{\Omega_T} u(\partial_t \phi)\, dx\, dt + \frac{1}{p}\int_{\operatorname{supp}\phi} |\nabla u|^p\, dx\, dt \leq \frac{Q}{p}\int_{\operatorname{supp}\phi} |\nabla(u-\phi)|^p\, dx\, dt, \tag{2.2}$$

for all test functions $\phi \in C_o^\infty(\Omega_T)$, where Ω_T denotes the parabolic cylinder $\Omega \times (0,T)$ and supp denotes the support of the function. In addition, for $g \in L^p(0,T;W^{1,p}(\Omega))$, we say that $u \in L^p(0,T;W^{1,p}(\Omega))$ is a (global) *parabolic Q-quasiminimizer with initial and boundary value g* if u satisfies (2.2),

$$u(\cdot,t) - g(\cdot,t) \in W_o^{1,p}(\Omega) \quad \text{for almost every } t \in (0,T), \tag{2.3}$$

and

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \int_\Omega |u-g|^2\, dx\, dt = 0. \tag{2.4}$$

Now we are ready to state our main results.

Theorem 2.1. *Let $T > 0$ and $N \geq 2$. For $p > 2N/(N+2)$, let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $\mathbb{R}^N \setminus \Omega$ is a uniformly p-thick. Let $p_i > 2N/(N+2)$ and $Q_i \geq 1$, $i = 1, 2, \ldots,$ be real numbers such that*

$$\lim_{i\to\infty} p_i = p \quad \textit{and} \quad \lim_{i\to\infty} Q_i = Q. \tag{2.5}$$

For $i \in \mathbb{N}$, let $u_i \in L^{p_i}(0,T;W^{1,p_i}(\Omega))$ be a parabolic Q_i-quasiminimizer with initial and boundary value $g \in C^1(\overline{\Omega_T})$ and suppose that there exists a measurable function u such that

$$\lim_{i\to\infty} u_i(x,t) = u(x,t) \quad \textit{for almost every } (x,t) \in \Omega_T. \tag{2.6}$$

Then $u \in L^p(0,T;W^{1,p}(\Omega))$ *and* u *satisfies*

$$u(\cdot,t) - g(\cdot,t) \in W_o^{1,p}(\Omega) \quad \text{for almost every } t \in (0,T), \tag{2.7}$$

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \int_\Omega |u-g|^2 \, dx\, dt = 0, \tag{2.8}$$

and moreover

$$-\int_{\Omega_T} u(\partial_t \phi)\, dx\, dt + \frac{1}{p}\int_{\operatorname{supp}\phi} |\nabla u|^p \, dx\, dt \le \frac{Q}{p}\int_{\operatorname{supp}\phi} |\nabla(u-\phi)|^p \, dx\, dt \tag{2.9}$$

for all $\phi \in C_o^\infty(\Omega_T)$.

Observe that since quasiminimizers are not unique, we have to assume some kind of convergence as in (2.6). Otherwise the limit function does not need to exist. If u_i converge to the minimizer as $i \to \infty$, then we obtain the strong convergence of u_i.

Theorem 2.2. *Assume the same conditions as in Theorem* 2.1*. If* $Q = 1$*, then*

$$u_i \to u \quad \text{in } L^p(0,T;W^{1,p}(\Omega)) \tag{2.10}$$

as $i \to \infty$.

3. Preliminary results

In this section we give some preliminary results on properties of quasiminimizers. In particular, we study regularity for parabolic quasiminimizers in time variable, the Sobolev space with zero boundary values, a Hardy type estimate for $u \in W_o^{1,p}(\Omega)$, and global higher integrability for the gradients of parabolic quasiminimizers.

3.1. **Regularity for parabolic quasiminimizers.**

We denote by $\langle \cdot,\cdot \rangle$ the paring between $(L^p(0,T;W_o^{1,p}(\Omega)))^*$ and $L^p(0,T;W_o^{1,p}(\Omega))$. We say that the function $v \in (L^p(0,T;W_o^{1,p}(\Omega)))^*$ is the weak derivative of the function $u \in L^p(0,T;W^{1,p}(\Omega))$ if

$$\langle v, \varphi \rangle = -\int_{\Omega_T} u(\partial_t \varphi)\, dx\, dt,$$

for all $\varphi \in C_o^\infty(\Omega_T)$, and v is denoted by $\partial_t u$. We first prove the following lemma, which improves regularity for parabolic quasiminimizers in the time variable. Similar phenomenon has been observed already in [22].

Lemma 3.1. *Let* $N \ge 1$*,* $\Omega \subset \mathbb{R}^N$ *be a bounded open set,* $p > 1$*,* $T > 0$*, and* $Q \ge 1$*. Let* $u \in L^p(0,T;W^{1,p}(\Omega))$ *be a parabolic* Q*-quasiminimizer. Then*

$$\partial_t u \in (L^p(0,T;W_o^{1,p}(\Omega)))^*$$

and

$$|\langle \partial_t u, \varphi\rangle| \le \frac{2^p Q}{p}\|u\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))}\|\nabla\varphi\|_{L^p(\Omega_T)},$$

for all $\varphi \in L^p(0,T;W_o^{1,p}(\Omega))$.

Proof. Since the desired inequality holds if $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} = 0$, we can assume without loss of generality that $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \neq 0$. Let $\varphi \in C_o^\infty(\Omega_T)$ be a test function satisfying $\|\nabla\varphi\|_{L^p(\Omega_T)} = 1$, and set $\phi = \|u\|_{L^p(0,T;W^{1,p}(\Omega))}\varphi$. Then, by (2.2) we have

$$
\begin{aligned}
&\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \int_0^T \int_\Omega u\, \partial_t \varphi \, dx\, dt \\
&\geq -\frac{Q}{p} \int_{\Omega_T} |\nabla(u - \|u\|_{L^p(0,T;W^{1,p}(\Omega))}\varphi)|^p \, dx\, dt \\
&\geq -\frac{2^{p-1}Q}{p} \int_{\Omega_T} (|\nabla u|^p + \|u\|^p_{L^p(0,T;W^{1,p}(\Omega))} |\nabla \varphi|^p) \, dx\, dt \\
&\geq -\frac{2^p Q}{p} \|u\|^p_{L^p(0,T;W^{1,p}(\Omega))}.
\end{aligned}
\tag{3.1}
$$

On the other hand, replacing ϕ by $-\phi$ in (2.2), we obtain

$$
\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \int_0^T \int_\Omega u \partial_t \varphi \, dx\, dt \leq \frac{2^p Q}{p} \|u\|^p_{L^p(0,T;W^{1,p}(\Omega))}. \tag{3.2}
$$

Since $\|\nabla\varphi\|_{L^p(\Omega_T)} = 1$, by (3.1) and (3.2) we have

$$
\left| \int_0^T \int_\Omega u\, \partial_t \varphi \, dx\, dt \right| \leq \frac{2^p Q}{p} \|u\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))} \|\nabla\varphi\|_{L^p(\Omega_T)}, \tag{3.3}
$$

for all $\varphi \in C_o^\infty(\Omega_T)$. Since $C_o^\infty(\Omega_T)$ is dense in $L^p(0,T;W_o^{1,p}(\Omega))$, by (3.3) we obtain $\partial_t u \in (L^p(0,T;W_o^{1,p}(\Omega)))^*$ and the desired inequality. This completes the proof. □

Next we consider a regularization of quasiminimizers with respect to the time variable. Let $\phi \in C_o^\infty(\Omega_T)$ with $\operatorname{supp}\phi = \widetilde{\Omega} \times [t_1, t_2] \Subset \Omega_T$. For $0 < \varepsilon < \min\{t_1, T - t_2\}$, we take a mollifier $\xi_\varepsilon \in C_o^\infty(\mathbb{R})$ satisfying $\operatorname{supp}\xi_\varepsilon \subset [-\varepsilon, \varepsilon]$, $\xi_\varepsilon \geq 0$, and $\|\xi_\varepsilon\|_{L^p(\mathbb{R})} = 1$ and denote

$$
[\phi]_\varepsilon(x,t) := \int_{-\varepsilon}^{\varepsilon} \phi(x, t-s)\xi_\varepsilon(s) ds \quad \text{in } \Omega_T. \tag{3.4}
$$

Since $\xi_\varepsilon(s)$ is an even function, by (2.2), after a change of variables and an integration by parts in the time derivative term, we obtain

$$
\begin{aligned}
&\int_{\Omega_T} (\partial_t [u]_\varepsilon)\phi \, dx\, dt + \frac{1}{p} \int_{\operatorname{supp}[\phi]_\varepsilon} |\nabla u|^p \, dx\, dt \\
&\qquad\qquad \leq \frac{Q}{p} \int_{\operatorname{supp}[\phi]_\varepsilon} |\nabla(u - [\phi]_\varepsilon)|^p \, dx\, dt,
\end{aligned}
\tag{3.5}
$$

for all $\phi \in C_o^\infty(\Omega_T)$. In the above inequality $[u]_\varepsilon$ denotes the smoothing of u according to (3.4). By Lemma 2, Corollary 1 and Remark 2 in [16] we know that, for a function $\psi \in L_c^p(0,T;W_o^{1,p}(\Omega))$ and for any $\varepsilon > 0$, there exists a function $\varphi \in \mathrm{Lip}_c(\Omega_T)$ and hence also a function $\varphi \in C_o^\infty(\Omega_T)$ such that

$$\|\psi - \varphi\|_{L^p(0,T;W^{1,p}(\Omega))} < \varepsilon, \quad \|\psi - \varphi\|_{L^2(\Omega_T)} < \varepsilon,$$
$$\text{and} \quad |\operatorname{supp}\varphi \setminus \operatorname{supp}\psi| < \varepsilon.$$

Using this density result, it is straightforward to show that inequality (3.5) also holds for any $\phi \in L^p_c(0,T;W_o^{1,p}(\Omega))$.

