THE DISCRETE MAXIMAL OPERATOR IN METRIC SPACES

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Abstract. We study the action of so-called discrete maximal operator on Newtonian, Hölder and BMO spaces on metric measure spaces equipped with a doubling measure and a Poincaré inequality. The discrete maximal operator has better regularity properties than the standard Hardy-Littlewood maximal operator and hence it is a more flexible tool in this context.

1. Introduction

By the maximal function theorem of Hardy, Littlewood and Wiener, the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbb{R}^n)$ when $1 < p \leq \infty$. The action of the maximal operator on some other function spaces is rather well understood as well. Indeed, Bennett, DeVore and Sharpley showed in [3] that the maximal operator is bounded on BMO (functions of bounded mean oscillation), provided it is not indentically infinity. It is also known that the maximal operator is bounded on the first order Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ when $1 < p \leq \infty$. For this, we refer to [17]. In particular, when $p = \infty$, this implies that the maximal operator is bounded on Lipschitz continuous functions and a similar argument shows that it is bounded on Hölder continuous functions as well. In [8], Chiarenza and Frasca showed that the maximal operator is bounded on Morrey spaces and they also gave a very elegant proof of the boundedness on BMO. For other related results see, for example, [2], [14], [23], [24] and [26].

In this work, we consider the action of the maximal function on Sobolev spaces, called Newtonian spaces, in metric measure spaces, see [31]. Many boundedness properties of the maximal operator follow from the fact that the maximal operator commutes with translations. It is clear that we do not have this property available in a metric space. Moreover, a slightly unexpected phenomenon was observed by Buckley in [7]. Indeed, he gave an example which shows that the Hardy-Littlewood maximal function of a Lipschitz continuous function

Date: February 7, 2009.

2000 *Mathematics Subject Classification.* 42B25, 35J60, 46E35.

may even fail to be continuous. Under an additional assumption on the measure, called the annular decay property, Buckley proved that the standard maximal operator maps Lipschitz continuous functions to Hölder continuous functions. Thus differentiablity properties are not preserved, in general. Under a similar but stronger condition MacManus showed in [28] that the maximal operator preserves Hajłasz type Sobolev spaces. A suitably modified version of his result holds for Newtonian spaces as well.

These phenomena clearly indicate that the standard Hardy-Littlewood maximal operator may not be the correct object to study in the point of view of regularity. In this work, we consider a discrete maximal operator, which seems to have better regularity properties. Its definition is based on the approximations of the function in terms of partitions of unity and Whitney type coverings. This kind of maximal function has been studied in [18] in connection with pointwise behaviour of Newtonian functions defined in the whole space. See also [1] and [22]. The main objective of this work is to focus on the case, when the maximal function is defined in a subdomain. Some of our results were announced and sketched already in [22], but here we provide detailed arguments. For almost all practical purposes, we can replace the standard maximal operator with the discrete maximal operator, because they are equivalent with two sided inequalities. We show that the discrete maximal operator preserves the Newtonian, Hölder and BMO spaces, if the measure is doubling and the space supports a Poincaré inequality. These are rather standard assumptions in analysis on metric measure spaces. In particular, we do not assume, for example, the annular decay property for the measure. Our results cover function spaces that are relevant in connection with Sobolev embedding theorems.

Acknowledgements. The authors are supported by the Finnish Academy of Science and Letters, the Vilho, Yrjö and Kalle Väisälä Foundation.

2. Preliminaries

2.1. **Doubling measures.** Let $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a Borel regular measure μ such that $0 < \mu(B(x, r)) < \infty$ for all open balls

$$B(x, r) = \{y \in X : d(y, x) < r\}$$

with $r > 0$.

The measure μ is said to be doubling, if there exists a constant $c_\mu \geq 1$, called the doubling constant of μ, such that

$$\mu(B(x, 2r)) \leq c_\mu \mu(B(x, r)),$$

for all $x \in X$ and $r > 0$. Note that an iteration of the doubling property implies, that if $B(x, R)$ is a ball in X, $y \in B(x, R)$ and $0 < r \le R < \infty$, then

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge c\Big(\frac{r}{R}\Big)^{Q} \tag{2.1}$$

for some $c = c(c_\mu)$ and $Q = \log c_\mu / \log 2$. The exponent Q serves as a counterpart of dimension related to the measure.

2.2. **Upper gradients.** A nonnegative Borel function g on X is said to be an upper gradient of a function $u : X \to [-\infty, \infty]$, if for all rectifiable paths $\gamma : [0, 1] \to X$ we have

$$|u(\gamma(0)) - u(\gamma(1))| \le \int_\gamma g\, ds, \tag{2.2}$$

whenever both $u(\gamma(0))$ and $u(\gamma(1))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. The assumption that g is a Borel function is needed in the definition of the path integral. If g is merely a μ-measurable function and (2.2) holds for p-almost every path (i.e. it fails only for a path family with zero p-modulus, $p \ge 1$), then g is said to be a p-weak upper gradient of u. If we redefine a p-weak upper gradient on a set of measure zero we obtain an upper gradient of the same function. If g is a p-weak upper gradient of u, then there is a sequence g_i, $i = 1, 2, \dots$, of upper gradients of u such that

$$\int_X |g_i - g|^p\, d\mu \to 0$$

as $i \to \infty$. Hence every p-weak upper gradient can be approximated by upper gradients in the $L^p(X)$-norm. If u has an upper gradient that belongs to $L^p(X)$ with $p > 1$, then it has a minimal p-weak upper gradient g_u in the sense that for every p-weak upper gradient g of u, $g_u \le g$ almost everywhere.

