HIGHER INTEGRABILITY FOR PARABOLIC SYSTEMS OF p-LAPLACIAN TYPE

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Abstract. We show that the gradient of a solution to a parabolic system of p-Laplacian type in $\mathbf{R}^n$ satisfies a reverse Hölder inequality provided $p > 2n/(n+2)$. In particular, this implies the local higher integrability of the gradient.

1. Introduction

In this work we study regularity of solutions to second order parabolic systems

$$\frac{\partial u_i}{\partial t} = \operatorname{div} A_i(x,t,\nabla u) + B_i(x,t,\nabla u), \qquad i = 1,\dots,N. \tag{1.1}$$

In particular, we are interested in systems of p-Laplacian type. We present more precise structural assumptions later, but the principal prototype that we have in mind is the p-parabolic system

$$\frac{\partial u_i}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u_i), \qquad i = 1,\dots,N,$$

with $1 < p < \infty$. As usual, solutions to (1.1) are taken in a weak sense and they are assumed to belong to a parabolic Sobolev space. A good source for the regularity theory is [D].

In the elliptic case when the system is

$$\operatorname{div} A_i(x,t,\nabla u) + B_i(x,t,\nabla u) = 0, \qquad i = 1,\dots,N, \tag{1.2}$$

it is known that solutions locally belong to a slightly higher Sobolev space than assumed a priori. This self improving property was first observed by Elcrat and Meyers in [ME] (see also [Gi] and [Str]). Their argument is based on reverse Hölder inequalities and a modification of Gehring's lemma [Ge] which originally was developed to study the higher integrability of the Jacobian of a quasiconformal mapping. In the elliptic case higher integrabilty results play a decisive role in studying the regularity of solutions, see [GM] and [Gi].

The purpose of this work is to obtain higher integrablity results in the p-parabolic setting. We prove that the gradient of a weak solution to (1.1) satisfies a reverse

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Hölder inequality for $p > 2n/(n+2)$. The critical exponent $2n/(n+2)$ occurs also in parabolic regularity theory, see [D]. We note that reverse Hölder inequalities and the local higher integrability for weak solutions were already proved for $p = 2$ by [GS] (see also [C]). Our result appears to be new even in the scalar case if $p \neq 2$.

One of the difficulties in proving our main result is that a solution does not remain a solution under multiplication by a constant which is not 0 nor 1. Since reverse Hölder inequalities are invariant under multiplication by a constant, we have to choose a class of cylinders whose side lengths depend on the size of the function in order to obtain a reverse Hölder inequality as in [Ge] and then higher integrability.

It seems to us that our results can be used to extend partial regularity results in [GM] for nonlinear elliptic systems to cover some parabolic systems. For $p = 2$ this was done in [GS], but our method applies also when $p \neq 2$.

2. Preliminaries

In order to be more precise about the structure and solutions of the system (1.1) we need some notation. Let $\Omega \subset \mathbf{R}^n$ be an open set and let $W^{1,p}(\Omega)$ denote the Sobolev space of real valued functions g such that $g \in L^p(\Omega)$ and the distributional first partial derivatives $\partial g/\partial x_i$, $i = 1, 2, \dots, n$, exist in Ω and belong to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|g\|_{1,p,\Omega} = \|g\|_{p,\Omega} + \sum_{i=1}^{n} \left\|\partial g/\partial x_i\right\|_{p,\Omega}.$$

Given $O \subset \mathbf{R}^n$ open, N a positive integer, $-\infty \le S < T \le \infty$, let

$$u = (u_1, \dots, u_N)\colon O \times (S,T) \to \mathbf{R}^N$$

and suppose that whenever $-\infty \le S < S_1 < T_1 < T \le \infty$ and $\overline{\Omega} \subset O$ we have

$$u \in L^2\big(\Omega \times [S_1, T_1]\big) \cap L^p\big([S_1, T_1]; W^{1,p}(\Omega)\big). \tag{2.1}$$

Here the notation $L^p\big([S_1,T_1]; W^{1,p}(\Omega)\big)$ means that for almost every t, $S_1 < t < T_1$, with respect to one dimensional Lebesgue measure, the function $x \mapsto u(x,t)$ is in $W^{1,p}(\Omega)$ componentwise and

$$|||u|||_{p,\Omega}^p = \|u\|_{p,\Omega\times(S_1,T_1)}^p + \int_{S_1}^{T_1} \sum_{i=1}^{N} \left\|u_i(\cdot,t)\right\|_{1,p,\Omega}^p \, dt < \infty. \tag{2.2}$$

Let ∇u denote the distributional gradient of u (taken componentwise) in the x variable only.

We suppose that $A = (A_1, \dots, A_N)$, where

$$A_i = A_i(x,t,\nabla u)\colon O \times (S,T) \times \mathbf{R}^{nN} \to \mathbf{R}^n,$$

and $B = (B_1, \dots, B_N)$, where

$$B_i = B_i(x,t,\nabla u)\colon O \times (S,T) \times \mathbf{R}^{nN} \to \mathbf{R},$$

are Lebesgue $(n+1)$-measurable functions on $O \times (S,T)$. This is the case, for example, if A_i and B_i, $i = 1,2,\ldots,N$, satisfy the well known Carathéodory type conditions. We assume that there exist positive constants c_i, $i = 1,2,3$, such that

$$|A_i| \le c_1 |\nabla u|^{p-1} + h_1 \tag{2.3}$$

$$|B_i| \le c_2 |\nabla u|^{p-1} + h_2, \tag{2.4}$$

and

$$\sum_{i=1}^{N} \langle A_i, \nabla u_i \rangle \ge c_3 |\nabla u|^p - h_3, \tag{2.5}$$

for $i = 1,2,\ldots,N$, and almost every $(x,t) \in O \times (S,T)$. Here $\langle \cdot , \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^n$ and h_i, $i = 1,2,3$, are measurable functions in $O \times (S,T)$ so that

$$\big\| (|h_1| + |h_2|)^{p/(p-1)} + |h_3| \big\|_{\widehat{q}, O \times (S,T)} = c_4 < \infty, \tag{2.6}$$

where $\widehat{q} > 1$.

Finally u satisfying (2.1) is said to be a weak solution in $O \times (S,T)$ to the nonlinear parabolic system

$$\frac{\partial u_i}{\partial t} = \operatorname{div} A_i(x,t,\nabla u) + B_i(x,t,\nabla u), \qquad i = 1,\ldots,N,$$

if the structural conditions (2.3)-(2.6) hold and

$$\int_S^T \int_O \sum_{i=1}^{N} \Big(- u_i \frac{\partial \phi_i}{\partial t} + \langle A_i, \nabla \phi_i \rangle - B_i \phi_i \Big)\, dx\, dt = 0 \tag{2.7}$$

for every test function $\phi = (\phi_1, \ldots, \phi_N) \in C_0^\infty(O \times (S,T))$.

The following theorem is our main result.

Theorem 2.8. *Let $p > 2n/(n+2)$ and suppose that u is a weak solution to (1.1). Then there exists $\varepsilon > 0$ such that*

$$u \in L^2\big(\Omega \times [S_1,T_1]\big) \cap L^{p+\varepsilon}\big([S_1,T_1]; W^{1,p+\varepsilon}(\Omega)\big),$$

where $\varepsilon > 0$ depends only on n, p, $\widehat{q}$, and c_i, for $i = 1,2,3$, while $|||u|||_{p+\varepsilon,\Omega}$ depends on these quantities as well as N, Ω, S_1, T_1 and c_4.