On the other hand, since $\|\xi_\varepsilon\|_{L^p(\mathbb{R}^N)} = 1$, by the Hölder inequality and the Fubini theorem we have

$$\begin{aligned}
\int_0^T \int_\Omega \big|[\phi]_\varepsilon(x,t)\big|^p \,dx\,dt &\le (2\varepsilon)^{p-1} \int_0^T \int_\Omega \int_{-\varepsilon}^{\varepsilon} \big|\phi(x,t-s)\big|^p \xi_\varepsilon^p(s)\,ds\,dx\,dt \\
&\le \int_{-\varepsilon}^{\varepsilon} \xi_\varepsilon^p(s) \Big[\int_0^T \int_\Omega \big|\phi(x,t-s)\big|^p \,dx\,dt\Big]\,ds \\
&\le \int_{-\varepsilon}^{T+\varepsilon} \int_\Omega \big|\phi(x,\tau)\big|^p \,dx\,d\tau, \\
&= \int_{\Omega_T} \big|\phi(x,t)\big|^p \,dx\,dt,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\int_0^T \int_\Omega |[\nabla\phi]_\varepsilon(x,t)|^p \,dx\,dt &\le \int_{\Omega_T} |\nabla\phi(x,t)|^p \,dx\,dt \\
&\le \|\phi\|^p_{L^p(0,T;W^{1,p}(\Omega))},
\end{aligned} \tag{3.7}$$

for all sufficiently small $\varepsilon > 0$. Using these properties, we prove the following lemma.

Lemma 3.2. *Let $N \ge 1$, $\Omega \subset \mathbb{R}^N$ be a bounded open set, $p > \frac{2N}{N+2}$, $T > 0$ and $Q \ge 1$. Let $\eta \in W^{1,\infty}(\Omega_T)$ be a function such that $\eta(x,t) = 0$ for a.e. (x,t) near $t = 0$ and $t = T$. Assume that $u \in L^p(0,T;W^{1,p}(\Omega))$ is a parabolic Q-quasiminimizer. Then $\partial_t(\eta u) \in (L^p(0,T;W_o^{1,p}(\Omega)))^*$ and*

$$\langle \partial_t v, \eta u\rangle = -\langle \partial_t(\eta u), v\rangle,$$

for every function $v \in L^p(0,T;W_o^{1,p}(\Omega))$ with $\partial_t v \in (L^p(0,T;W_o^{1,p}(\Omega)))^*$.

Proof. For simplicity, we set $X = L^p(0,T;W^{1,p}(\Omega_T))$. Let $\{\eta_n\}_{n=1}^\infty \in C^\infty(\Omega_T)$ be a sequence such that

$$\|\eta_n - \eta\|_{W^{1,\infty}(\Omega_T)} \to 0 \quad \text{as } n \to \infty.$$

Since $\eta_n\varphi \in C_o^\infty(\Omega_T)$ for all $\varphi \in C_o^\infty(\Omega_T)$, by Lemma 3.1 we have

$$\begin{aligned}
|\langle \partial_t u, \eta_n\varphi\rangle| &\le \frac{2^pQ}{p}\|u\|_X^{p-1}\|\nabla(\eta_n\varphi)\|_{L^p(\Omega_T)} \\
&\le \frac{2^pQ}{p}\|u\|_X^{p-1}\|\eta_n\|_{W^{1,\infty}(\Omega_T)}\|\varphi\|_X,
\end{aligned} \tag{3.8}$$

for all $\varphi \in C_o^\infty(\Omega_T)$. Moreover, since

$$\langle \partial_t u, \eta_n \varphi \rangle = -\int_{\Omega_T} u(\varphi \partial_t \eta_n + \eta_n \partial_t \varphi)\, dx\, dt$$

and

$$\begin{aligned}\left| \int_{\Omega_T} u\varphi(\partial_t \eta_n)\, dx\, dt \right| &\le \|\partial_t \eta_n\|_{L^\infty(\Omega_T)} \|u\|_{L^2(\Omega_T)} \|\varphi\|_{L^2(\Omega_T)} \\ &\le C_1 \|\partial_t \eta_n\|_{L^\infty(\Omega_T)} \|u\|_X \|\varphi\|_X,\end{aligned}$$

which follows from the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, as $p > 2N/(N+2)$, by (3.8) we obtain

$$\begin{aligned}\left| \int_{\Omega_T} \eta_n u(\partial_t \varphi)\, dx\, dt \right| &\le C_1 \|\partial_t \eta_n\|_{L^\infty(\Omega_T)} \|u\|_X \|\varphi\|_X \\ &\quad + \frac{2^p Q}{p} \|u\|_X^{p-1} \|\eta_n\|_{W^{1,\infty}(\Omega_T)} \|\varphi\|_X.\end{aligned}$$

Here, C_1 is the constant which depends on N, p and on the domain Ω. Then, letting $n \to \infty$, by the Poincaré inequality we see that there exists a constant C_2, depending only on p, N, and Q and $\|\eta\|_{W^{1,\infty}(\Omega_T)}$, such that

$$\left| \int_{\Omega_T} \eta u(\partial_t \varphi)\, dx\, dt \right| \le C_2 \left(\|u\|_X + \|u\|_X^{p-1} \right) \|\varphi\|_X, \tag{3.9}$$

for all $\varphi \in C_o^\infty(\Omega_T)$. This implies that $\partial_t(\eta u) \in (L^p(0,T; W_o^{1,p}(\Omega_T)))^*$.

Let $\{v_n\}_{n=1}^\infty \subset C_o^\infty(\Omega_T)$ be a sequence such that $\|v_n - v\|_{L^p(0,T;W^{1,p}(\Omega_T))} \to 0$ as $n \to \infty$. Letting $\varepsilon > 0$ be a sufficiently small constant, we have $[\eta u]_\varepsilon \in C_o^\infty(\Omega_T)$ since $\eta = 0$ a.e. near $t = 0$ and $t = T$. Then we get by integration by parts that

$$\langle \partial_t v_n, [\eta u]_\varepsilon \rangle = -\langle v_n, \partial_t [\eta u]_\varepsilon \rangle,$$

and passing to the limit $\varepsilon \to 0$, by Lemma 3.1 we obtain

$$\langle \partial_t v_n, \eta u \rangle = \int_{\Omega_T} (\partial_t v_n) \eta u dx dt = -\langle \partial_t(\eta u), v_n \rangle. \tag{3.10}$$

Furthermore, by (3.9) we have

$$|\langle \partial_t v_n - \partial_t v, \eta u \rangle| \le C_2 \left(\|u\|_X + \|u\|_X^{p-1} \right) \|v_n - v\|_X.$$

This together with Lemma 4.1 and (3.10) yields

$$\langle \partial_t v, \eta u \rangle = -\langle \partial_t(\eta u), v \rangle,$$

and we conclude the proof of Lemma 3.2. $\square$

3.2. **Sobolev space with zero boundary values.** In this subsection we recall some properties of the Sobolev space with zero boundary values and a Hardy type estimate. We begin with a stability result for Sobolev spaces with zero boundary values under suitable assumptions on regularity of the boundary. For the proof of Proposition 3.3, we refer to [3].

Proposition 3.3. *Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded open set, and $p > 1$. Assume that $\mathbb{R}^N \setminus \Omega$ is uniformly p-thick. Then there exists a positive constant ε such that*

$$W_o^{1,p-\varepsilon}(\Omega) \cap W^{1,p}(\Omega) = W_o^{1,p}(\Omega).$$

We also have the following Hardy type estimate, see [1] and [10].

Proposition 3.4. *Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $p > 1$. Assume that $\mathbb{R}^N \setminus \Omega$ is uniformly p-thick. Then there exists a positive constant C, depending only on N, p and μ, such that*

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, \mathbb{R}^N \setminus \Omega)} \right)^p dx \leq C \|u\|^p_{W^{1,p}(\Omega)},$$

for all $u \in W_o^{1,p}(\Omega)$.

Furthermore, the following proposition gives a sufficient condition for a function $u \in W^{1,p}(\Omega)$ such that $u \in W_o^{1,p}(\Omega)$. For the proof of Proposition 3.5, see [4].

Proposition 3.5. *Let $N \geq 1$, $p > 1$, and Ω be an open subset of $\mathbb{R}^N$. If $u \in W^{1,p}(\Omega)$ satisfies*

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, \mathbb{R}^N \setminus \Omega)} \right)^p dx < \infty,$$

then $u \in W_o^{1,p}(\Omega)$.

3.3. **Global higher integrability.** One of the fundamental ingredients for our stability proofs is global higher integrability of the gradient ∇u on the domain Ω_T, which follows from the parabolic Q–quasiminimizing property. For the proof of Proposition 3.6, see [2].