2.3. **Newtonian spaces.** We define the first order Sobolev spaces on the metric space X using the p-weak upper gradients. These spaces are called Newtonian spaces. For $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p\, d\mu + \inf_g \int_X g^p\, d\mu\Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. The same definition applies to subsets of X as well. The notion of a p-weak upper gradient is used to prove that $N^{1,p}(X)$ is a Banach space. For the properties of Newtonian spaces we refer to [31], [32] and [5].

2.4. **Capacity.** The p-capacity of a set $E \subset X$ is the number

$$\operatorname{cap}_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E. We say that a property regarding points in X holds p-quasieverywhere (p-q.e.) if the set of points for which the property does not hold has capacity zero. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ p-q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$. Hence, the capacity is the correct gauge for distinguishing between two Newtonian functions.

To be able to compare the boundary values of Sobolev functions we need a Sobolev space with zero boundary values. Let E be a measurable subset of X. The Sobolev space with zero boundary values is the space

$$N^{1,p}_0(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ } p\text{-q.e. in } X \setminus E\}.$$

The space $N^{1,p}_0(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space.

2.5. **Poincaré inequality.** We say that X supports a weak $(1,p)$-Poincaré inequality if there exist constants $c > 0$ and $\tau \geq 1$ such that for all balls $B(x,r) \subset X$, for all locally integrable functions u on X and for all p-weak upper gradients g of u,

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq cr \Big(\fint_{B(x,\tau r)} g^p \, d\mu \Big)^{1/p}, \tag{2.3}$$

where we denote

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Note that since p-weak upper gradients can be approximated by upper gradients in the $L^p(X)$-norm, it would be enough to require the Poincaré inequality for upper gradients only.

By the Hölder inequality it is easy to see that if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. If X is complete and μ doubling, then it is shown in [15] that a weak $(1,p)$-Poincaré inequality implies a weak $(1,q)$-Poincaré inequality for some $q < p$. Hence $(1,p)$-Poincaré inequality is a self improving condition.

2.6. **General assumptions.** Throughout the work, we assume that X is complete, μ is doubling and X supports a weak $(1,p)$-Poincaré inequality. This implies, for example, that Lipschitz functions are dense in $N^{1,p}(X)$ and that the Sobolev embedding theorem holds, see [5].

3. The global maximal function

This section is devoted to the definition and basic properties of the discrete maximal function defined on the whole space. The definition is based on the following four ingredients.

3.1. **Covering of the space.** Let $r > 0$. Since the measure is doubling there are balls $B(x_i, r)$, $i = 1, 2, \dots$, such that

$$X = \bigcup_{i=1}^{\infty} B(x_i, r)$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i, 6r)} \le N < \infty.$$

This means that the dilated balls $B(x_i, 6r)$, $i = 1, 2, \dots$, are of bounded overlap. The constant N depends only on the doubling constant and, in particular, it is independent of r.

3.2. **Partition of unity.** We construct a partition of unity subordinate to the covering $B(x_i, r)$, $i = 1, 2, \dots$, of X. Indeed, there is a family of functions ψ_i, $i = 1, 2, \dots$, such that $0 \le \psi_i \le 1$, $\psi_i = 0$ in $X \setminus B(x_i, 6r)$, $\psi_i \ge \nu$ in $B(x_i, 3r)$, ψ_i is Lipschitz with constant L/r_i with ν and L depending only on the covering, and

$$\sum_{i=1}^{\infty} \psi_i(x) = 1$$

for every $x \in X$. The partition of unity can be constructed by first choosing auxiliary cutoff functions φ_i so that $0 \le \varphi_i \le 1$, $\varphi_i = 0$ on $X \setminus B(x_i, 6r)$, $\varphi_i = 1$ in $B(x_i, 3r)$ and each φ_i is Lipschitz continuous with constant $1/r$. For example, we can take

$$\varphi_i(x) = \begin{cases} 1, & x \in B(x_i, 3r), \\ 2 - \dfrac{d(x, x_i)}{3r}, & x \in B(x_i, 6r) \setminus B(x_i, 3r), \\ 0, & x \in X \setminus B(x_i, 6r). \end{cases}$$

Then we define the functions ψ_i, $i = 1, 2, \dots$, in the partition of unity by

$$\psi_i(x) = \frac{\varphi_i(x)}{\sum_{j=1}^{\infty} \varphi_j(x)}.$$

It is not difficult to see that the defined functions satisfy the required properties.

3.3. **Discrete convolution.** Let $f \in L^1_{\mathrm{loc}}(X)$. Now we are ready to define the approximation of f at the scale of $3r$ by setting

$$f_r(x) = \sum_{i=1}^{\infty} \psi_i(x) f_{B(x_i,3r)}$$

for every $x \in X$. The function f_r is called the discrete convolution of f. The partition of unity and the discrete convolution are standard tools in harmonic analysis on homogeneous spaces, see for example [9] and [27].