The proof of our main result follows from two propositions in Section 4.

3. Fundamental estimates

In this section we state and outline the proofs of some Sobolev and Caccioppoli type lemmas which will be used in the proof of the main result. To do this we shall need some notation. Given r, $s > 0$, $(x, t) \in \mathbf{R}^{n+1}$, let

$$D_r(x) = \{y \in \mathbf{R}^n : |y - x| < r\}$$

denote the open ball in $\mathbf{R}^n$ and

$$Q_{r,s}(x, t) = D_r(x) \times (t - s, t + s)$$

a cylinder in $\mathbf{R}^{n+1}$. Let $|E|$ denote the Lebesgue $(n+1)$-measure of the measurable set E and if f is integrable on E with $0 < |E| < \infty$, then the integral average of f over E is

$$\fint_E f\,dx\,dt = \frac{1}{|E|}\int_E f\,dx\,dt.$$

If $Q_{\rho,s}(z, \tau) \subset O \times (S, T)$, then

$$I_\rho(t) = I_\rho(t, u, z, \tau) = m(D_\rho(z))^{-1}\int_{D_\rho(z)} u(x, t)\,dx,$$

whenever $\tau - s < t < \tau + s$. Here m denotes Lebesgue measure in $\mathbf{R}^n$ and the integral is taken componentwise.

Lemma 3.1. *Suppose that u is a weak solution to* (1.1)*. If $Q_{4\rho,s}(z, \tau) \subset O \times (S, T)$, then there exists $\widehat{\rho}$, $\rho < \widehat{\rho} < 2\rho$, and a constant c depending on p, n, c_1 and c_2, such that*

$$|I_{\widehat{\rho}}(t_2) - I_{\widehat{\rho}}(t_1)| \le cs\rho^{-1}\fint_{Q_{2\rho,s}(z,\tau)} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\,dx\,dt$$

for almost all t_i, $\tau - s < t_i < \tau + s$, $i = 1, 2$.

Proof. To prove this lemma, let δ, $\eta > 0$ be small, $\rho < \widehat{\rho} < 2\rho$, $t_1 < t_2$, $\psi_1 \in C_0^\infty(t_1 - \eta, t_2 + \eta)$ with $\psi_1 = 1$ on the interval (t_1, t_2), and let $\psi_2 \in C_0^\infty(D_{\widehat{\rho}+\delta}(z))$ be a radial function with $\psi_2 = 1$ on $D_{\widehat{\rho}}(z)$. For fixed $j = 1, 2, \dots, N$, we put $\phi_j = \psi_1\psi_2$ and $\phi_i = 0$ otherwise. Using (2.7) and letting first $\eta \to 0$ and then $\delta \to 0$ we get from well known Sobolev type arguments that for almost every t_1, t_2 and $\widehat{\rho}$, as above,

$$\begin{aligned} m(D_{\widehat{\rho}}(z))\big(I_{\widehat{\rho}}(t_2, u_j) - I_{\widehat{\rho}}(t_1, u_j)\big) &= \int_{\partial D_{\widehat{\rho}}(z)\times(t_1,t_2)} \langle x - z, A_j(x, t)\rangle\,|x - z|^{-1}\,d\sigma(x)\,dt \\ &\quad + \int_{D_{\widehat{\rho}}(z)\times(t_1,t_2)} B_j\,dx\,dt. \end{aligned}$$

Here σ denotes $(n-1)$-dimensional surface area on $\partial D_{\widehat{\rho}}(z)$. Choose $\widehat{\rho}$, $\rho < \widehat{\rho} < 2\rho$, so that

$$\begin{aligned}\int_{\partial D_{\widehat{\rho}}(z)\times(t_1,t_2)} &\big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\, d\sigma\, dt \\ &\le 100\rho^{-1} \int_{D_{2\rho}(z)\times(t_1,t_2)} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\, dx\, dt.\end{aligned}$$

Using this choice, (2.3), (2.4) in the above inequality, and summing over $j = 1, 2, \dots, N$, we deduce the claim. □

The following lemma is a Caccioppoli type estimate for parabolic systems of $p-$Laplacian type. For short we write

$$h^p = (|h_1| + |h_2|)^{p/(p-1)} + |h_3|.$$

Lemma 3.2. *Let u be a weak solution to* (1.1) *and $a = (a_1, \dots, a_N) \in \mathbf{R}^N$. Then there exists a constant c depending on n, N, p, c_i for $i = 1, 2, 3, 4$, such that if $Q_{\rho_1,s_1}(z,\tau) \subset Q_{\rho_2,s_2}(z,\tau) \subset O \times (S,T)$ with $0 < \rho_2, s_2 < 1$, then we have*

$$\begin{aligned}\int_{Q_{\rho_1,s_1}(z,\tau)} &|\nabla u|^p\, dx\, dt + \operatorname*{ess\,sup}_{t\in(\tau-s_1,\tau+s_1)} \int_{D_{\rho_1}(z)} |u-a|^2\, dx \\ \le& c\,(s_2 - s_1)^{-1} \int_{Q_{\rho_2,s_2}(z,\tau)} |u-a|^2\, dx\, dt \\ &+ c\,(\rho_2-\rho_1)^{-p} \int_{Q_{\rho_2,s_2}(z,\tau)} |u-a|^p\, dx\, dt + c\int_{Q_{\rho_2,s_2}(z,\tau)} h^p\, dx\, dt.\end{aligned}$$

Proof. Lemma 3.2 follows from a standard Caccioppoli type estimate obtained from (2.7) by formally choosing test functions of the form

$$\phi_i = (u-a)_i\, \psi_i^p, \qquad i = 1, 2, \dots, N,$$

where $\psi_i \in C_0^\infty(Q_{\rho_2,s_2}(z,\tau))$ is a cutoff function with $\psi_i = 1$ on $Q_{\rho_1,s_1}(z,\tau)$, $0 \le \psi_i \le 1$ and

$$(\rho_2-\rho_1)^{-1}\big\||\nabla\psi_i|\big\|_\infty + (s_2-s_1)^{-1}\Big\|\frac{\partial\psi_i}{\partial t}\Big\|_\infty < 1000.$$

There is a difficulty with the test functions ϕ_i since the solution usually has a very modest degree of regularity with respect to the time variable. We refer the reader to [D, p. 24–27] for an argument to overcome this difficulty. □

Next we prove a Sobolev type inequality.