Proposition 3.6. *Let $N \geq 2$, $p > 2N/(N+2)$, and $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $\mathbb{R}^N \setminus \Omega$ is uniformly p-thick with parameters μ and r_0. For $Q \geq 1$ and $g \in C^1(\overline{\Omega_T})$, let $u \in L^p(0,T; W^{1,p}(\Omega))$ be a parabolic Q-quasiminimizer satisfying (2.3) and (2.4). Then there exists a positive constant δ, depending only on N, p, Q, μ, and r_0, such that*

$$u \in L^{p+\delta}(0,T; W^{1,p+\delta}(\Omega)).$$

Furthermore

$$\int_{\Omega_T} |\nabla u|^{p+\delta}\, dx\, dt$$

is bounded from above by some positive constant depending only on N, p, Q, μ, r_0, δ, g, and $\|\nabla u\|_{L^p(\Omega_T)}$.

4. Uniform estimate for quasiminimizers

In this section we study a uniform estimate of $\|u_i\|_{L^{p_i}(0,T;W^{1,p_i}(\Omega))}$ with respect to $i \in \mathbb{N}$. We first study the Caccioppoli type estimate, and prove the following lemma, which is an extension of Lemma 3.1 of [9].

Lemma 4.1. *Assume the same conditions as in Theorem* 2.1. *Then, for any $\bar{\delta} > 0$, there exists a positive constant C, depending only on N, $\bar{\delta}$ and the upper bound $\bar{p}$ of $\{p_i\}$, and in particular independent of i, such that*

$$\begin{aligned}
&\sup_{t\in(0,T)} \int_\Omega |u_i(\cdot,t) - g(\cdot,t)|^2\,dx + \int_{\Omega_T} |\nabla u_i|^{p_i}\,dx\,dt \\
&\quad \le C\int_{\Omega_T} |\partial_t g|^{p_i/(p_i-1)}\,dx\,dt + \bar{Q}\int_{\Omega_T} |\nabla g|^{p_i}\,dx\,dt + \bar{\delta}\int_{\Omega_T} |u_i - g|^{p_i}\,dx\,dt,
\end{aligned}$$

for all $i \in \mathbb{N}$. Here $\bar{Q}$ denotes the upper bound of $\{Q_i\}$.

Proof. For $0 < s < T$ and $0 < h < \min\{s/4, (T-s)/2\}$, we set

$$\chi_s^h(t) := \begin{cases} 0, & 0 \le t \le h, \\ (t-h)/h, & h < t \le 2h, \\ 1, & 2h < t \le s-2h, \\ (s-t-h)/h, & s-2h < t \le s-h, \\ 0, & s-h < t \le T. \end{cases}$$

Then, since the support of the function χ_s^h is compact in $(0,T)$ and by the lateral boundary condition (2.3), we can take the test function

$$\phi_\varepsilon^h(x,t) := \chi_s^h([u_i]_\varepsilon - [g]_\varepsilon) \in \mathrm{Lip}_c(0,T;W_o^{1,p_i}(\Omega))$$

in (3.5), and obtain

$$\begin{aligned}
-\int_{\Omega_T} [u_i]_\varepsilon \partial_t \phi_\varepsilon^h\,dx\,dt + &\frac{1}{p_i}\int_{\mathrm{supp}[\phi_\varepsilon^h]_\varepsilon} |\nabla u_i|^{p_i}\,dx\,dt \\
&\le \frac{Q_i}{p_i}\int_{\mathrm{supp}[\phi_\varepsilon^h]_\varepsilon} \left|\nabla\big(u_i - [\phi_\varepsilon^h]_\varepsilon\big)\right|^{p_i}\,dx\,dt,
\end{aligned} \tag{4.1}$$

where $\varepsilon > 0$ is a sufficiently small constant and $[\,\cdot\,]_\varepsilon$ is defined by (3.4). Set

$$\begin{aligned}
&\int_{\Omega_T} [u_i]_\varepsilon \partial_t \phi_\varepsilon^h\,dx\,dt \\
&\quad = \int_{\Omega_T} ([u_i]_\varepsilon - [g]_\varepsilon)\partial_t \phi_\varepsilon^h\,dx\,dt + \int_{\Omega_T} [g]_\varepsilon \partial_t \phi_\varepsilon^h\,dx\,dt \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{4.2}$$

By an integration by parts we have

$$
\begin{aligned}
I_1 &= \int_{\Omega_T} \left[|[u_i]_\varepsilon - [g]_\varepsilon|^2 \partial_t \chi_s^h + \chi_s^h ([u_i]_\varepsilon - [g]_\varepsilon) \partial_t ([u_i]_\varepsilon - [g]_\varepsilon) \right] dx\, dt \\
&= \int_{\Omega_T} |[u_i]_\varepsilon - [g]_\varepsilon|^2 \partial_t \chi_s^h \, dx\, dt + \frac{1}{2} \int_{\Omega_T} \chi_s^h \partial_t |[u_i]_\varepsilon - [g]_\varepsilon|^2 \, dx\, dt \\
&= \frac{1}{2} \int_{\Omega_T} |[u_i]_\varepsilon - [g]_\varepsilon|^2 \partial_t \chi_s^h \, dx\, dt.
\end{aligned}
$$

Thus, letting $\varepsilon \to 0$ and thereafter $h \to 0$, we obtain, using also the initial condition (2.4):

$$
\begin{aligned}
I_1 &\xrightarrow{\varepsilon \to 0} -\frac{1}{2h} \int_{s-2h}^{s-h} \int_\Omega |u_i(x,t) - g(x,t)|^2 \, dx\, dt \\
&\qquad + \frac{1}{2h} \int_h^{2h} \int_\Omega |u_i(x,t) - g(x,t)|^2 \, dx\, dt \\
&\xrightarrow{h \to 0} - \int_\Omega |u_i(\cdot, s) - g(\cdot, s)|^2 \, dx,
\end{aligned}
\tag{4.3}
$$

for almost all $s \in (0,T)$.

Furthermore, by integration by parts and the Young inequality, for any $\delta' > 0$, we can find a positive constant $C_{\delta'}$, independent of i, such that

$$
\begin{aligned}
I_2 &= -\int_h^{T-h} \int_\Omega \chi_s^h ([u_i]_\varepsilon - [g]_\varepsilon) \partial_t [g]_\varepsilon \, dx\, dt \le \int_{\Omega_T} |[u_i]_\varepsilon - [g]_\varepsilon| |\partial_t [g]_\varepsilon| \, dx\, dt \\
&\le \delta' \int_{\Omega_T} |[u_i]_\varepsilon - [g]_\varepsilon|^{p_i} \, dx\, dt + C_{\delta'} \int_{\Omega_T} |\partial_t [g]_\varepsilon|^{p_i/(p_i-1)} \, dx\, dt.
\end{aligned}
$$

Since $g \in C^1(\overline{\Omega_T})$, we obtain

$$
\lim_{\varepsilon, h \to 0} I_2 \le \delta' 2^{\bar{p}-1} \int_{\Omega_T} |u_i - g|^{p_i} \, dx\, dt + C_{\delta'} \int_{\Omega_T} |\partial_t g|^{p_i/(p_i-1)} \, dx\, dt. \tag{4.4}
$$

On the other hand, since

$$
\lim_{\varepsilon, h \to 0} \int_{\operatorname{supp}[\phi_\varepsilon^h]_\varepsilon} \left| \nabla \big(u_i - [\phi_\varepsilon^h]_\varepsilon \big) \right|^{p_i} dx\, dt \le \int_{\operatorname{supp}(u_i - g)} |\nabla g|^{p_i} \, dx\, dt,
$$

by (4.1)–(4.4) there exists a positive constant C, independent of i and δ', such that

$$
\begin{aligned}
&\int_\Omega |u_i(\cdot, s) - g(\cdot, s)|^2 \, dx + \int_{\Omega_T} |\nabla u_i|^{p_i} \, dx\, dt \\
&\le \bar{p}\, C_{\delta'} \int_{\Omega_T} |\partial_t g|^{p_i/(p_i-1)} \, dx\, dt + C \int_{\Omega_T} |\nabla g|^{p_i} \, dx\, dt + C\delta' \int_{\Omega_T} |u_i - g|^{p_i} \, dx\, dt,
\end{aligned}
$$

which holds for almost all $s \in (0,T)$. Here the constant $C_{\delta'}$ depends on $\bar{p}$ and δ'. Therefore, for any $\bar{\delta} > 0$, taking a sufficiently small $\delta' > 0$ satisfying $C\delta' < \bar{\delta}$ and passing over to the supremum over $s \in (0,T)$ on the left–hand–side, we obtain the desired inequality. □

As a corollary of Lemma 4.1, we obtain a uniform estimate of the norm $\|u_i\|_{L^{p_i}(0,T;W^{1,p_i}(\Omega))}$.

Corollary 4.2. *Assume the same conditions as in Theorem 2.1. Then*
(4.5)
$$\sup_{i\in\mathbb{N}}\left(\sup_{t\in(0,T)}\int_\Omega |u_i(\cdot,t)|^2\,dx+\int_{\Omega_T}|u_i|^{p_i}\,dx\,dt+\int_{\Omega_T}|\nabla u_i|^{p_i}\,dx\,dt\right)<\infty.$$
Furthermore there exists a positive constant δ such that
$$\sup_{i\in\mathbb{N}}\int_{\Omega_T}\left(|u_i|^{p+\delta}+|\nabla u_i|^{p+\delta}\right)dx\,dt<\infty. \tag{4.6}$$

Proof. Corollary 4.2 can be proved by the same argument as Corollary 3.2 in [9] with the aid of Lemma 4.1, and hence we omit the details of the proof. Additionally we note that the uniform bound of the sup-Term in the case $p<2$ can also be obtained via the Caccioppoli–type inequality in Lemma 4.1 and the fact that $g\in C^1(\overline{\Omega}_T)$. □

By Corollary 4.2 we can obtain the strong convergence of u_i and weak convergence of the derivatives in $L^{p+\delta}(\Omega_T)$, and moreover also in the case $p<2$ the strong convergence in L^2.