3.4. **The global maximal function.** Let r_j, $j = 1, 2, \dots$, be an enumeration of the positive rationals. For every radius r_j we choose balls $B(x_i, r_j)$, $i = 1, 2, \dots$, of X as above. Observe that for each radius there are many possible choices for the covering but we simply take one of those. We define the discrete maximal function in X by

$$M^* f(x) = \sup_j |f|_{r_j}(x)$$

for every $x \in X$. Observe that the defined maximal operator depends on the chosen coverings. However, this is not a serious matter, since we obtain estimates which are independent of the chosen coverings. Indeed, by Lemma 3.1 in [18] there is a constant $c = c(c_\mu)$ such that

$$c^{-1} M f(x) \le M^* f(x) \le c M f(x) \tag{3.1}$$

for every $x \in X$. Here Mf is the standard centered Hardy-Littlewood maximal function

$$M f(x) = \sup \fint_{B(x,r)} |f| \, d\mu,$$

where the supremum is taken over all positive radii r.

4. Local maximal function

The definition of the local maximal function in a subdomain of X is rather similar to that of the global maximal function. The main difference is in the covering argument.

4.1. **Whitney type covering.** To define a discrete maximal function in an open subset of X we apply the following Whitney type covering lemma. A similar covering result has been used in [9] and [27].

Lemma 4.1. *Let Ω be an open subset of X and we assume that the complement of Ω is non-empty. Let $t \in (0,1)$ be a scaling parameter. Then there are balls $B(x_i, r_i)$, $i = 1, 2, \dots$, such that*

$$\Omega = \bigcup_{i=1}^{\infty} B(x_i, r_i)$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i,6r_i)} \le N < \infty$$

in Ω, where N depends only on the doubling constant. Moreover, for every point $x \in B(x_i, 6r_i)$ we have

$$\kappa_1 r_i \le t \operatorname{dist}(x, X \setminus \Omega) \le \kappa_2 r_i.$$

The constants κ_1 and κ_2 are independent of the scale t.

Proof. Fix $s > 1$, for example $s = 2$ will do. For every $x \in \Omega$ let

$$\rho_x = \frac{t(s-1)}{30(s+1)} \operatorname{dist}(x, X \setminus \Omega).$$

It is clear that the union of the balls $B(x, \rho_x)$ with $x \in \Omega$, covers the set Ω. By a covering theorem we get countably many pairwise disjoint balls $B(x_i, \rho_i)$, $i = 1, 2, \dots$, for which the dilated balls $B(x_i, 5\rho_i)$ form a covering of Ω.

Let $r_i = 5\rho_i$. For any ball $B(x_i, r_i)$ in the covering and any $x \in B(x_i, 6r_i)$ we have

$$\frac{12}{s-1} r_i \le t \operatorname{dist}(x, X \setminus \Omega) \le \frac{12s}{s-1} r_i.$$

This gives us the constants κ_1 and κ_2.

To prove the boundedness of the overlap, let $x \in \Omega$. For any ball $B(x_i, r_i)$ for which $x \in B(x_i, 6r_i)$ we have

$$r_i \ge \frac{(s-1)t \operatorname{dist}(x, X \setminus \Omega)}{12s}$$

and

$$B(x_i, r_i) \subset B(x, (s-1)t \operatorname{dist}(x, X \setminus \Omega)).$$

Since the balls $B(x_i, \rho_i)$ are pairwise disjoint and contained in a ball of comparable size (at most $60s$ times the radius of any of them) we conclude that the number of the balls is bounded by a constant only depending on s and on the doubling constant. □

4.2. **The local maximal function.** Let Ω be an open subset of X with $X \setminus \Omega \neq \emptyset$ and assume that $f \in L^1_{\text{loc}}(\Omega)$. Let $0 < t < 1$ be a rational number and consider a Whitney type decomposition of Ω. We construct a partition of unity and the discrete convolution related to the Whitney balls exactly in the same way as in the global case. Let t_j, $j = 1, 2, \dots$, be an enumeration of the positive rationals of the interval $(0, 1)$. For every scale t_j we choose a Whitney type covering as in Lemma 4.1 and construct a discrete convolution $|f|_{t_j}$. We define the local discrete maximal function in Ω by

$$M^*_\Omega f(x) = \sup_j |f|_{t_j}(x)$$

for every $x \in X$. Again the defined maximal operator depends on the chosen coverings.

4.3. **Basic properties.** As a supremum of continuous functions, the discrete maximal function is lower semicontinuous and hence measurable. It is also clear from the definition that the discrete maximal operator is homogeneous in the sense that if $\alpha \in \mathbb{R}$, then

$$M^*_\Omega(\alpha f)(x) = |\alpha| M^*_\Omega f(x)$$

for every $x \in \Omega$. Moreover, the discrete maximal operator is sublinear, which means that

$$M^*_\Omega(f + g)(x) \le M^*_\Omega f(x) + M^*_\Omega g(x)$$

for every $x \in \Omega$.