Lemma 3.3. *Let $1 \le \nu < \infty$ and suppose that $u \in L^\nu\big((\tau-2s,\tau+2s); W^{1,\nu}(D_{2\rho}(z))\big)$. Then there is a constant c depending on n and ν such that*

$$\int_{Q_{\rho,s}(z,\tau)} |u(x,t)-I_\rho(t)|^{\nu(1+2/n)}\,dx\,dt \le c\int_{Q_{2\rho,2s}(z,\tau)} |\nabla u(x,t)|^\nu\,dx\,dt \cdot\Big(\operatorname*{ess\,sup}_{t\in(\tau-2s,\tau+2s)}\int_{D_{2\rho}(z)} |u(x,t)-I_\rho(t)|^2\,dx\Big)^{\nu/n}.$$

Proof. Let t, $\tau-2s<t<\tau+2s$, be such that $x\mapsto u(x,t)$ belongs to $W^{1,\nu}(D_{2\rho}(z))$ and denote $\rho^*=2\rho$. Let $\psi\in C_0^\infty(Q_{2\rho,2s}(z,\tau))$ be a cutoff function such that $\psi=1$ on $Q_{\rho,s}(z,\tau)$ and $|\nabla\psi|\le 10/\rho$. Let

$$v(x,t)=|u(x,t)-I_\rho(t)|\psi(x,t).$$

Hölder's inequality implies that

$$\begin{aligned}J&=\int_{D_{\rho^*}(z)} v(x,t)^{\nu(1+2/n)}\,dx\\ &\le\Big(\int_{D_{\rho^*}(z)} v(x,t)^2\,dx\Big)^{1/n}\Big(\int_{D_{\rho^*}(z)} v(x,t)^{(\nu+(2/n)(\nu-1))n/(n-1)}\,dx\Big)^{(n-1)/n}.\end{aligned}$$

We use Sobolev's theorem for functions in $W^{1,1}(D_{\rho^*}(z))$ to deduce that there is constant $c=c(n)$ such that

$$\begin{aligned}&\Big(\int_{D_{\rho^*}(z)} v(x,t)^{(\nu+(2/n)(\nu-1))n/(n-1)}\,dx\Big)^{(n-1)/n}\\ &\qquad\le c\int_{D_{\rho^*}(z)} v(x,t)^{(\nu-1)(1+2/n)}|\nabla v(x,t)|\,dx\\ &\qquad\le c\Big(\int_{D_{\rho^*}(z)} v(x,t)^{\nu(1+2/n)}\,dx\Big)^{(\nu-1)/\nu}\Big(\int_{D_{\rho^*}(z)} |\nabla v(x,t)|^\nu\,dx\Big)^{1/\nu}.\end{aligned}$$

Thus

$$J\le cJ^{(\nu-1)/\nu}\Big(\int_{D_{\rho^*}(z)} |\nabla v(x,t)|^\nu\,dx\Big)^{1/\nu}\Big(\int_{D_{\rho^*}(z)} v(x,t)^2\,dx\Big)^{1/n}.$$

Clearly

$$\Big(\int_{D_{\rho^*}(z)} |\nabla v(x,t)|^\nu\,dx\Big)^{1/\nu}\le c{\rho^*}^{-1}\Big(\int_{D_{\rho^*}(z)} |u(x,t)-I_\rho(t)|^\nu\,dx\Big)^{1/\nu} +\Big(\int_{D_{\rho^*}(z)} |\nabla u(x,t)|^\nu\,dx\Big)^{1/\nu}.$$

Poincaré's inequality in $W^{1,\nu}(D_{\rho^*}(z))$ implies that

$$\begin{aligned}\Big(\int_{D_{\rho^*}(z)} &|u(x,t)-I_\rho(t)|^\nu\,dx\Big)^{1/\nu}\\ &\le\Big(\int_{D_{\rho^*}(z)}|u(x,t)-I_{\rho^*}(t)|^\nu\,dx\Big)^{1/\nu}+|I_{\rho^*}(t)-I_\rho(t)|\,m(D_{\rho^*}(z))^{1/\nu}\\ &\le c\Big(\int_{D_{\rho^*}(z)}|u(x,t)-I_{\rho^*}(t)|^\nu\,dx\Big)^{1/\nu}\\ &\le c\rho^*\Big(\int_{D_{\rho^*}(z)}|\nabla u(x,t)|^\nu\,dx\Big)^{1/\nu}\end{aligned}$$

and hence

$$\Big(\int_{D_{\rho^*}(z)}|\nabla v(x,t)|^\nu\,dx\Big)^{1/\nu}\le c\Big(\int_{D_{\rho^*}(z)}|\nabla u(x,t)|^\nu\,dx\Big)^{1/\nu}.$$

The same argument as above gives

$$\begin{aligned}\Big(\int_{D_{\rho^*}(z)}v(x,t)^2\,dx\Big)^{1/n}&\le c\Big(\int_{D_{\rho^*}(z)}|u(x,t)-I_\rho(t)|^2\,dx\Big)^{1/n}\\ &\le c\Big(\int_{D_{\rho^*}(z)}|u(x,t)-I_{\rho^*}(t)|^2\,dx\Big)^{1/n}.\end{aligned}$$

Collecting the obtained estimates we arrive at

$$J\le c\int_{D_{\rho^*}(z)}|\nabla u(x,t)|^\nu\,dx\Big(\int_{D_{\rho^*}(z)}|u(x,t)-I_{\rho^*}(t)|^2\,dx\Big)^{\nu/n}.$$

The claim follows by integrating this inequality with respect to t over the interval $(\tau-2s,\tau+2s)$. Observe that the proof applies to the case $n=1$ as well. □

The following two lemmas are essential tools in proving our main result. We divide the discussion into two parts depending on whether $p\ge 2$ or $2n/(n+2)<p<2$.

Lemma 3.4. *Let u be a weak solution to* (1.1) *with $p\ge 2$. Suppose that $\lambda>0$, $s=\lambda^{2-p}\rho^2$, and $Q_{40\rho,40^2s}(z,\tau)\subset O\times(S,T)$. Denote $Q=Q_{\rho,s}(z,\tau)$, $Q'=Q_{4\rho,4^2s}(z,\tau)$ and $Q''=Q_{20\rho,20^2s}(z,\tau)$. If there is $c_5\ge 1$ such that*

$$\begin{aligned}c_5^{-1}\lambda^p&\le\fint_Q\big(|\nabla u|^p+h^p\big)\,dx\,dt\\ &\le c_5\fint_{Q''}\big(|\nabla u|^p+h^p\big)\,dx\,dt\le c_5^2\lambda^p,\end{aligned}$$

then there is $c\ge 1$ such that

$$\fint_{Q''}|\nabla u|^p\,dx\,dt\le c\Big(\fint_{Q'}|\nabla u|^q\,dx\,dt\Big)^{p/q}+c\fint_{Q'}h^p\,dx\,dt,$$

where $q = \max\{p-1, pn/(n+2)\}$. The constant c has the same dependence as the constant in Lemma 3.2, except that it also depends on c_5.

Proof. First suppose that $p \geq 2$. From Lemma 3.2 with $\rho_1 = \rho$, $s_1 = s$, $\rho_2 = 2\rho$ and $s_2 = 2s$, we have

$$\begin{aligned}
&\fint_{Q_{\rho,s}(z,\tau)} |\nabla u|^p \,dx\,dt \leq cs^{-1} \fint_{Q_{2\rho,2s}(z,\tau)} |u-a|^2 \,dx\,dt \\
&\qquad + c\rho^{-p} \fint_{Q_{2\rho,2s}(z,\tau)} |u-a|^p \,dx\,dt + c \fint_{Q_{2\rho,2s}(z,\tau)} h^p \,dx\,dt \\
&\qquad = T_1 + T_2 + c \fint_{Q_{2\rho,2s}(z,\tau)} h^p \,dx\,dt.
\end{aligned} \tag{3.5}$$

Since $p \geq 2$ we may estimate T_1 in terms of T_2 using Hölder's and Young's inequalities and the assumption that $s = \lambda^{2-p}\rho^2$ as

$$\begin{aligned}
T_1 &\leq c\lambda^{p-2}\rho^{-2} \fint_{Q_{2\rho,2s}(z,\tau)} |u-a|^2 \,dx\,dt \\
&\leq \lambda^p/(4c_5) + cT_2,
\end{aligned} \tag{3.6}$$

where $c \geq 1$ has the same dependence as c in Lemma 3.2 except that it also depends on c_5.