Lemma 4.3. *Assume the same conditions as in Theorem 2.1. Then there exist a subsequence $\{u_i\}_{i=1}^\infty$ and a positive constant δ such that we have $u\in L^{p+\delta}(0,T;W^{1,p+\delta}(\Omega))$ and*
$$u_i\to u\quad \textit{in}\ L^{p+\delta}(\Omega_T)\cap L^2(\Omega_T), \tag{4.7}$$
$$\nabla u_i\rightharpoonup\nabla u\quad \textit{weakly in}\ L^{p+\delta}(\Omega_T), \tag{4.8}$$
as $i\to\infty$. Furthermore $\partial_t u\in(L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega)))^*$ and
$$\partial_t u_i\overset{*}{\rightharpoonup}\partial_t u\quad \textit{in the weak-}{*}\ \textit{topology on}\ (L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega)))^* \tag{4.9}$$
as $i\to\infty$.

Proof. We can prove (4.7) and (4.8) by the similar argument as in Lemma 3.3 in [9]. Since $\lim_{i\to\infty}p_i=p$ and $\lim_{i\to\infty}Q_i=Q$, by Lemma 3.1 there exist positive constants C and δ, independent of i, such that
$$\begin{aligned}|\langle\partial_t u_i,\phi\rangle|&\le\frac{2^{p_i}Q_i}{p_i}\|u_i\|^{p_i-1}_{L^{p_i}(0,T;W^{1,p_i}(\Omega))}\|\nabla\phi\|_{L^{p_i}(\Omega_T)}\\&\le\frac{2^{p_i}Q_i}{p_i}\|u_i\|^{p_i-1}_{L^{p+\delta}(0,T;W^{1,p+\delta}(\Omega))}\|\nabla\phi\|_{L^{p+\delta}(\Omega_T)}\cdot(|\Omega|T)^{1/p_i-1/(p+\delta)}\\&\le C\|u_i\|^{p_i-1}_{L^{p+\delta}(0,T;W^{1,p+\delta}(\Omega))}\|\nabla\phi\|_{L^{p+\delta}(\Omega_T)},\end{aligned}$$
for all $\phi\in C_o^\infty(\Omega_T)$ and sufficiently large $i\in\mathbb{N}$. This together with (4.6) implies that
$$\sup_{i\in\mathbb{N}}\|\partial_t u_i\|_{(L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega_T))^*}<\infty. \tag{4.10}$$

Then, in view of the Rellich-Kondrachov theorem together with (4.6) and (4.10), by taking a subsequence if necessary we can find $\widetilde{u} \in L^{p+\delta}(\Omega_T)$ with $\nabla\widetilde{u} \in L^{p+\delta}(\Omega_T)$ such that

$$\begin{aligned} u_i \to \widetilde{u} \quad &\text{in } \ L^{p+\delta}(\Omega_T), \\ \nabla u_i \rightharpoonup \nabla\widetilde{u} \quad &\text{weakly in } \ L^{p+\delta}(\Omega_T), \end{aligned}$$

as $i \to \infty$ (see [20] and [21]). Since $u_i \to u$ almost everywhere in Ω as $i \to \infty$, we have $\widetilde{u} = u$, and obtain (4.7) and (4.8).

Moreover, applying [21, Corollary 8], with the choices $X := W^{1,p}(\Omega)$, $B := L^2(\Omega)$ and $Y := W^{-1,p'}(\Omega) = \big(W_o^{1,p}(\Omega)\big)^*$ and having the inclusions

$$W^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega),$$

where the first inclusion is compact, since $p > 2N/(N+2)$, the uniform bounds (4.6) and (4.10) allow us to conclude also the strong convergence $u_i \to u$ in $L^2(\Omega_T)$ (in fact $u_i \to u$ in $L^q(0,T;L^2(\Omega))$ for any $q < \infty$).

Next we prove (4.9). Taking a subsequence if necessary, we see that there exists a function $v \in (L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega_T))^*$ such that

$$\partial_t u_i \overset{*}{\rightharpoonup} v \quad \text{in the weak-}* \text{ topology on } \ (L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega_T))^*, \tag{4.11}$$

as $i \to \infty$. On the other hand, since it follows from (4.7) that

$$\lim_{i\to\infty} \langle \partial_t u_i, \phi\rangle = -\lim_{i\to\infty} \int_{\Omega_T} u_i \partial_t \phi \, dx \, dt = -\int_{\Omega_T} u \partial_t \phi \, dx \, dt,$$

for all $\phi \in C_o^\infty(\Omega_T)$, by (4.11) we have

$$\langle v, \phi\rangle = \lim_{i\to\infty} \langle \partial_t u_i, \phi\rangle = -\int_{\Omega_T} u \partial_t \phi \, dx \, dt = \langle \partial_t u, \phi\rangle$$

and

$$|\langle \partial_t u, \phi\rangle| \le \|v\|_{(L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega_T))^*} \|\nabla\phi\|_{L^{p+\delta}(\Omega_T)},$$

for all $\phi \in C_o^\infty(\Omega_T)$. This implies that $\partial_t u \in (L^{p+\delta}(0,T;W_o^{1,p+\delta}(\Omega_T))^*$ and $v = \partial_t u$. Thus we obtain (4.9). □

5. Proof of Theorem 2.1

In this section we complete the proof of our first main result.

Proof of Theorem 2.1**.** We first prove (2.7) and (2.8). Let $\varepsilon > 0$ be a sufficiently small constant to be chosen later. Taking a sufficiently large i, we have $p - \varepsilon < p_i < p + \varepsilon$ and $u_i(t) - g(t) \in W_o^{1,p-\varepsilon}(\Omega)$ for almost all

$t \in (0,T)$, and by Proposition 3.4 we obtain

$$
\begin{aligned}
&\int_{\Omega_T} \left(\frac{|u_i(x,t)-g(x,t)|}{\operatorname{dist}(x,\mathbb{R}^N\setminus\Omega)}\right)^{p-\varepsilon} dx\,dt \\
&\le C_1 \int_0^T \|u_i(t)-g(t)\|^{p-\varepsilon}_{W^{1,p-\varepsilon}(\Omega)} dt \qquad (5.1)\\
&\le C_2 \int_0^T \left(\|u_i(t)\|^{p-\varepsilon}_{W^{1,p-\varepsilon}(\Omega)} + \|g\|^{p-\varepsilon}_{W^{1,p-\varepsilon}(\Omega)}\right) dt.
\end{aligned}
$$

At this point we use the Caccioppoli type estimate in terms of Lemma 4.1 to control the $W^{1,p-\varepsilon}$–norm of u_i as follows

$$
\begin{aligned}
\int_{\Omega_T} |\nabla u_i|^{p-\varepsilon}\,dx\,dt &\le |\Omega_T|^{1-(p-\varepsilon)/p_i}\left(\int_{\Omega_T} |\nabla u_i|^{p_i}\,dx\,dt\right)^{(p-\varepsilon)/p_i} \\
&\le |\Omega_T|^{1-(p-\varepsilon)/p_i}\biggl[C\int_{\Omega_T} |\partial_t g|^{p_i/(p_i-1)}\,dx\,dt + C\int_{\Omega_T} |\nabla g|^{p_i}\,dx\,dt \\
&\qquad + \bar\delta \int_{\Omega_T} |u_i-g|^{p_i}\,dx\,dt\biggr]^{(p-\varepsilon)/p_i} \\
&\le C\,|\Omega_T|^{1-(p-\varepsilon)/p_i}\biggl[\|u_i\|^{p-\varepsilon}_{L^{p_i}(\Omega_T)} + \|\partial_t g\|^{(p-\varepsilon)/(p_i-1)}_{L^{p_i/(p_i-1)}(\Omega_T)} \\
&\qquad + \|g\|^{p-\varepsilon}_{L^{p_i}(0,T;W^{1,p_i}(\Omega))}\biggr],
\end{aligned}
$$

for a constant C which depends on N, $\bar p$ and $\bar\delta$ and for arbitrary $\bar\delta > 0$. Combining this with (5.1) and using once again Hölder's inequality, we arrive at

$$
\begin{aligned}
&\int_{\Omega_T} \left(\frac{|u_i(x,t)-g(x,t)|}{\operatorname{dist}(x,\mathbb{R}^N\setminus\Omega)}\right)^{p-\varepsilon} dx\,dt \\
&\quad\le C\left[1 + \|u_i\|^{p-\varepsilon}_{L^{p+\varepsilon}(\Omega_T)} + \|g\|^{p-\varepsilon}_{L^{p+\varepsilon}(0,T;W^{1,p+\varepsilon}(\Omega))} + \|\partial_t g\|^{(p-\varepsilon)/(p_i-1)}_{L^{(p-\varepsilon)/(p-\varepsilon-1)}(\Omega_T)}\right].
\end{aligned}
$$

Here we used in the last step also that $(p-\varepsilon)/(p-\varepsilon-1) > p_i/(p_i-1)$. The above estimate holds for all $i \in \mathbb{N}$, and C is a positive constant which depends only on N, $\bar p$, $\bar\delta$ and $|\Omega_T|$ and blows up as $|\Omega_T| \to \infty$. Moreover, the right–hand side of the above inequality is uniformly bounded, since $p_i > 2N/(N+2) > 1$ and the uniform bound of $\|u_i\|_{L^{p+\varepsilon}(\Omega_T)}$.