Let $\sigma \ge 1$. The restricted Hardy-Littlewood maximal function $M_{\sigma,\Omega} f$ is defined as

$$M_{\sigma,\Omega} f(x) = \sup \fint_{B(x,r)} |f| \, d\mu$$

where the supremum is taken over all radii r for which

$$0 < \sigma r < \operatorname{dist}(x, X \setminus \Omega).$$

The word restricted refers to the fact that we restrict the radii of the admissible balls. If $\sigma = 1$, we denote $M_{\sigma,\Omega} f = M_\Omega f$. Moreover, if $\Omega = X$, then $M_\Omega f = Mf$.

Next we show the local counterpart of inequality (3.1).

Lemma 4.2. *Let $\Omega \subset X$ be open and $f \in L^1_{\mathrm{loc}}(\Omega)$. Then there exists a constant $c = c(c_\mu)$ such that*

$$c^{-1} M_{\sigma,\Omega} f(x) \le M^*_\Omega f(x) \le c M_\Omega f(x)$$

for every $x \in \Omega$. Here $\sigma = \kappa_2$, where κ_2 is the constant in Lemma 4.1.

Proof. Fix $x \in \Omega$ and $r > 0$ with $0 < \kappa_2 r < \operatorname{dist}(x, X \setminus \Omega)$. Choose a scale $t \in \mathbb{Q} \cap (0, 1)$ such that

$$\frac{9}{10} \kappa_2 r \le t \operatorname{dist}(x, X \setminus \Omega) \le \kappa_2 r.$$

Take the Whitney type covering $B(x_i, r_i)$, $i = 1, 2, \dots$ of Ω with the parameter t. Then x belongs to some ball $B(x_i, r_i)$ of the covering and $B(x, r) \subset B(x_i, 3r_i)$. Thus

$$\begin{aligned} \fint_{B(x,r)} |f| \, d\mu \le c \fint_{B(x_i,3r_i)} |f| \, d\mu \le c\psi_i(x) \fint_{B(x_i,3r_i)} |f| \, d\mu \\ \le c|f|_t(x) \le cM_\Omega^* f(x), \end{aligned}$$

where we used the fact that $\psi_i \ge \nu > 0$ in $B(x_i, 3r_i)$. The first inequality follows by taking the supremum since the right hand side is independent of the radius r.

Then we prove the second inequality. Fix $x \in \Omega$ and a scale $t \in \mathbb{Q} \cap (0, 1)$. If $x \in B(x_i, 6r_i)$, we have

$$B(x_i, 3r_i) \subset B(x, 9r_i) \subset B(x_i, 24r_i).$$

This implies that

$$\begin{aligned} |f|_t(x) &= \sum_{i=1}^{\infty} \psi_i(x) \fint_{B(x_i,3r_i)} |f| \, d\mu \\ &\le \sum_{i=1}^{\infty} \psi_i(x) \frac{\mu(B(x, 9r_i))}{\mu(B(x_i, 3r_i))} \fint_{B(x,9r_i)} |f| \, d\mu \le cM_\Omega f(x). \end{aligned}$$

Since the right hand side is independent of the scale t, the second inequality follows. □

By the Hardy-Littlewood maximal function theorem for doubling measures (see [9]) we see that the Hardy-Littlewood maximal operator is bounded on $L^p(X)$ when $1 < p \le \infty$ and maps $L^1(X)$ into the weak $L^1(X)$. Since the maximal operators are comparable by Lemma 4.2 we conclude that the same results hold for the discrete maximal operator. In particular, there is a constant $c = c(p, c_\mu) > 0$ such that

$$\begin{aligned} \|M_\Omega^* f\|_{L^p(\Omega)} &\le c\|M_\Omega f\|_{L^p(\Omega)} \le c\|M(f\chi_\Omega)\|_{L^p(X)} \\ &\le c\|f\chi_\Omega\|_{L^p(X)} = c\|f\|_{L^p(\Omega)} \end{aligned} \tag{4.3}$$

whenever $p > 1$. If $p = 1$ there is a constant $c = c(c_\mu)$ such that the weak type estimate

$$\begin{aligned} \mu(\{x \in \Omega : M_\Omega^* f(x) > \lambda\}) &\le \mu(\{x \in \Omega : cM_\Omega f(x) > \lambda\}) \\ &\le \mu(\{x \in X : cM(f\chi_\Omega)(x) > \lambda\}) \\ &\le \frac{c}{\lambda} \int_X |f|\chi_\Omega \, d\mu = \frac{c}{\lambda} \int_\Omega |f| \, d\mu \end{aligned} \tag{4.4}$$

holds for every $\lambda > 0$.

Next we show that the discrete convolution approximates the function almost everywhere.