Hence it is enough to estimate T_2. By Lemma 3.1 we choose $\widehat{\rho}$, $2\rho < \widehat{\rho} < 4\rho$, so that

$$|I_{\widehat{\rho}}(t) - I_{\widehat{\rho}}(\xi)| \leq cs\rho^{-1} \fint_{Q'} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big) \,dx\,dt, \tag{3.7}$$

for almost every ξ, $\tau - 2s < \xi < \tau + 2s$. Let $\widehat{Q} = Q_{\widehat{\rho},2s}(z,\tau)$ and in (3.5) take

$$a = a(\widehat{Q}) = (a_1(\widehat{Q}), \dots, a_N(\widehat{Q})), \quad \text{where} \quad a_i(\widehat{Q}) = \fint_{\widehat{Q}} u_i \,dx\,dt,$$

for $i = 1, 2, \dots, N$. Then we have

$$T_2 \leq c\rho^{-p} \fint_{\widehat{Q}} |u - I_{\widehat{\rho}}(t)|^p \,dx\,dt + c\rho^{-p} \operatorname*{ess\,sup}_{t \in (\tau-2s,\tau+2s)} |I_{\widehat{\rho}}(t) - a(\widehat{Q})|^p. \tag{3.8}$$

We begin with estimating the second term on the right side of (3.8). Using (3.7) we have

$$\begin{aligned}
|I_{\widehat{\rho}}(t) - a(\widehat{Q})| &\leq (4s)^{-1} \int_{\tau-2s}^{\tau+2s} |I_{\widehat{\rho}}(t) - I_{\widehat{\rho}}(\xi)| \,d\xi \\
&\leq cs\rho^{-1} \fint_{Q'} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big) \,dx\,dt
\end{aligned} \tag{3.9}$$

and hence using the definition of λ we obtain

$$
\begin{aligned}
c\rho^{-p} &\operatorname*{ess\,sup}_{t\in(\tau-2s,\tau+2s)} |I_{\widehat{\rho}}(t) - a(\widehat{Q})|^p \\
&\le c\lambda^{p(2-p)}\Big(\fint_{Q'} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\,dx\,dt\Big)^p \\
&\le c\Big(\fint_{Q'} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\,dx\,dt\Big)^{p/(p-1)} \\
&\le c\Big(\fint_{Q'} |\nabla u|^{p-1}\,dx\,dt\Big)^{p/(p-1)} + c\fint_{Q'} h^p\,dx\,dt.
\end{aligned}
\tag{3.10}
$$

Observe that the assumption $p \ge 2$ is used in the second inequality above.

Next we estimate the first term on the right side of (3.8). Lemma 3.3 implies that

$$
\begin{aligned}
&\fint_{\widehat{Q}} |u - I_{\widehat{\rho}}(t)|^p\,dx\,dt \\
&\quad\le c\fint_{\widetilde{Q}} |\nabla u|^q\,dx\,dt\,\Big(\operatorname*{ess\,sup}_{t\in(\tau-4s,\tau+4s)} \int_{D_{\widetilde{\rho}}(z)} |u - I_{\widetilde{\rho}}(t)|^2\,dx\Big)^{q/n},
\end{aligned}
\tag{3.11}
$$

where $q = pn/(n+2)$, $\widetilde{Q} = Q_{2\widehat{\rho},4s}(z,\tau)$ and $\widetilde{\rho} = 2\widehat{\rho}$.

We estimate the essential supremum on the right side of (3.11). Let $Q^* = Q_{10\rho,10s}(z,\tau)$. Clearly

$$
\begin{aligned}
&\int_{D_{\widetilde{\rho}}(z)} |u - I_{\widetilde{\rho}}(t)|^2\,dx \\
&\quad\le c\int_{D_{\widetilde{\rho}}(z)} |u - a(Q^*)|^2\,dx + c\,m(D_{\widetilde{\rho}}(z))\,|a(Q^*) - I_{\widetilde{\rho}}(t)|^2 \\
&\quad\le c\int_{D_{\widetilde{\rho}}(z)} |u - a(Q^*)|^2\,dx
\end{aligned}
$$

and hence using Lemma 3.2 with $a = a(Q^*)$ we have

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(\tau-4s,\tau+4s)} \int_{D_{\widetilde{\rho}}(z)} |u - I_{\widetilde{\rho}}(t)|^2\,dx \\
&\quad\le c \operatorname*{ess\,sup}_{t\in(\tau-4s,\tau+4s)} \int_{D_{\widetilde{\rho}}(z)} |u - a(Q^*)|^2\,dx \\
&\quad\le cs^{-1}\int_{Q^*} |u - a(Q^*)|^2\,dx\,dt + c\rho^{-p}\int_{Q^*} |u - a(Q^*)|^p\,dx\,dt,
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
s^{-1}\int_{Q^*} |u - a(Q^*)|^2\,dx\,dt &\le cs^{-1}\int_{Q^*} |u - I_{10\rho}(t)|^2\,dx\,dt \\
&\quad + cs^{-1}|Q^*| \operatorname*{ess\,sup}_{t\in(\tau-10s,\tau+10s)} |I_{10\rho}(t) - a(Q^*)|^2
\end{aligned}
\tag{3.13}
$$

and

$$\rho^{-p}\int_{Q^*}|u-a(Q^*)|^p\,dx\,dt\le c\rho^{-p}\int_{Q^*}|u-I_{10\rho}(t)|^p\,dx\,dt \\ +c\rho^{-p}|Q^*|\operatorname*{ess\,sup}_{t\in(\tau-10s,\tau+10s)}|I_{10\rho}(t)-a(Q^*)|^p. \tag{3.14}$$

By Poincaré's inequality in $W^{1,2}(D_{10\rho}(z))$ we have

$$\begin{aligned} s^{-1}\int_{Q^*}|u-I_{10\rho}(t)|^2\,dx\,dt&=s^{-1}\int_{\tau-10s}^{\tau+10s}\int_{D_{10\rho}(z)}|u-I_{10\rho}(t)|^2\,dx\,dt\\ &\le cs^{-1}\rho^2\int_{Q^*}|\nabla u|^2\,dx\,dt\le cs^{-1}\rho^2\Big(\fint_{Q^*}|\nabla u|^p\,dx\,dt\Big)^{2/p}|Q^*|\\ &\le c\rho^{n+2}\lambda^2. \end{aligned}\tag{3.15}$$

Here we used the assumption that $p\ge 2$ again. Exactly the same argument gives

$$\rho^{-p}\int_{Q^*}|u-I_{10\rho}(t)|^p\,dx\,dt\le c\int_{Q^*}|\nabla u|^p\,dx\,dt\le c\rho^{n+2}\lambda^2. \tag{3.16}$$