Since $u_i \to u$ for almost every $(x,t) \in \Omega_T$, by (4.6), (5.1), and the Fatou lemma we obtain

$$
\int_0^T \int_\Omega \left(\frac{|u(x,t)-g(x,t)|}{\operatorname{dist}(x,\mathbb{R}^N\setminus\Omega)}\right)^{p-\varepsilon} dx\,dt < \infty.
$$

This implies that

$$
\int_\Omega \left(\frac{|u(x,t)-g(x,t)|}{\operatorname{dist}(x,\mathbb{R}^N\setminus\Omega)}\right)^{p-\varepsilon} dx < \infty,
$$

for almost every $t \in (0,T)$, and by Proposition 3.5 we obtain $u(t) - g(t) \in W_o^{1,p-\varepsilon}(\Omega)$ for almost every $t \in (0,T)$. On the other hand, since $\mathbb{R}^N \setminus \Omega$ is uniformly p-thick, we can apply Proposition 3.3 to obtain

$$W_o^{1,p-\varepsilon'}(\Omega) \cap W^{1,p}(\Omega) = W_o^{1,p}(\Omega),$$

for some $\varepsilon' > 0$. Now, choosing ε small enough, for example $\varepsilon := \varepsilon'/2$, we conclude that $u(t) - g(t) \in W_o^{1,p}(\Omega)$ for almost every $t \in (0,T)$, and obtain (2.7).

In order to prove (2.8), for $\tau \in (0,T)$, we take parameters $h, k > 0$ such that $2h < \tau - k < \tau$ and set

$$\chi_{0,\tau}^{h,k}(t) = \begin{cases} 0, & 0 \le t \le h, \\ (t-h)/h, & h < t \le 2h, \\ 1, & 2h < t \le \tau - k, \\ (\tau - t)/k, & \tau - k < t \le \tau, \\ 0, & \tau < t \le T. \end{cases}$$

Consider the function $\chi_{0,\tau}^{h,k}([u_i]_\varepsilon - [g]_\varepsilon)$ in (3.5), and let $\varepsilon \to 0$. Then, by the similar argument as in Lemma 4.1 we obtain

$$\begin{aligned} &\frac{1}{k}\int_{\tau-k}^{\tau}\int_\Omega |u_i - g|^2\,dx\,dt - \frac{1}{h}\int_h^{2h}\int_\Omega |u_i - g|^2\,dx\,dt \\ &\quad \le C \sup_{i\in\mathbb{N}} \int_o^\tau \int_\Omega \Big(|\nabla u_i|^{p_i} + |u_i - g|^{p_i} + |\partial_t g|^{p_i/(p_i-1)} + |\nabla g|^{p_i}\Big)\,dx\,dt, \end{aligned}$$

where C is a positive constant.

We first pass to the limit $h \to 0$ in the inequality above, and then pass to the limit $i \to \infty$. Since $u_i \to u$ in $L^{p+\delta}(\Omega_T)$ and in $L^2(\Omega_T)$ as $i \to \infty$ by (4.7) and $u_i|_{t=0} = g$, we obtain

$$\begin{aligned} &\frac{1}{k}\int_{\tau-k}^{\tau}\int_\Omega |u - g|^2\,dx\,dt \\ &\quad \le C \sup_{i\in\mathbb{N}} \int_0^\tau \int_\Omega \Big(|\nabla u_i|^{p_i} + |u_i - g|^{p_i} + |\partial_t g|^{p_i/(p_i-1)} + |\nabla g|^{p_i}\Big)\,dx\,dt. \end{aligned} \tag{5.2}$$

Furthermore, since

$$\lim_{k\to 0}\frac{1}{k}\int_{\tau-k}^{\tau}\int_\Omega |u-g|^2\,dx\,dt = \int_\Omega |u(x,\tau) - g(x,\tau)|^2\,dx,$$

by (4.6) and (5.2) we obtain

$$\lim_{\tau\to 0}\int_\Omega |u(x,\tau) - g(x,\tau)|^2\,dx = 0.$$

This implies that

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int_\Omega |u(x,t) - g(x,t)|^2\,dx\,dt = 0,$$

and we conclude that (2.8) holds.

We next prove (2.9). Fix $\alpha > 0$. Let $\phi \in C_o^\infty(\Omega_T)$ and set $K = \operatorname{supp} \phi$. Since K is a compact subset of Ω_T, we can take open sets O_1 and O_2 such that $K \Subset O_1 \Subset O_2 \Subset \Omega_T$ and

$$\int_{O_2 \setminus K} |\nabla u|^{p+\delta} \, dx \, dt < \alpha, \tag{5.3}$$

where $\delta > 0$ is the constant given in Lemma 4.3. Let $\varepsilon > 0$ be a sufficiently small constant, and take a test function

$$\varphi_{i,\varepsilon} := \phi + \eta([u_i]_\varepsilon - [u]_\varepsilon),$$

where $\eta \in C_o^\infty(\Omega_T)$ is a cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in a neighborhood of K, and $\eta \equiv 0$ in $\Omega_T \setminus O_1$. Then, since $\varphi_{i,\varepsilon}$ is a valid test function in (3.5), we obtain

$$\begin{aligned} -\int_{\Omega_T} [u_i]_\varepsilon \partial_t(\varphi_{i,\varepsilon}) \, dx \, dt + \frac{1}{p_i} \int_{\operatorname{supp}[\varphi_{i,\varepsilon}]_\varepsilon} |\nabla u_i|^{p_i} \, dx \, dt& \\ \le \frac{Q_i}{p_i} \int_{\operatorname{supp}[\varphi_{i,\varepsilon}]_\varepsilon} |\nabla (u_i - [\varphi_{i,\varepsilon}]_\varepsilon)|^{p_i} \, dx \, dt&. \end{aligned} \tag{5.4}$$

Let $\beta > 0$. By (4.8) we have

$$\begin{aligned} \int_K |\nabla u|^{p-\beta} \, dx \, dt &\le \liminf_{i \to \infty} \int_K |\nabla u_i|^{p-\beta} \, dx \, dt \\ &\le \liminf_{i \to \infty} \left[\left(\int_K |\nabla u_i|^{p_i} \, dx \, dt \right)^{(p-\beta)/p_i} (|\Omega| T)^{1-(p-\beta)/p_i} \right] \\ &\le (|\Omega| T)^{\beta/p} \left(\liminf_{i \to \infty} \int_K |\nabla u_i|^{p_i} \, dx \, dt \right)^{(p-\beta)/p}. \end{aligned}$$

Since $\beta > 0$ is arbitrary and $K \subset \operatorname{supp}[\varphi_{i,\varepsilon}]_\varepsilon$ for every i, we obtain

$$\int_K |\nabla u|^p \, dx \, dt \le \liminf_{i \to \infty} \int_{\operatorname{supp}[\varphi_{i,\varepsilon}]_\varepsilon} |\nabla u_i|^{p_i} \, dx \, dt. \tag{5.5}$$

Let

$$I_{i,\varepsilon} := -\int_{\Omega_T} [u_i]_\varepsilon (\partial_t(\varphi_{i,\varepsilon} - \phi)) \, dx \, dt = -\int_{\Omega_T} [u_i]_\varepsilon \partial_t(\eta [u_i - u]_\varepsilon) \, dx \, dt.$$

Then we have

$$\begin{aligned} -I_{i,\varepsilon} &= \int_{\Omega_T} [u_i - u]_\varepsilon \partial_t(\eta [u_i - u]_\varepsilon) \, dx \, dt + \int_{\Omega_T} [u]_\varepsilon \partial_t(\eta [u_i - u]_\varepsilon) \, dx \, dt \\ &=: J_{i,\varepsilon} + K_{i,\varepsilon}. \end{aligned}$$

Since $\eta \in C_o^\infty(\Omega_T)$, we have

$$
\begin{aligned}
J_{i,\varepsilon} &= \int_{\Omega_T} (\partial_t \eta)[u_i - u]_\varepsilon^2 \, dx \, dt + \frac{1}{2} \int_{\Omega_T} \eta \frac{\partial}{\partial t} [u_i - u]_\varepsilon^2 \, dx \, dt \\
&= \int_{\Omega_T} (\partial_t \eta)[u_i - u]_\varepsilon^2 \, dx \, dt - \frac{1}{2} \int_{\Omega_T} (\partial_t \eta)[u_i - u]_\varepsilon^2 \, dx \, dt \\
&= \frac{1}{2} \int_{\Omega_T} (\partial_t \eta)[u_i - u]_\varepsilon^2 \, dx \, dt,
\end{aligned}
$$

and by (4.7) we obtain

$$
\lim_{i\to\infty} \lim_{\varepsilon\to 0} J_{i,\varepsilon} = 0. \tag{5.6}
$$

Furthermore, putting

$$
\begin{aligned}
K_{i,\varepsilon} &= \int_{\Omega_T} (\partial_t \eta)[u]_\varepsilon [u_i - u]_\varepsilon \, dx \, dt + \int_{\Omega_T} (\partial_t [u_i - u]_\varepsilon) \eta [u]_\varepsilon \, dx \, dt \\
&= \int_{\Omega_T} (\partial_t \eta)[u]_\varepsilon [u_i - u]_\varepsilon \, dx \, dt \\
&\qquad - \int_{\Omega_T} (\partial_t [u_i - u]_\varepsilon) \eta (u - [u]_\varepsilon) \, dx \, dt + \int_{\Omega_T} (\partial_t [u_i - u]_\varepsilon) \eta u \, dx \, dt \\
&=: K_{i,\varepsilon}^1 + K_{i,\varepsilon}^2 + K_{i,\varepsilon}^3,
\end{aligned}
$$

by (3.6), (3.7), and Lemma 3.1 we can find a positive constant C, independent of ε and i, such that

$$
|K_{i,\varepsilon}^1| \le \sup_{\Omega_T} |\partial_t \eta| \cdot \|u\|_{L^2(\Omega_T)} \|u_i - u\|_{L^2(\Omega_T)}
$$

and

$$
|K_{i,\varepsilon}^2| \le C \|u_i - u\|_{L^p(0,T;W^{1,p}(\Omega))}^{p-1} \|\nabla([u]_\varepsilon - u)\|_{L^p(\Omega_T)}.
$$