Lemma 4.5. *Let Ω be an open subset of X and assume that $f \in L^1_{\mathrm{loc}}(\Omega)$. Then $f_t \to f$ almost everywhere in Ω as $t \to 0$.*

Proof. Since the measure μ is doubling almost every point $x \in \Omega$ is a Lebesgue point of a locally integrable function f. Let $x \in \Omega$ be a Lebesgue point of f. Let $t \in (0,1)$ and let $B(x_i, r_i)$, $i = 1, 2, \dots$ be the Whitney type balls given by Lemma 4.1. Define

$$r(t,x) = \sup\left\{d(x,y) : y \in B(x_i, 3r_i), x \in B(x_i, 6r_i),\ i = 1, 2, \dots\right\}.$$

Now for any i with $x \in B(x_i, 6r_i)$ we have $B(x_i, r_i) \subset B(x, r(t,x))$ and thus

$$\frac{\mu(B(x, r(t,x))}{\mu(B(x_i, 3r_i))} \le c,$$

where c depends only on the doubling constant of μ. This gives

$$\fint_{B(x_i,3r_i)} |f - f(x)|\, d\mu \le c \fint_{B(x,r(t,x))} |f - f(x)|\, d\mu$$

from which it follows that

$$\begin{aligned} |f_t(x) - f(x)| &\le \sum_{i=1}^{\infty} \psi_i(x) |f_{B(x_i,3r_i)} - f(x)| \\ &\le c \fint_{B(x,r(t,x))} |f - f(x)|\, d\mu. \end{aligned}$$

Since x is a Lebesgue point,

$$\lim_{t\to 0} |f_t(x) - f(x)| = 0,$$

where we also used the fact that $r(t,x)$ tends to zero as $t \to 0$. This proves the pointwise convergence. □

Remark 4.6. (1) Lemma 4.5 implies that

$$|f(x)| = \lim_{t\to 0} |f|_t(x) \le M^*_\Omega f(x) \tag{4.7}$$

for almost every $x \in \Omega$.

(2) We observe that if $f \in L^p(\Omega)$ for some $1 < p < \infty$, the discrete convolution approximates f in the norm. Indeed, let $f \in L^p(\Omega)$ for some $p > 1$. Then by Lemma 4.2 and the maximal function theorem we have $(M^* f)^p \in L^1(\Omega)$. By definition $f_t^p \le (M^* f)^p$ and thus Lebesgue's dominated convergence theorem gives the claim.

Next we show that, if a maximal function is finite in one point of X, it is finite almost everywhere. As far as we know, the first proof of this fact in the Euclidean case is by Wik [34]. See also [3] and [13]. We state the result only for the global Hardy-Littlewood maximal function, but since the maximal functions are equivalent, the same result also holds for the global discrete maximal function.

Lemma 4.8. *Assume Mf is finite at some $x_0 \in X$. Then Mf is finite almost everywhere.*

Proof. Let $k = 1, 2, \dots$. By sublinearity of the maximal operator

$$Mf(x) \le M(f\chi_{B(x_0,2k)})(x) + M(f\chi_{X\setminus B(x_0,2k)})(x)$$

for every $x \in X$. Since $f\chi_{B(x_0,2k)}$ is integrable, the first term on the right hand side is finite almost everywhere by the weak type estimate (4.4). For the second term let $x \in B(x_0, k)$. Take any ball $B(y, r)$ such that $x \in B(y, r)$ and $B(y, r)$ intersects the complement of the ball $B(x_0, 2k)$. Since $r \ge k$ we have $B(x_0, r) \subset B(y, 3r)$. From this we conclude that

$$\begin{aligned} \fint_{B(y,r)} |f|\chi_{X\setminus B(x_0,2k)}\, d\mu &\le c \fint_{B(x_0,3r)} |f|\chi_{X\setminus B(x_0,2k)}\, d\mu \\ &\le cM(f\chi_{X\setminus B(x_0,2k)})(x_0) \le cMf(x_0). \end{aligned}$$

By taking supremum on the left hand side we have

$$M(f\chi_{X\setminus B(x_0,2k)})(x) \le cMf(x_0)$$

for every $x \in B(x_0, k)$. This implies that for every $k = 1, 2, \dots$ the maximal function is finite almost everywhere in $B(x_0, k)$. Since X is a countable union of such balls the claim follows. □

5. The discrete maximal function and Newtonian spaces

Our goal is to show that the discrete maximal operator preserves the smoothness of the function in the sense that it is a bounded operator in the Newtonian space. The global case has been studied in [18] and here we concentrate on the local case. We begin by proving the corresponding result for the discrete convolution in a fixed scale.

Lemma 5.1. *Suppose that $u \in N^{1,p}(\Omega)$ with $p > 1$ and let $0 < t < 1$. Then $u_t \in N^{1,p}(\Omega)$ and there is a constant $c = c(c_\mu, p)$ and $q < p$ such that $c(M_\Omega g^q)^{1/q}$ is a p-weak upper gradient of u_t in Ω whenever g is a p-weak upper gradient of u in Ω.*

Proof. By (4.7) we have $|u_t| \le M^*_\Omega u$ almost everywhere and from (4.3) we conclude that $u_t \in L^p(\Omega)$.