Using Lemma 3.1 we choose $\widetilde{\rho}$, $10\rho<\widetilde{\rho}<20\rho$, such that

$$|I_{\widetilde{\rho}}(t)-I_{\widetilde{\rho}}(\xi)|\le cs\rho^{-1}\fint_{Q''}\big(|\nabla u|^{p-1}+|h_1|+|h_2|\big)\,dx\,dt,$$

when $\tau-10s<\xi<\tau+10s$. This implies that

$$\begin{aligned} s^{-1}|Q^*|&\operatorname*{ess\,sup}_{t\in(\tau-10s,\tau+10s)}|I_{10\rho}(t)-a(Q^*)|^2\\ &\le cs^2\rho^{n-2}\Big(\fint_{Q''}\big(|\nabla u|^p+h^p\big)\,dx\,dt\Big)^{2(p-1)/p}\le c\rho^{n+2}\lambda^2. \end{aligned}\tag{3.17}$$

A similar argument (see (3.10)) also gives

$$\begin{aligned} \rho^{-p}|Q^*|&\operatorname*{ess\,sup}_{t\in(\tau-10s,\tau+10s)}|I_{10\rho}(t)-a(Q^*)|^p\\ &\le c\rho^n s\lambda^{p(2-p)}\Big(\fint_{Q''}\big(|\nabla u|^{p-1}+|h_1|+|h_2|\big)\,dx\,dt\Big)^p\le c\rho^{n+2}\lambda^2. \end{aligned}\tag{3.18}$$

Using (3.12)–(3.18) we conclude that

$$\operatorname*{ess\,sup}_{t\in(\tau-4s,\tau+4s)}\int_{D_{\widetilde{\rho}}(z)}|u-I_{\widetilde{\rho}}(t)|^2\,dx\le c\rho^{n+2}\lambda^2.$$

By (3.11) and Young's inequality we see that the first term on the right side of (3.8) can be estimated as

$$\begin{aligned} c\rho^{-p}\fint_{\widehat{Q}}|u-I_{\widehat{\rho}}(t)|^p\,dx\,dt&\le c\lambda^{2q/n}\fint_{\widetilde{Q}}|\nabla u|^q\,dx\,dt\\ &\le c\Big(\fint_{\widetilde{Q}}|\nabla u|^q\,dx\,dt\Big)^{p/q}+\lambda^p/(4c_5). \end{aligned}$$

Finally using (3.5), (3.6), (3.8) and (3.10) we have

$$\begin{aligned}\fint_{Q} |\nabla u|^p \, dx\, dt &\le \lambda^p/(2c_5) + c\Big(\fint_{Q'} |\nabla u|^q \, dx\, dt\Big)^{p/q} \\ &\quad + c\Big(\fint_{Q'} |\nabla u|^{p-1} \, dx\, dt\Big)^{p/(p-1)} + c\fint_{Q'} h^p \, dx\, dt.\end{aligned} \tag{3.19}$$

The claim follows from this estimate by absorbing the term containing λ^p into the left side. □

Next we prove an analogue of Lemma 3.4 for $2n/(n+2) < p < 2$.

Lemma 3.20. *Let u be a weak solution to* (1.1) *with $2n/(n+2) < p < 2$. Suppose that $\lambda > 0$, $s = \lambda^{2-p}\rho^2$, and $Q_{40\rho,40^2s}(z,\tau) \subset O \times (S,T)$. Denote $Q = Q_{\rho,s}(z,\tau)$, $Q' = Q_{4\rho,4^2s}(z,\tau)$ and $Q'' = Q_{20\rho,20^2s}(z,\tau)$. If there is $c_6 \ge 1$ such that*

$$\begin{aligned}c_6^{-1}\lambda^p &\le \fint_{Q} \big(|\nabla u|^p + s^{-1}|u - a(Q)|^2 + h^p\big)\, dx\, dt \\ &\le c_6 \fint_{Q''} \big(|\nabla u|^p + s^{-1}|u - a(Q'')|^2 + h^p\big)\, dx\, dt \le c_6^2\lambda^p,\end{aligned}$$

then there is $c \ge 1$ such that

$$\fint_{Q''} |\nabla u|^p \, dx\, dt \le c\Big(\fint_{Q'} |\nabla u|^q \, dx\, dt\Big)^{p/q} + c\fint_{Q'} h^p \, dx\, dt,$$

where $q = 2n/(n+2)$. The constant c has the same dependence as the constant in Lemma 3.2, except that it also depends on c_6.

Proof. We use the same notation as in the proof of Lemma 3.4. Clearly (3.5) also holds this case. Since $p < 2$ we use Hölder's inequality to estimate T_2 in (3.5) in terms of T_1 and obtain

$$\begin{aligned}T_2 &\le c\Big(\rho^{-2}\fint_{Q_{2\rho,2s}(z,\tau)} |u - a|^2 \, dx\, dt\Big)^{p/2} \\ &\le c\lambda^{(1-p/2)p}\, T_1^{p/2} \le \lambda^p/(4c_6) + c\,T_1.\end{aligned} \tag{3.21}$$

To estimate T_1 by Lemma 3.1 we choose $\widehat{\rho}$, $2\rho < \widehat{\rho} < 4\rho$, so that (3.7) holds. Let $\widehat{Q} = Q_{\widehat{\rho},2s}(z,\tau)$. Using the same argument which led to (3.8) we see that

$$T_1 \le cs^{-1}\fint_{\widehat{Q}} |u - I_{\widehat{\rho}}(t)|^2 \, dx\, dt + cs^{-1} \operatorname*{ess\,sup}_{t\in(\tau-2s,\tau+2s)} |I_{\widehat{\rho}}(t) - a(\widehat{Q})|^2. \tag{3.22}$$

Using (3.9), the definition of λ and Young's inequality, we obtain

$$\begin{aligned}&s^{-1} \operatorname*{ess\,sup}_{t\in(\tau-2s,\tau+2s)} |I_{\widehat{\rho}}(t) - a(\widehat{Q})|^2 \\ &\le c\lambda^{2-p}\Big(\fint_{Q'} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\, dx\, dt\Big)^2 \\ &\le \lambda^p/(4c_6) + c\Big(\fint_{Q'} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\, dx\, dt\Big)^{p/(p-1)} \\ &\le \lambda^p/(4c_6) + c\Big(\fint_{Q'} |\nabla u|^{p-1} \, dx\, dt\Big)^{p/(p-1)} + c\fint_{Q'} h^p \, dx\, dt.\end{aligned} \tag{3.23}$$

To estimate the first term on the right side of (3.22) we use Lemma 3.3 and argue first as in (3.11) to get

$$
\begin{aligned}
&\fint_{\widehat{Q}} |u - I_{\widehat{\rho}}(t)|^2 \, dx \, dt \\
&\leq c \fint_{\widetilde{Q}} |\nabla u|^q \, dx \, dt \Big(\operatorname*{ess\,sup}_{t \in (\tau - 4s, \tau + 4s)} \int_{D_{\widetilde{\rho}}(z)} |u - I_{\widetilde{\rho}}(t)|^2 \, dx \Big)^{q/n},
\end{aligned}
\tag{3.24}
$$

where $q = 2n/(n+2)$, $\widetilde{Q} = Q_{2\widehat{\rho},4s}(z,\tau)$ and $\widetilde{\rho} = 2\widehat{\rho}$. The essential supremum on the right side of (3.24) is then estimated as in (3.12) and we obtain