By (4.7) we obtain

$$
\lim_{i\to\infty} \lim_{\varepsilon\to 0} \left(|K_{i,\varepsilon}^1| + |K_{i,\varepsilon}^2| \right) = 0. \tag{5.7}
$$

On the other hand, Lemma 3.2 implies that

$$
K_{i,\varepsilon}^3 = \langle \partial_t(\eta u), [u_i - u]_\varepsilon \rangle \to \langle \partial_t(\eta u), u_i - u \rangle = -\langle \partial_t(u_i - u), \eta u \rangle,
$$

as $\varepsilon \to 0$ and, consequently, by (4.9) we have

$$
\lim_{i\to\infty} \lim_{\varepsilon\to 0} K_{i,\varepsilon}^3 = 0.
$$

This together with (5.6) and (5.7) implies that

$$
\lim_{i\to\infty} \lim_{\varepsilon\to 0} - \int_{\Omega_T} [u_i]_\varepsilon \partial_t \varphi_{i,\varepsilon} \, dx \, dt = - \int_{\Omega_T} u \partial_t \phi \, dx \, dt. \tag{5.8}
$$

Therefore, letting $\varepsilon \to 0$ in (5.4), by (5.5) and (5.8) we obtain

$$
\begin{aligned}
&-\int_{\Omega_T} u\partial_t\phi\,dx\,dt + o(1) + \frac{1}{p}\int_K |\nabla u|^p\,dx\,dt \\
&\qquad\qquad \le \frac{Q_i}{p_i}\int_{O_1} |\nabla(u_i-\phi-\eta(u_i-u))|^{p_i}\,dx\,dt,
\end{aligned} \tag{5.9}
$$

for all sufficiently large $i \in \mathbb{N}$. By the definition of η and K we have $\eta \equiv 1$ in a neighborhood of K and $\phi \equiv 0$ in $\Omega_T \setminus K$, and obtain

$$
\begin{aligned}
&\int_{O_1} |\nabla(u_i-\phi-\eta(u_i-u))|^{p_i}\,dx\,dt \\
&\quad = \int_K |\nabla(u-\phi)|^{p_i}\,dx\,dt + \int_{O_1\setminus K} |\nabla(u_i-\eta(u_i-u))|^{p_i}\,dx\,dt.
\end{aligned} \tag{5.10}
$$

Since

$$
|\nabla(u_i-\eta(u_i-u))| \le (1-\eta)|\nabla u_i| + |\nabla\eta||u_i-u| + \eta|\nabla u|
$$

in $O_1 \setminus K$, we have

$$
\begin{aligned}
&\int_{O_1\setminus K} |\nabla(u_i-\eta(u_i-u))|^{p_i}\,dx\,dt \\
&\le C_1\int_{O_1\setminus K} \left((1-\eta)^{p_i}|\nabla u_i|^{p_i} + |\nabla\eta|^{p_i}|u_i-u|^{p_i} + \eta^{p_i}|\nabla u|^{p_i}\right)\,dx\,dt,
\end{aligned} \tag{5.11}
$$

where C_1 is a positive constant independent of i. By the Hölder inequality we have

$$
\int_{O_1\setminus K} \eta^{p_i}|\nabla u|^{p_i}\,dx\,dt \le \left(\int_{O_1\setminus K} |\nabla u|^{p+\delta}\,dx\,dt\right)^{p_i/(p+\delta)} |O_1\setminus K|^{1-p_i/(p+\delta)}.
$$

This together with (5.3) implies that there exists a positive constant C_2 such that

$$
\limsup_{i\to\infty}\int_{O_1\setminus K} \eta^{p_i}|\nabla u|^{p_i}\,dx\,dt < C_2\alpha^{p/(p+\delta)}. \tag{5.12}
$$

Furthermore, by the Hölder inequality and (4.7) there exists a positive constant C such that

$$
\int_{O_1\setminus K} |\nabla\eta|^{p_i}|u_i-u|^{p_i}\,dx\,dt \le C\|u_i-u\|_{L^{p+\delta}(\Omega_T)}^{p/(p+\delta)} \to 0, \tag{5.13}
$$

as $i \to \infty$. Note that the constant C in the above estimate depends also on $\nabla\eta$ and therefore also on ϕ, but it is independent of i.

In order to finish the proof of (2.9), it remains to estimate the quantity

$$
\lim_{i\to\infty}\int_{O_1\setminus K} (1-\eta)^{p_i}|\nabla u_i|^{p_i}\,dz.
$$

For this aim we modify the arguments of [7] and [14], and prove the following auxiliary result.

Lemma 5.1. *Assume the same conditions as in Theorem* 2.1*. Let D be a compact subset of Ω_T. For any $s \in (0, s_0)$, define*

$$D(s) := \{z \in \Omega_T : \operatorname{dist}_p(z, D) < s\},$$

where dist_p *denotes the parabolic distance in $\mathbb{R}^{N+1}$ and $s_0 := \operatorname{dist}_p(D, \mathbb{R}^N \times (0,\infty) \setminus \Omega_T)$. Then there exists a positive constant C such that*

$$\limsup_{i\to\infty} \int_{D(s)} |\nabla u_i|^{p_i}\,dx\,dt \le C \int_{D(s)} |\nabla u|^p\,dx\,dt,$$

for almost every $s \in (0, s_0)$.

Proof. Let $0 < \sigma < s < s_0$ and $\eta \in C_o^\infty(\Omega_T)$ be a cut-off function such that

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \quad \text{in } D(\sigma), \qquad \eta \equiv 0 \quad \text{in } \Omega_T \setminus D(s). \tag{5.14}$$

Let $\varepsilon > 0$ be a sufficiently small constant, and take the test function

$$\phi_{i,\varepsilon} = \eta([u_i]_\varepsilon - [u]_\varepsilon).$$

Then, by (3.5) we have

$$\begin{aligned} -\int_{\Omega_T} [u_i]_\varepsilon(\partial_t\phi_{i,\varepsilon})\,dx\,dt + \frac{1}{p_i}\int_{\operatorname{supp}[\phi_{i,\varepsilon}]_\varepsilon} |\nabla u_i|^{p_i}\,dx\,dt& \\ \le \frac{Q_i}{p_i}\int_{\operatorname{supp}[\phi_{i,\varepsilon}]_\varepsilon} |\nabla(u_i - [\phi_{i,\varepsilon}]_\varepsilon)|^{p_i}\,dx\,dt.& \end{aligned} \tag{5.15}$$

By the same argument as above we see that

$$\lim_{i\to\infty}\lim_{\varepsilon\to 0} -\int_{\Omega_T} [u_i]_\varepsilon \partial_t\phi_{i,\varepsilon}\,dx\,dt = 0. \tag{5.16}$$

To proceed with the argument, we denote the integral in the line above with $I_\varepsilon^{(i)}$ and moreover $I^{(i)} := \lim_{\varepsilon\to 0} I_\varepsilon^{(i)}$. With this short–hand notation (5.16) writes as $\lim_{i\to\infty} I^{(i)} = \lim_{i\to\infty}\lim_{\varepsilon\to 0} I_\varepsilon^{(i)} = 0$.

Therefore, with this notation, letting $\varepsilon \to 0$ in (5.15), we have

$$\int_{D(s)} |\nabla u_i|^{p_i}\,dx\,dt + I^{(i)} \le 2Q\int_{D(s)} |\nabla(u_i - \phi_i)|^{p_i}\,dx\,dt,$$

for all sufficiently large $i \in \mathbb{N}$, where $\phi_i := \eta(u_i - u)$. This together with (5.14) implies that

$$\begin{aligned} &\int_{D(\sigma)} |\nabla u_i|^{p_i}\,dx\,dt + I^{(i)} \le 2Q\int_{D(s)} |\nabla(u_i - \phi_i)|^{p_i}\,dx\,dt \\ &\le C\int_{D(s)} \left((1-\eta)^{p_i}|\nabla u_i|^{p_i} + |\nabla\eta|^{p_i}|u_i - u|^{p_i} + \eta^{p_i}|\nabla u|^{p_i}\right)dx\,dt \\ &\le C\int_{D(s)\setminus D(\sigma)} |\nabla u_i|^{p_i}\,dx\,dt + C\int_{D(s)} \left(|\nabla\eta|^{p_i}|u_i - u|^{p_i} + \eta^{p_i}|\nabla u|^{p_i}\right)dx\,dt, \end{aligned}$$

for some constant C which is independent of i.