Then we consider the upper gradient. We write $u_t(x)$ as

$$u_t(x) = u(x) + \sum_{i=1}^{\infty} \psi_i(x)\big(u_{B(x_i,3r_i)} - u(x)\big).$$

Observe that at each point the sum is only over finitely many balls so that the convergence of the series is clear. Note also that $u(x) < \infty$

for almost every $x \in \Omega$ by (4.3). Let $g \in L^p(\Omega)$ be a p-weak upper gradient of u in Ω. By the basic properties of the upper gradients, we have that

$$\Big(\frac{L}{r_i}|u - u_{B(x_i,3r_i)}| + g\Big)\chi_{B(x_i,6r_i)}$$

is a p-weak upper gradient of $\psi_i(u_{B(x_i,3r_i)} - u)$. Here L is the Lipschitz constant of the partition of unity. This implies that

$$g_t = g + \sum_{i=1}^{\infty} \Big(\frac{L}{r_i}|u - u_{B(x_i,3r_i)}| + g\Big)\chi_{B(x_i,6r_i)}$$

is a p-weak upper gradient of u_t. Then we prove an estimate for g_t in terms of the local maximal function. Let $x \in B(x_i, 6r_i)$. Then $B(x_i, 3r_i) \subset B(x, 9r_i)$ and

$$|u(x) - u_{B(x_i,3r_i)}| \le |u(x) - u_{B(x,9r_i)}| + |u_{B(x,9r_i)} - u_{B(x_i,3r_i)}|.$$

We estimate the second term on the right side by the weak $(1, q)$-Poincaré inequality and the doubling condition as

$$\begin{aligned}
|u_{B(x,9r_i)} - u_{B(x_i,3r_i)}| &\le \fint_{B(x_i,3r_i)} |u - u_{B(x,9r_i)}|\, d\mu \\
&\le c \fint_{B(x,9r_i)} |u - u_{B(x,9r_i)}|\, d\mu \\
&\le c r_i \Big(\fint_{B(x,9\tau' r_i)} g^q\, d\mu\Big)^{1/q} \le c r_i (M_\Omega g^q(x))^{1/q}.
\end{aligned}$$

Observe that here we used the self improving property of the Poincaré inequality proved by Keith and Zhong in [15]. The first term on the right side is estimated by a standard telescoping argument. Since almost every point is a Lebesgue point of u, we have

$$\begin{aligned}
|u(x) - u_{B(x,9r_i)}| &\le \sum_{j=0}^{\infty} |u_{B(x,3^{2-j}r_i)} - u_{B(x,3^{1-j}r_i)}| \\
&\le c \sum_{j=0}^{\infty} \fint_{B(x,3^{2-j}r_i)} |u - u_{B(x,3^{2-j}r_i)}|\, d\mu \\
&\le c \sum_{j=0}^{\infty} 3^{2-j} r_i \Big(\fint_{B(x,3^{2-j}\tau' r_i)} g^q\, d\mu\Big)^{1/q} \\
&\le c r_i (M_\Omega g^q(x))^{1/q}
\end{aligned}$$

for almost every $x \in \Omega$. Here we used the Poincaré inequality and the doubling condition again. Hence we have

$$|u(x) - u_{B(x_i,3r_i)}| \le c r_i (M_\Omega g^q(x))^{1/q}$$

for almost every $x \in B(x_i, 6r_i)$. From this we conclude that

$$g_t(x) = g(x) + \sum_{i=1}^{\infty} \Big(\frac{L}{r_i} |u(x) - u_{B(x_i,3r_i)}| + g \Big) \chi_{B(x_i,6r_i)}$$
$$\leq c(M_\Omega g^q(x))^{1/q}$$

for almost every $x \in \Omega$. This implies that $c(M_\Omega g^q)^{1/q}$ is a p-weak upper gradient of u_t. By (4.3) we have

$$\int_\Omega (M_\Omega g^q)^{p/q} \, d\mu \leq c \int_\Omega g^p \, d\mu$$

and hence $(M_\Omega g^q)^{1/q} \in L^p(\Omega)$. This completes the proof. □

Remark 5.2. If $u \in N^{1,p}(\Omega)$ with $p > 1$, then by the previous lemma $u_t \in N^{1,p}(\Omega)$ for every t, $0 < t < 1$. By Remark 4.6 we see that $u_t \to u$ in $L^p(\Omega)$ and by Lemma 4.5 $u_t \to u$ almost everywhere as $t \to 0$. However, one dimensional examples show that u_t does not, in general, converge to u as $t \to 0$ in the Newtonian space $N^{1,p}(\Omega)$. This can be seen by considering such partitions of unity in the construction of the maximal function that every component at all scales is constant in a set of large measure.

Remark 5.3. During the proof of the previous theorem some of the balls are enlarged. This implies that some Whitney coverings may induce discrete convolutions for which the previous proof is false. To avoid this technical problem we assume that the Whitney balls are well inside the subdomain Ω. In detail, given $p > 1$ we have $1 \leq q < p$ and $\tau' \geq 1$ so that X supports the weak $(1, q)$-Poincaré inequality. By choosing $s \leq \frac{4}{3\tau'} + 1$ in the proof of Lemma 4.1 we can guarantee that the enlarged balls lie inside Ω.

Now we are ready to conclude that the discrete maximal operator preserves Newtonian spaces. We use the following simple fact in the proof: Suppose that u_i, $i = 1, 2, \dots$, are functions and g_i, $i = 1, 2, \dots$, are p-weak upper gradients of u_i, respectively. Let $u = \sup_i u_i$ and $g = \sup_i g_i$. If u is finite almost everywhere, then g is a p-weak upper gradient of u. For the proof, we refer to [5].