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t \in (\tau - 4s, \tau + 4s)} \int_{D_{\widetilde{\rho}}(z)} |u - I_{\widetilde{\rho}}(t)|^2 \, dx &\leq c s^{-1} \int_{Q^*} |u - a(Q^*)|^2 \, dx \, dt \\
&\quad + c\rho^{-p} \int_{Q^*} |u - a(Q^*)|^p \, dx \, dt,
\end{aligned}
\tag{3.25}
$$

where $Q^* = Q_{10\rho,10s}(z,\tau)$ as before. Using the assumption of the lemma and remembering that $s = \lambda^{2-p}\rho^2$, we have

$$
s^{-1} \int_{Q^*} |u - a(Q^*)|^2 \, dx \, dt \leq c\lambda^p |Q^*| \leq c\rho^{n+2}\lambda^2.
$$

The second term on the right side of (3.25) can be estimated exactly the same way as in the case $p \geq 2$, see (3.14),(3.16) and (3.18). We conclude that

$$
\operatorname*{ess\,sup}_{t \in (\tau - 4s, \tau + 4s)} \int_{D_{\widetilde{\rho}}(z)} |u - I_{\widetilde{\rho}}(t)|^2 \, dx \leq c\rho^{n+2}\lambda^2.
$$

By (3.24) and Young's inequality we arrive at

$$
\begin{aligned}
c s^{-1} \fint_{\widehat{Q}} |u - I_{\widehat{\rho}}(t)|^2 \, dx \, dt &\leq c\lambda^{p-q} \fint_{\widetilde{Q}} |\nabla u|^q \, dx \, dt \\
&\leq c \Big(\fint_{\widetilde{Q}} |\nabla u|^q \, dx \, dt \Big)^{p/q} + \lambda^p/(4c_6).
\end{aligned}
\tag{3.26}
$$

The claim now follows from using (3.22), (3.23) and (3.26) as before by absorbing the term containing λ^p into the left side. This completes the proof. □

3.27. Remark. We record for the future reference that the constant c in Lemmas 3.4 and 3.20 remain bounded above if p is in a compact subset of $(2n/(n+2), \infty)$. This is easily seen by analyzing the constants in Lemmas 3.1, 3.2 and 3.3.

4. Reverse Hölder inequalities

In this section we show that gradients of weak solutions of (1.1) satisfy a reverse Hölder inequality provided $p > 2n/(n+2)$. As in the previous section, slightly different arguments are needed to handle the cases $p \geq 2$ and $2n/(n+2) < p < 2$. First we study the case $p \geq 2$.

Proposition 4.1. *Let u be a weak solution to* (1.1) *when $p \geq 2$ and suppose that $Q_{4R,(4R)^p}(z,\tau) \subset O \times (S,T)$, where $0 < R < 1$. Then there exist $\varepsilon > 0$ and $c \geq 1$ having the same dependence as the corresponding constants in Theorem 2.8 with*

$$\Big(\fint_{Q_{R,R^p}(z,\tau)} |\nabla u|^{p+\varepsilon}\,dx\,dt\Big)^{1/(p+\varepsilon)} \leq cR^{\sigma p-1}\Big(\fint_{Q_{2R,(2R)^p}(z,\tau)} |\nabla u|^p\,dx\,dt\Big)^{\sigma} + cR^{-1} + c\Big(\fint_{Q_{2R,(2R)^p}(z,\tau)} h^{p+\varepsilon}\,dx\,dt\Big)^{1/(p+\varepsilon)},$$

where $\sigma = (2+\varepsilon)/(2(p+\varepsilon))$.

Proof. To prove Proposition 4.1 we assume, as we may, that $R = 1$ and $(z,\tau) = (0,0)$, since otherwise we consider

$$v(x,t) = u(z+Rx, \tau+R^p t)$$

for $(x,t) \in Q_{4,4^p}(0,0)$. It is easily seen that v is a weak solution to a partial differential equation similar to (1.1) and with the same structure. Proving Proposition 4.1 for v with $R = 1$ relative to (0,0) and then transforming back we get Proposition 4.1 for u.

For short we denote $\widetilde{Q} = Q_{2,2^p}(0,0)$. To begin the proof of Proposition 4.1 we divide $\widetilde{Q}$ into Whitney type cylinders

$$Q_i = Q_{r_i,r_i^2}(z_i,\tau_i), \qquad i = 1,2,\dots,$$

so that r_i is comparable to the parabolic distance of Q_i to the boundary of $\widetilde{Q}$. Let us recall that the parabolic distance of sets E, $F \subset \mathbf{R}^{n+1}$ is

$$\inf\big\{|x-y| + |t-s|^{1/2} \colon (x,t) \in E \text{ and } (y,s) \in F\big\}.$$

Moreover the cylinders Q_i, $i = 1,2,\dots$, are of bounded overlap and

$$Q_{5r_i,(5r_i)^2}(z_i,\tau_i) \subset \widetilde{Q}.$$

Next for $(x,t) \in \widetilde{Q}$ we define

$$g(x,t) = \big(|\nabla u| + h\big)(x,t)$$

and

$$f(x,t) = \widehat{c}^{\,-1} \min\big\{|Q_i|^{1/2} \colon (x,t) \in Q_i\big\}\, g(x,t),$$

where $\widehat{c} \geq 1$ will be chosen later.

Let

$$\lambda_0^2 = \fint_{\widetilde{Q}} |g|^p\,dx\,dt$$

and $\lambda > \max\{\lambda_0, 1\} = \lambda_0'$. Suppose that $(x,t) \in Q_i$ with $|f(x,t)| > \lambda$. We set

$$\alpha = |Q_i|^{-1} \qquad \text{and} \qquad \gamma = \alpha^{1-p/2}\lambda^{2-p}.$$

If $r_i/20 \le r \le r_i$, then for $\widehat{c}$ large enough we have

$$\fint_{Q_{r,\gamma r^2}(x,t)} |g|^p \,dx\,dt \le c\gamma^{-1}\alpha \fint_{\widetilde{Q}} |g|^p \,dx\,dt \le \widehat{c}^p\lambda^p\alpha^{p/2}. \tag{4.2}$$

By Lebesgue's differentiation theorem, we have for almost every such (x,t) that

$$\lim_{r\to 0} \fint_{Q_{r,\gamma r^2}(x,t)} |g|^p \,dx\,dt > \widehat{c}^p\lambda^p\alpha^{p/2}. \tag{4.3}$$

From (4.2), (4.3) and continuity of the integral we see that there exists ρ, $0 < \rho < r_i/20$, such that

$$\fint_{Q_{\rho,\gamma\rho^2}(x,t)} |g|^p \,dx\,dt = \widehat{c}^p\lambda^p\alpha^{p/2} \tag{4.4}$$

and

$$\fint_{Q_{r,\gamma r^2}(x,t)} |g|^p \,dx\,dt \le \widehat{c}^p\lambda^p\alpha^{p/2} \tag{4.5}$$

for $\rho \le r \le r_i$.