Then, adding

$$C\int_{D(\sigma)} |\nabla u_i|^{p_i}\,dx\,dt$$

to the both hand sides of the above inequality, we obtain

$$\begin{aligned}(1+C)\int_{D(\sigma)} &|\nabla u_i|^{p_i}\,dx\,dt + I^{(i)} \\ &\le C\int_{D(s)} \left(|\nabla u_i|^{p_i} + |\nabla\eta|^{p_i}|u_i-u|^{p_i} + \eta^{p_i}|\nabla u|^{p_i}\right)\,dx\,dt.\end{aligned} \tag{5.17}$$

For $s\in(0,s_0)$, we set

$$\Psi(s) := \limsup_{i\to\infty}\int_{D(s)} |\nabla u_i|^{p_i}\,dx\,dt.$$

Since $D(s)\subset D(s')$ if $s<s'$, the function Ψ is nondecreasing and finite because of the higher integrability of ∇u_i, and we conclude that the set of points of discontinuity of Ψ is at most countable.

Let $s\in(0,s_0)$ be a point of continuity of Ψ. Then, by (5.17) we have

$$\begin{aligned}(1+C)\Psi(\sigma)\le C\Psi(s) + \limsup_{i\to\infty}\int_{D(s)} &|\nabla\eta|^{p_i}|u_i-u|^{p_i}\,dx\,dt \\ &+C\int_{D(s)}|\nabla u|^{p}\,dx\,dt.\end{aligned}$$

Note that to obtain the preceding estimate, we have to see that

$$\int_{D(s)} |\nabla u|^{p_i}\,dx\,dt \to \int_{D(s)} |\nabla u|^{p}\,dx\,dt, \tag{5.18}$$

as $i\to\infty$. To see this, we use the elementary estimate

$$\big||\xi|^a - |\xi|^b\big| \le \left[\frac{1}{\varepsilon}|\xi|^{\max\{a,b\}+\varepsilon} + \frac{1}{\varepsilon}\Big(\frac{1}{a}+\frac{1}{b}\Big)\right]|a-b|,$$

for all $\xi\in\mathbb{R}^k$, $a,b>0$ and $\varepsilon>0$, which we take from [9, Proof of Th. 3.4]. We let i be large enough to have $p_i\le p+\delta/2$ and apply the preceding estimate with the choices $\xi = Du$, $a=p_i$, $b=p$ and $\varepsilon=\delta/2$, where $\delta>0$ denotes the higher integrability exponent Proposition 3.6, to obtain

$$\int_{D(s)} \big||\nabla u|^{p_i} - |\nabla u|^{p}\big|\,dx\,dt \le \left[\frac{2}{\delta}\int_{D(s)}|\nabla u|^{p+\delta}\,dx\,dt + C\right]|p_i-p|,$$

where the constant C depends on $\bar p$ and $|\Omega_T|$. Letting $i\to\infty$, the right–hand side tends to zero and therefore we conclude the desired convergence (5.18).

On the other hand, as in (5.13), we can obtain

$$\limsup_{i\to\infty}\int_{D(s)} |\nabla\eta|^{p_i}|u_i-u|^{p_i}\,dx\,dt = 0.$$

Therefore we have

$$(1+C)\Psi(\sigma) \le C\Psi(s) + C\int_{D(s)} |\nabla u|^p \,dx\,dt$$

for all $\sigma \in (0,s)$. Since s is a point of continuity of Ψ, we obtain

$$(1+C)\Psi(s) \le C\Psi(s) + C\int_{D(s)} |\nabla u|^p \,dx\,dt,$$

which gives

$$\Psi(s) \le C\int_{D(s)} |\nabla u|^p \,dx\,dt.$$

This completes the proof of Lemma 5.1. □

We continue the proof of Theorem 2.1. Since $\eta \equiv 1$ in a neighborhood of K, we can take a compact set $D \subset \overline{O_1} \setminus K$ satisfying

$$\int_{O_1\setminus K} (1-\eta)^{p_i} |\nabla u_i|^{p_i} \,dx\,dt \le \int_D |\nabla u_i|^{p_i} \,dx\,dt. \tag{5.19}$$

Then we see that $D(s) \subset O_2 \setminus K$ for all sufficiently small $s > 0$, and by Lemma 5.1 we can take constants C_3 and $s > 0$ such that

$$\limsup_{i\to\infty} \int_{D(s)} |\nabla u_i|^{p_i} \,dx\,dt \le C_3 \int_{D(s)} |\nabla u|^p \,dx\,dt.$$

This together with (5.3) and (5.19) implies that

$$\begin{aligned}
\limsup_{i\to\infty} \int_{O_1\setminus K} (1-\eta)^{p_i} |\nabla u_i|^{p_i} \,dx\,dt &\le \limsup_{i\to\infty} \int_{D(s)} |\nabla u_i|^{p_i} \,dx\,dt \\
&\le C_3 \int_{D(s)} |\nabla u|^p \,dx\,dt \\
&\le C_3 \left(\int_{O_2\setminus K} |\nabla u|^{p+\delta} \,dx\,dt \right)^{p/(p+\delta)} (|\Omega|T)^{1-p/(p+\delta)} \\
&\le C_3 (|\Omega|T)^{1-p/(p+\delta)} \alpha^{p/(p+\delta)},
\end{aligned}$$

and by (5.11) we have

$$\int_{O_1\setminus K} |\nabla(u_i - \eta(u_i - u))|^{p_i} \,dx\,dt \le C_1C_2\alpha^{p/(p+\delta)} + C_1C_3(|\Omega|T)^{1-p/(p+\delta)}\alpha^{p/(p+\delta)},$$

where all the appearing constants are independent of α. Therefore, by the arbitrariness of $\alpha > 0$ and (5.9) we obtain

$$-\int_{\Omega_T} u(\partial_t \phi) \,dx\,dt + \frac{1}{p}\int_K |\nabla u|^p \,dx\,dt \le \frac{Q}{p}\int_K |\nabla(u-\phi)|^p \,dx\,dt$$

for all $\phi \in C_o^\infty(\Omega_T)$ with $K = \operatorname{supp}\phi$. Thus we obtain (2.9), and complete the proof of Theorem 2.1. □

Finally we prove our second main result.

Proof of Theorem 2.2. Let $h > 0$ and $\varepsilon > 0$ be sufficiently small constants, and take the test function

$$\phi_\varepsilon^h(x,t) := \chi^h[u_i - u]_\varepsilon$$

in (3.5) with

$$\chi^h(t) := \begin{cases} 0, & 0 \le t \le h, \\ (t-h)/h, & h < t \le 2h, \\ 1, & 2h < t \le T-2h, \\ (T-t-h)/h, & T-2h < t \le T-h, \\ 0, & T-h < t \le T. \end{cases}$$

Then, by (3.5) we obtain

$$\begin{aligned} -\int_{\Omega_T} [u_i]_\varepsilon \partial_t \phi_\varepsilon^h \, dx\, dt + \frac{1}{p_i} \int_{\operatorname{supp}[\phi_\varepsilon^h]_\varepsilon} |\nabla u_i|^{p_i} \, dx\, dt \\ \le \frac{Q_i}{p_i} \int_{\operatorname{supp}[\phi_\varepsilon^h]_\varepsilon} \left|\nabla\left(u_i - [\phi_\varepsilon^h]_\varepsilon\right)\right|^{p_i} dx\, dt. \end{aligned} \tag{5.20}$$

Denote

$$\begin{aligned} \int_{\Omega_T} [u_i]_\varepsilon \partial_t \phi_\varepsilon^h \, dx\, dt &= \int_{\Omega_T} [u_i - u]_\varepsilon \partial_t \phi_\varepsilon^h \, dx\, dt + \int_{\Omega_T} [u]_\varepsilon \partial_t \phi_\varepsilon^h \, dx\, dt \\ &=: I_1 + I_2. \end{aligned}$$

As in the proof of Lemma 4.1, since $u_i|_{t=0} = u|_{t=0} = g$, we obtain

$$\lim_{h\to 0} \lim_{\varepsilon \to 0} I_1 \le 0. \tag{5.21}$$

Furthermore, we have

$$\begin{aligned} I_2 &= -\int_{\Omega_T} (\partial_t [u]_\varepsilon) \chi^h [u_i - u]_\varepsilon \, dx\, dt = -\langle \partial_t [u]_\varepsilon, \chi^h [u_i - u]_\varepsilon \rangle \\ &= -\langle \partial_t [u]_\varepsilon - \partial_t u, \chi^h [u_i - u]_\varepsilon \rangle \\ &\quad - \langle \partial_t u, \chi^h [u_i - u]_\varepsilon - \chi^h (u_i - u) \rangle - \langle \partial_t u, \chi^h (u_i - u) \rangle, \end{aligned}$$

and it follows from Lemma 3.1, (3.6), and (3.7) that there exists a constant $C > 0$ such that

$$\begin{aligned} &|\langle \partial_t([u]_\varepsilon - u), \chi^h [u_i - u]_\varepsilon \rangle| \\ &\qquad \le C \|[u]_\varepsilon - u\|_{L^p(0,T;W^{1,p}(\Omega))}^{p-1} \|\nabla (u_i - u)\|_{L^p(\Omega_T)} \to 0 \end{aligned}$$

and

$$\begin{aligned} &|\langle \partial_t u, \chi^h [u_i - u]_\varepsilon - \chi^h (u_i - u) \rangle| \\ &\qquad \le C \|u\|_{L^p(0,T;W^{1,p}(\Omega))}^{p-1} \|\nabla \{(u_i - u) - [u_i - u]_\varepsilon\}\|_{L^p(\Omega_T)} \to 0, \end{aligned}$$

as $\varepsilon \to 0$. These together with Lemma 3.2 imply that

$$\lim_{\varepsilon \to 0} I_2 = -\langle \partial_t u, \chi^h (u_i - u) \rangle = \langle \partial_t \{\chi^h (u_i - u)\}, u \rangle. \tag{5.22}$$