Theorem 5.4. *If $u \in N^{1,p}(\Omega)$ with $p > 1$, then $M^*_\Omega u \in N^{1,p}(\Omega)$. In addition, there is a constant $c = c(c_\mu, p)$ and $q < p$ such that the function $c(Mg^q)^{1/q}$ is a p-weak upper gradient of $M^*_\Omega u$ whenever g is a p-weak upper gradient of u in Ω.*

Proof. By (4.3) we see that $M^*_\Omega u \in L^p(\Omega)$ and, in particular, $M^*_\Omega u < \infty$ almost everywhere in Ω. Since

$$M^*_\Omega u(x) = \sup_j |u|_{t_j}(x)$$

and by the preceding lemma $c(M_\Omega g^q)^{1/q}$ is an upper gradient of $|u|_{t_j}$ for every j, we conclude that it is an upper gradient of $M^*_\Omega u$. Here we also used the fact that every p-weak upper gradient of u will do as a p-weak upper gradient of $|u|$ as well. The claim follows from the maximal function theorem and Lemma 4.2. □

Remark 5.5. By Theorem 5.4 and the maximal function theorem we can conclude that the local discrete maximal operator M^*_Ω is bounded in $N^{1,p}(\Omega)$ if $p > 1$. Indeed, there is a constant $c = c(c_\mu, p)$ such that

$$\|M^*_\Omega u\|_{N^{1,p}(\Omega)} \le c\|u\|_{N^{1,p}(\Omega)}$$

for every $u \in N^{1,p}(\Omega)$.

The next result shows that the discrete maximal operator also preserves the boundary values in the Newtonian sense. For the Euclidean case we refer to [21].

Theorem 5.6. *Let $\Omega \subset X$ be an open set and assume that $u \in N^{1,p}(\Omega)$ with $p > 1$. Then*

$$|u| - M^*_\Omega u \in N^{1,p}_0(\Omega).$$

Proof. Let g_u be the minimal p-weak upper gradient of u. Let $0 < t < 1$ and consider the discrete convolution $|u|_t$. Let $x \in \Omega$ with $x \in B(x_i, 6r_i)$. Using the same telescoping argument as in the proof of Lemma 5.1 and the properties of the Whitney balls we have

$$\begin{aligned} \big||u|_{B(x_i,3r_i)} - |u(x)|\big| &\le cr_i(M_\Omega g_u^q(x))^{1/q} \\ &\le ct \operatorname{dist}(x, X\setminus\Omega)(M_\Omega g_u^q(x))^{1/q} \end{aligned}$$

for almost every $x \in B(x_i, 6r_i)$. It follows that

$$\begin{aligned} \big||u|_t(x) - |u(x)|\big| &= \Big|\sum_{i=1}^\infty \psi_i(x)\big(|u|_{B(x_i,3r_i)} - |u(x)|\big)\Big| \\ &\le \sum_{i=1}^\infty \psi_i(x)\big||u|_{B(x_i,3r_i)} - |u(x)|\big| \\ &\le ct \operatorname{dist}(x, X\setminus\Omega)(M_\Omega g_u^q(x))^{1/q}. \end{aligned}$$

For every $x \in \Omega$ there is a sequence t_j, $j = 1, 2, \dots$, of scales such that

$$M^*_\Omega u(x) = \lim_{j\to\infty} |u|_{t_j}(x)$$

This implies that

$$\begin{aligned} \big||u(x)| - M^*_\Omega u(x)\big| &= \lim_{j\to\infty} ||u(x) - |u|_{t_j}(x)|| \\ &\le c \operatorname{dist}(x, X\setminus\Omega)(M_\Omega g_u^q(x))^{1/q}, \end{aligned}$$

where we used the fact that $t_j \leq 1$. Hence by the maximal function theorem we conclude that

$$\int_\Omega \left(\frac{\big||u(x)| - M^*_\Omega u(x)\big|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p d\mu(x) \leq c \int_\Omega (M_\Omega g_u^q(x))^{p/q} \, d\mu(x)$$

$$\leq c \int_\Omega |g_u(x)|^p \, d\mu(x).$$

This implies that

$$\frac{|u(x)| - M^*_\Omega u(x)}{\operatorname{dist}(x, X \setminus \Omega)} \in L^p(\Omega)$$

and from Theorem 5.1 in [16] we conclude that $|u| - M^*_\Omega u \in N^{1,p}_0(\Omega)$. □

Remark 5.7. The previous theorem implies that, in particular, the discrete maximal operator preserves Newtonian spaces with zero boundary values: If $u \in N^{1,p}_0(\Omega)$ with $p > 1$, then $M^*_\Omega u \in N^{1,p}_0(\Omega)$.

6. The discrete maximal function and Hölder continuity

The next result shows that the global discrete maximal function $M^* f$ is Hölder continuous with the same exponent as f. In particular, if f is Lipschitz continuous, then $M^* f$ is also Lipschitz continuous.

Theorem 6.1. *Let f be a Hölder continuous function in X with the exponent $0 < \alpha \leq 1$, i.e. there exists a constant $L_1 \geq 0$ such that*

$$|f(x) - f(y)| \leq L_1 d(x, y)^\alpha$$

for every x, y in X. Then there is a constant L_2, which depends only on L_1, such that

$$|M^* f(x) - M^* f(y)| \leq L_2 d(x, y)^\alpha$$

for every x, y in X, provided $M^* f$ is not identically infinity in X.