Let $s = \gamma\rho^2$ and denote

$$Q = Q_{\rho,s}(x,t), \quad Q' = Q_{4\rho,4^2 s}(x,t), \quad \text{and} \quad Q'' = Q_{20\rho,20^2 s}(x,t).$$

Since λ, $\alpha > 1$ and $p \ge 2$ we have $\gamma \le 1$. This implies that $Q'' \subset \widetilde{Q}$.

Now (4.4) and (4.5) imply that there is a constant $c \ge 1$ such that

$$c^{-1}\lambda^p\alpha^{p/2} \le \fint_{Q} |g|^p \,dx\,dt \le c \fint_{Q''} |g|^p \,dx\,dt \le c^2\lambda^p\alpha^{p/2}. \tag{4.6}$$

Observe that

$$s = \gamma\rho^2 = \left(\lambda\alpha^{1/2}\right)^{2-p}\rho^2.$$

Thus we can apply Lemma 3.4 with λ replaced by $\lambda\alpha^{1/2}$. Note that c_5 in this case depends only on n and p. From Lemma 3.4 we conclude for

$$q = \max\{p-1, pn/(n+2)\}$$

and $c \ge 1$ that

$$\fint_{Q''} |\nabla u|^p \,dx\,dt \le c\left(\fint_{Q'} |\nabla u|^q \,dx\,dt\right)^{p/q} + c\fint_{Q'} h^p \,dx\,dt. \tag{4.7}$$

Using (4.6) and (4.7) we have

$$\begin{aligned} c^{-1}\lambda^p \le \fint_{Q''} |f|^p \,dx\,dt &\le c\left(\fint_{Q'} |f|^q \,dx\,dt\right)^{p/q} + c\fint_{Q'} k \,dx\,dt \\ &\le c^2 \fint_{Q'} |f|^p \,dx\,dt \le c^3\lambda^p, \end{aligned} \tag{4.8}$$

where

$$k(x,t) = \min\left\{|Q_i|^{p/2} \colon (x,t) \in Q_i\right\} h^p(x,t).$$

Let $G(\lambda) = \left\{(x,t) \in \widetilde{Q} \colon |f(x,t)| > \lambda\right\}$ and $\eta > 0$. Then by (4.8) we obtain

$$\begin{aligned}\Big(\fint_{Q'} |f|^q\,dx\,dt\Big)^{p/q} &\le c\eta^p\lambda^p + \Big(|Q'|^{-1}\int_{Q'\cap G(\eta\lambda)} |f|^q\,dx\,dt\Big)^{p/q} \\ &\le c\eta^p \Big(\fint_{Q'} |f|^q\,dx\,dt\Big)^{p/q} + c\eta^p \fint_{Q'} k\,dx\,dt \\ &\quad + c\lambda^{p-q}|Q'|^{-1}\int_{Q'\cap G(\eta\lambda)} |f|^q\,dx\,dt.\end{aligned} \tag{4.9}$$

A similar argument gives

$$\begin{aligned}\fint_{Q'} k\,dx\,dt &\le c\eta^p \Big(\fint_{Q'} |f|^q\,dx\,dt\Big)^{p/q} + c\eta^p \fint_{Q'} k\,dx\,dt \\ &\quad + c\,|Q'|^{-1}\int_{Q'\cap G(\eta\lambda)} k\,dx\,dt.\end{aligned} \tag{4.10}$$

Choosing $\eta > 0$ small enough in (4.9), (4.10) and absorbing terms we arrive at

$$\begin{aligned}&\Big(\fint_{Q'} |f|^q\,dx\,dt\Big)^{p/q} + \fint_{Q'} k\,dx\,dt \\ &\quad\le c\lambda^{p-q}|Q'|^{-1}\int_{Q'\cap G(\eta\lambda)} |f|^q\,dx\,dt + c\,|Q'|^{-1}\int_{Q'\cap G(\eta\lambda)} k\,dx\,dt.\end{aligned} \tag{4.11}$$

An examination of the proof of the well known Vitali type covering lemma shows that we can choose pairwise disjoint cylinders

$$Q_i' = Q_{4\rho_i,\gamma(4\rho_i)^2}(x_i,t_i), \qquad i = 1,2,\dots,$$

such that almost everywhere

$$G(\lambda) \subset \bigcup_{i=1}^{\infty} Q_i'' \subset \widetilde{Q},$$

where

$$Q_i'' = Q_{20\rho_i,\gamma(20\rho_i)^2}(x_i,t_i), \qquad i = 1,2,\dots$$

From (4.8) and (4.11) we deduce that for some small $\eta > 0$ we have

$$\begin{aligned}c^{-1}\lambda^p &\le \fint_{Q_i''} |f|^p\,dx\,dt \\ &\le c\lambda^{p-q}\fint_{Q_i'} |f|^q\,dx\,dt + c\fint_{Q_i'} k\,dx\,dt \\ &\le c^2\lambda^{p-q}\,|Q_i'|^{-1}\int_{Q_i'\cap G(\eta\lambda)} |f|^q\,dx\,dt + c^2\,|Q_i'|^{-1}\int_{Q_i'\cap G(\eta\lambda)} k\,dx\,dt \\ &\le c^3\lambda^p.\end{aligned} \tag{4.12}$$

Multiplying (4.12) by $|Q'|$ and summing over i we get from (4.12) and disjointness of the cylinders Q_i', $i=1,2,\dots$, that

$$
\begin{aligned}
\int_{G(\lambda)} |f|^p\,dx\,dt &\le \sum_i \int_{Q_i''} |f|^p\,dx\,dt \\
&\le c\lambda^{p-q} \sum_i \int_{Q_i'\cap G(\eta\lambda)} |f|^q\,dx\,dt + \sum_i \int_{Q_i'\cap G(\eta\lambda)} k\,dx\,dt \qquad (4.13)\\
&\le c\lambda^{p-q} \int_{G(\eta\lambda)} |f|^q\,dx\,dt + c\int_{G(\eta\lambda)} k\,dx\,dt.
\end{aligned}
$$

We can now apply a standard argument to complete the proof of Proposition 4.1. For completeness we sketch it. Using Fubini's theorem and (4.13) we have

$$
\begin{aligned}
\int_{G(\lambda_0')} |f|^{p+\varepsilon}\,dx\,dt &= \varepsilon \int_{\lambda_0'}^{\infty} \lambda^{\varepsilon-1}\Big(\int_{G(\lambda)} |f|^p\,dx\,dt\Big)\,d\lambda \\
&\quad + {\lambda_0'}^{\varepsilon} \int_{G(\lambda_0')} |f|^p\,dx\,dt \\
&\le c\varepsilon \int_{\lambda_0'}^{\infty} \lambda^{\varepsilon-1+p-q}\Big(\int_{G(\eta\lambda)} |f|^q\,dx\,dt\Big)\,d\lambda \\
&\quad + c\varepsilon \int_{\lambda_0'}^{\infty} \lambda^{\varepsilon-1+p-q}\Big(\int_{G(\eta\lambda)} k\,dx\,dt\Big)\,d\lambda + {\lambda_0'}^{\varepsilon} \int_{G(\lambda_0')} |f|^p\,dx\,dt \\
&\le \frac{c\varepsilon}{\varepsilon+p-q} \int_{G(\lambda_0')} |f|^{p+\varepsilon}\,dx\,dt + c\int_{G(\lambda_0')} |f|^{\varepsilon} k\,dx\,dt \\
&\quad + {\lambda_0'}^{\varepsilon} \int_{G(\lambda_0')} |f|^p\,dx\,dt.
\end{aligned}
$$