Since

$$
\begin{aligned}
&|\langle \partial_t\{\chi^h(u_i-u)\}, u\rangle - \langle \partial_t(u_i-u), u\rangle| \\
&\qquad\qquad \le C\|(1-\chi^h)(u_i-u)\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))}\|\nabla u\|_{L^p(\Omega_T)} \to 0,
\end{aligned}
$$

as $h \to 0$, by (4.9) and (5.22) we obtain

$$
\lim_{h\to 0}\lim_{\varepsilon\to 0} I_2 = \langle \partial_t(u_i-u), u\rangle \to 0 \tag{5.23}
$$

as $i \to \infty$. Therefore, applying the similar argument as in the proof of Theorem 2.1, by (5.20)–(5.23) we conclude that

$$
\begin{aligned}
\limsup_{i\to\infty}\int_{\Omega_T}|\nabla u_i|^{p_i}\,dx\,dt &\le \limsup_{i\to\infty} Q_i\int_{\Omega_T}|\nabla u|^{p_i}\,dx\,dt \\
&= \int_{\Omega_T}|\nabla u|^{p}\,dx\,dt.
\end{aligned} \tag{5.24}
$$

This is the only point, where $Q_i \to 1$ comes into play.

Let $\gamma > 0$ be a sufficiently small constant and $i \in \mathbb{N}$ be sufficiently large such that $|p_i - p| < \gamma$. Then, by the Hölder inequality we have

(5.25)

$$
\begin{aligned}
&\int_{\Omega_T}|\nabla u_i|^{p}\,dx\,dt = \int_{\Omega_T}|\nabla u_i|^{p-\gamma}|\nabla u_i|^{\gamma}\,dx\,dt \\
&\le \left(\int_{\Omega_T}|\nabla u_i|^{p_i}\,dx\,dt\right)^{(p-\gamma)/p_i}\left(\int_{\Omega_T}|\nabla u_i|^{\gamma p_i/(p_i-p+\gamma)}\,dx\,dt\right)^{(p_i-p+\gamma)/p_i}.
\end{aligned}
$$

Let q_i be the constant such that

$$
\frac{\gamma p_i}{p_i-p+\gamma}q_i = p+\delta, \quad \text{that is,} \quad q_i = (p+\delta)\frac{p_i-p+\gamma}{\gamma p_i},
$$

where $\delta > 0$ is given in Lemma 4.3. Then, since $q_i > 1$ for sufficiently large i, by the Hölder inequality we have

$$
\begin{aligned}
&\left(\int_{\Omega_T}|\nabla u_i|^{\gamma p_i/(p_i-p+\gamma)}\,dx\,dt\right)^{(p_i-p+\gamma)/p_i} \\
&\quad \le \left(\int_{\Omega_T}|\nabla u_i|^{\gamma p_i q_i/(p_i-p+\gamma)}\,dx\,dt\right)^{(p_i-p+\gamma)/(p_i q_i)}(|\Omega|T)^{(p_i-p+\gamma)(1-1/q_i)/p_i} \\
&\quad = \left(\int_{\Omega_T}|\nabla u_i|^{p+\delta}\,dx\,dt\right)^{\delta/(p+\delta)}(|\Omega|T)^{(p_i-p+\gamma)/p_i-\delta/(p+\delta)}.
\end{aligned}
$$

This together with (4.6) implies that there exists a positive constant C, which is independent of γ, such that

$$
\begin{aligned}
&\limsup_{i\to\infty}\left(\int_{\Omega_T}|\nabla u_i|^{\gamma p_i/(p_i-p+\gamma)}\,dx\,dt\right)^{(p_i-p+\gamma)/p_i} \\
&\qquad\qquad \le C^{\gamma}(|\Omega|T)^{\gamma(1/p-1/(p+\delta))}.
\end{aligned} \tag{5.26}
$$

Then, by (5.24), (5.25), and (5.26) we obtain

$$\limsup_{i\to\infty}\int_{\Omega_T}|\nabla u_i|^p\,dx\,dt \le \left(\int_{\Omega_T}|\nabla u|^p\,dx\,dt\right)^{(p-\gamma)/p} C^{\gamma}(|\Omega|T)^{\gamma(1/p-1/(p+\gamma))}.$$

Since $\gamma > 0$ is arbitrary, we conclude that

$$\limsup_{i\to\infty}\int_{\Omega_T}|\nabla u_i|^p\,dx\,dt \le \int_{\Omega_T}|\nabla u|^p\,dx\,dt. \tag{5.27}$$

On the other hand, since it follows from corollary 4.2 that

$$\nabla u_i \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega_T) \tag{5.28}$$

as $i\to\infty$, we have

$$\int_{\Omega_T}|\nabla u|^p\,dx\,dt \le \liminf_{i\to\infty}\int_{\Omega_T}|\nabla u_i|^p\,dx\,dt,$$

and by (5.27) we obtain

$$\lim_{i\to\infty}\|\nabla u_i\|_{L^p(\Omega_T)} = \|\nabla u\|_{L^p(\Omega_T)}.$$

This together with (5.28) implies that $\nabla u_i \to \nabla u$ in $L^p(\Omega_T)$ as $i\to\infty$. Thus we complete the proof of Theorem 2.2. □

References

[1] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$, J. London Math. Soc. **34** (1986), 274–290.

[2] J. Habermann, Global gradient estimates for non–quadratic vector–valued parabolic quasi–minimizers, submitted for publication.

[3] L. I. Hedberg and T. Kilpeläinen, On the stability of Sobolev spaces with zero boundary values, Math. Scand. **85** (1999), 245–258.

[4] T. Kipeläinen, J. Kinnunen, and O. Martio, Sobolev spaces with zero boundary values on metric spaces, Potential Anal. **12** (2000), 233–247.

[5] J. Kinnunen, N. Marola, M. Miranda jr., and F. Paronetto, Harnack's inequality for parabolic De Giorgi classes in metric spaces, Adv. Differential Equations, **17** (2012), 801–832.

[6] J. Kinnunen, N. Marola and O. Martio, Harnack's principle for quasiminimizers, Ric. Mat. **56** (2007), 73-88.

[7] J. Kinnunen and O. Martio, Potential theory of quasiminimizers, Ann. Acad. Sci. Fenn. Math. **28** (2003), 459–490.

[8] J. Kinnunen and M. Masson, Parabolic comparison principle and quasiminimizers in metric measure spaces, to appear in Proc. Amer. Math. Soc.

[9] J. Kinnunen and M. Parviainen, Stability for degenerate parabolic equations, Adv. Calculus of Variations **3** (2010), 29–48.

[10] J. L. Lewis, Uniformly fat sets, Trans. Amer. Math. Soc. **308** (1988), 177–196.

[11] G. Li and O. Martio, Stability of solutions of varying degenerate elliptic equations, Indiana Univ. Math. J., **47** (1998) (3), 873–891.

[12] P. Lindqvist, Stability for the solutions of $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ with varying p, J. Math. Anal. Appl., **127** (1987)(1), 93–102.

[13] P. Lindqvist, On nonlinear Rayleigh quotients, Potential Anal. **2**(1993)(3), 199–218.

[14] O. E. Maasalo and A. Zatorska-Goldstein, Stability of quasiminimizers of the p-Dirichlet integral with varying p on metric spaces, J. London Math. Soc. (2) **77** (2008), 771–788.

[15] N. Marola and M. Masson, On the weak Harnack inequality for parabolic minimizers in metric measure spaces, to appear in Tohoku Math. J.
[16] M. Masson, M. Miranda jr, F. Paronetto and M. Parviainen, Local higher integrability for parabolic quasiminimizers in metric spaces, to appear in Ric. Mat., DOI:10.1007/s11587-013-0150-z.
[17] M. Masson and M. Parviainen, Global higher integrability for parabolic quasiminimizers in metric measure spaces, to appear in J. Anal. Math.
[18] M. Masson and J. Siljander, Hölder regularity for parabolic De Giorgi classes in metric measure spaces, to appear in Manuscripta Math., DOI: 10.1007/s00229-012-0598-2.
[19] M. Parviainen, Global higher integrability for parabolic quasiminimizers in nonsmooth domains, Calc. Var. Partial Differential Equations **31** (2008), 75–98.
[20] R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Mathematical Surveys and Monographs **49** (1997), American Mathematical Society, Providence, RI.
[21] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. **146** (1987), 65–96.
[22] W. Wieser, Parabolic Q-minima and minimal solutions to variational flows, Manuscripta Math. **59** (1987) (1), 63–107.
[23] V.V. Zhikov, Meyer-type estimates for solving the nonlinear Stokes system, Differ. Uravn. **33** (1997), 107–114, 143; translation in Differential equations, **33** (1997), 108–115.
[24] V.V. Zhikov, On some variational problems, Russian J. Math. Phys. **5** (1998), 105–116.
[25] S. Zhou, On the local behaviour of parabolic Q-minima, J. Partial Differential Equations **6** (1993), 255–272.
[26] S. Zhou, Parabolic Q-minima and their application, J. Partial Differential Equations **7** (1994), 289–322.

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