By Young's inequality we obtain

$$
\int_{G(\lambda_0')} |f|^{\varepsilon} k\,dx\,dt \le \varepsilon \int_{G(\lambda_0')} |f|^{\varepsilon+p}\,dx\,dt + c\int_{G(\lambda_0')} k^{1+\varepsilon/p}\,dx\,dt.
$$

Choosing $\varepsilon>0$ small enough we may absorb the integrals involving $|f|^{p+\varepsilon}$ into the left side and we obtain

$$
\int_{G(\lambda_0')} |f|^{p+\varepsilon}\,dx\,dt \le c{\lambda_0'}^{\varepsilon} \int_{G(\lambda_0')} |f|^p\,dx\,dt + c\int_{G(\lambda_0')} k^{1+\varepsilon/p}\,dx\,dt.
$$

Observe that there is a difficulty in moving terms to the left side since they may be infinite. This technical problem can be treated, for example, by truncating the function f. To be more precise, let $\Lambda > \lambda_0'$ and denote $f_\Lambda = \min\{|f|,\Lambda\}$. If $|f|$ is replaced by f_Λ in the definition of $G(\lambda)$, we see that (4.13) holds with $|f|$ replaced by f_Λ, and we can go through the above argument. Now all absorbed terms are finite and we obtain the claim passing to the limit as $\Lambda\to\infty$. Thus

$$
\begin{aligned}
\int_{\widetilde{Q}} |f|^{p+\varepsilon}\,dx\,dt &\le {\lambda_0'}^{\varepsilon} \int_{\widetilde{Q}\setminus G(\lambda_0')} |f|^p\,dx\,dt + \int_{G(\lambda_0')} |f|^{p+\varepsilon}\,dx\,dt \\
&\le c{\lambda_0'}^{\varepsilon} \int_{\widetilde{Q}} |f|^p\,dx\,dt + c\int_{\widetilde{Q}} k^{1+\varepsilon/p}\,dx\,dt.
\end{aligned}
$$

Proposition 4.1 follows easily from this inequality. □

Then we prove a counterpart of Proposition 4.1 when $2n/(n+2) < p < 2$.

Proposition 4.14. *Let u be a weak solution to (1.1) when $2n/(n+2) < p < 2$ and suppose that $Q_{4R,(4R)^p}(z,\tau) \subset O \times (S,T)$, where $0 < R < 1$. Then there exist $\varepsilon > 0$ and $c \geq 1$ having the same dependence as the corresponding constants in Theorem 2.8 with*

$$\Big(\fint_{Q_{R,R^p}(z,\tau)} |\nabla u|^{p+\varepsilon}\, dx\, dt\Big)^{1/(p+\varepsilon)}$$
$$\leq cR^{-1}\Big(\int_{Q_{2R,(2R)^p}(z,\tau)} |u - a(Q_{2R,(2R)^p}(z,\tau))|^2\, dx\, dt\Big)^{\nu}$$
$$+ cR^{-1} + c\Big(\fint_{Q_{2R,(2R)^p}(z,\tau)} h^{p+\varepsilon}\, dx\, dt\Big)^{1/(p+\varepsilon)},$$

where $\nu = \big(2\varepsilon + (n+2)p - 2n\big)/\big((p+\varepsilon)((n+2)p - 2n)\big)$.

Proof. Again we divide $\widetilde{Q} = Q_{2,2^p}(0,0)$ into Whitney type cylinders Q_i, $i = 1, 2, \ldots,$ exactly in the same way as in the proof of Proposition 4.1. Next for $(x,t) \in \widetilde{Q}$ put

$$g(x,t) = \big(|\nabla u| + h\big)(x,t).$$

Let

$$\lambda_0^{((n+2)p-2n)/2} = \fint_{\widetilde{Q}} |u - a(\widetilde{Q})|^2\, dxdt, \tag{4.15}$$

and $\lambda > \max\{\lambda_0, 1\} = \lambda_0'$. For $(x,t) \in \widetilde{Q}$ we define

$$f(x,t) = \widehat{c}^{\,-1} \min\big\{|Q_i|^{\sigma} \colon (x,t) \in Q_i\big\}\, g(x,t),$$

where

$$\sigma = \frac{2n+8}{(n+2)\big((n+2)p - 2n\big)}$$

and $\widetilde{c} \geq 1$ will be chosen later. Suppose $(x,t) \in Q_i$ with $|f(x,t)| > \lambda$. Put $\alpha = |Q_i|^{-\sigma}$ and let $\gamma = (\lambda\alpha)^{2-p}$. Again by Lebesgue's differentiation theorem, we have for almost every such (x,t) that

$$\lim_{r \to 0} \fint_{Q_{r,\gamma r^2}(x,t)} |g|^p\, dx\, dt > \widehat{c}^{\,p} \lambda^p \alpha^p.$$

Also if $r = (\lambda\alpha)^{p/2-1} r_i'$ with $r_i/20 \leq r_i' \leq r_i$, then $Q_{r,\gamma r^2}(x,t) \subset \widetilde{Q}$ and for $\widehat{c}$ large enough we have from Lemma 3.2 and Hölder's inequality

$$\fint_{Q_{r,\gamma r^2}(x,t)} \big(|g|^p + \gamma^{-1} r^{-2} |u - a(Q_{r,\gamma r^2}(x,t))|^2\big)\, dx\, dt$$
$$\leq c(n) r_i^{-(n+4)} (\lambda\alpha)^{(1-p/2)n} \fint_{\widetilde{Q}} \big(h^p + 1 + |u - a(\widetilde{Q})|^2\big)\, dx\, dt$$
$$< \widehat{c}^{\,p} \lambda^p \alpha^p,$$

since $p > 2n/(n+2)$. We again use continuity of the integral to find ρ, $0 < \rho < (\lambda\alpha)^{p/2-1} r_i/20$ such that

$$\fint_{Q_{\rho,\gamma\rho^2}(x,t)} \big(|g|^p + \gamma^{-1}\rho^{-2}|u - a(Q_{\rho,\gamma\rho^2}(x,t))|^2\big)\,dx\,dt = \widehat{c}^p\lambda^p\alpha^p \tag{4.16}$$

and

$$\fint_{Q_{r,\gamma r^2}(x,t)} \big(|g|^p + \gamma^{-1}r^{-2}|u - a(Q_{r,\gamma r^2}(x,t))|^2\big)\,dx\,dt \le \widehat{c}^p\lambda^p\alpha^p \tag{4.17}$$

for $\rho \le r \le (\lambda\alpha)^{p/2-1} r_i$. From (4.16) and (4.17) we see that Lemma 3.20 can be applied with $Q = Q_{\rho,\gamma\rho}(x,t)$ and λ replaced by $\lambda\alpha$. We can now repeat the proof of Proposition 4.1 essentially verbatim from (4.8) on to get Proposition 4.14. We omit the details. □

The proof of Theorem 2.8 follows easily from Proposition 4.1, Proposition 4.14 and Lemma 3.3.

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