Proof. Fix a scale $r > 0$ and let $x, y \in X$. We begin by proving that the discrete convolution f_r is Hölder continuous. We consider two cases. First we assume that $d(x, y) > r$. By the definition of the discrete convolution we have

$$|f_r(x) - f_r(y)| \leq |f(x) - f(y)| + \sum_{i=1}^\infty \psi_i(x) |f_{B(x_i, 3r)} - f(x)|$$

$$+ \sum_{i=1}^\infty \psi_i(y) |f_{B(x_i, 3r)} - f(y)|.$$

The terms in the sums are non-zero only if $x \in B(x_i, 6r)$ or $y \in B(x_i, 6r)$ for some i. If $x \in B(x_i, 6r)$ for some i, then by Hölder continuity of f we have

$$|f_{B(x_i,3r)} - f(x)| \leq \fint_{B(x_i,3r)} |f(z) - f(x)|\, d\mu(z) \leq cr^\alpha.$$

Similarly, if $y \in B(x_i, 6r)$ for some i, then

$$|f_{B(x_i,3r)} - f(y)| \leq cr^\alpha.$$

Since the balls $B(x_i, 6r)$, $i = 1, 2, \dots$, are of bounded overlap and f is Hölder continuous, we arrive at

$$|f_r(x) - f_r(y)| \leq cd(x, y)^\alpha + cr^\alpha$$

Since $d(x, y) > r$, we have

$$|f_r(x) - f_r(y)| \leq cd(x, y)^\alpha$$

and we are done.

Then we assume that $d(x, y) \leq r$. By the definition of the discrete convolution we have

$$|f_r(x) - f_r(y)| \leq \sum_{i=1}^{\infty} |\psi_i(x) - \psi_i(y)||f_{B(x_i,3r)} - f(x)|.$$

The term in the sum is non-zero only if $x \in B(x_i, 6r)$ or $y \in B(x_i, 6r)$ for some i. If $x \in B(x_i, 6r)$, then

$$|f_{B(x_i,3r)} - f(x)| \leq cr^\alpha$$

as above. On the other hand, if $y \in B(x_i, 6r)$, then $x \in B(x_i, 7r)$ because $d(x, y) \leq r$ and we again have

$$|f_{B(x_i,3r)} - f(x)| \leq cr^\alpha.$$

Since there are only a bounded number indices for which the term in the sum is non-zero we arrive at

$$\sum_{i=1}^{\infty} |\psi_i(x) - \psi_i(y)||f_{B(x_i,3r)} - f(x)| \leq cd(x, y)r^{\alpha-1} \leq cd(x, y)^\alpha.$$

Here we also used Lipschitz continuity of ψ_i. This shows that f_r is Hölder continuous.

Let us prove now that the global discrete maximal function preserves Hölder continuity. Without loss of generality we may assume that $M^*f(x) \geq M^*f(y)$.

Let $\varepsilon > 0$. Choose $r_\varepsilon > 0$ so that

$$|f|_{r_\varepsilon}(x) > M^*f(x) - \varepsilon.$$

Then

$$M^*f(x) - M^*f(y) \leq |f|_{r_\varepsilon}(x) - |f|_{r_\varepsilon}(y) + \varepsilon \leq cd(x, y)^\alpha + \varepsilon.$$

Since the left hand side is independent of ε the theorem follows by letting $\varepsilon \to 0$. $\square$

Remark 6.2. The proof of the previous theorem shows that the discrete maximal operator is bounded in the space of Hölder continuous functions.

Now we modify the previous argument for the local discrete maximal function. The main difference lies in the fact that the balls in the Whitney type covering differ in size.

Theorem 6.3. *Let f be a Hölder continuous function in Ω with the exponent $0 < \alpha \leq 1$, i.e. there exists a constant $L_1 \geq 0$ such that*

$$|f(x) - f(y)| \leq L_1 d(x,y)^\alpha$$

for every x, y in Ω. Then there is a constant L_2, which depends only on L_1 and the constants in the Whitney type covering, such that

$$|M^*_\Omega f(x) - M^*_\Omega f(y)| \leq L_2 d(x,y)^\alpha$$

for every x, y in Ω, provided $M^* f$ is finite almost everywhere in Ω.

Proof. Fix a scale $t \in (0,1)$ and let $x, y \in \Omega$. Again we show that the local discrete convolution f_t is Hölder continous. We consider two cases. First we assume that

$$\frac{t}{c_2} \max\{\operatorname{dist}(x, X \setminus \Omega), \operatorname{dist}(y, X \setminus \Omega)\} < d(x,y), \tag{6.4}$$

where c_2 is the constant in Lemma 4.1. In the same way as in the proof of Theorem 6.1 we obtain

$$|f_t(x) - f_t(y)| \leq c d(x,y)^\alpha + c \sum_{i=1}^\infty \psi_i(x) r_i^\alpha + c \sum_{i=1}^\infty \psi_i(y) r_i^\alpha. \tag{6.5}$$

By the properties of the Whitney type covering given by Lemma 4.1 and (6.4) we have

$$\sum_{i=1}^\infty \psi_i(x) r_i^\alpha \leq \Big(\frac{t}{\kappa_1} \operatorname{dist}(x, X \setminus \Omega)\Big)^\alpha < \Big(\frac{\kappa_2}{\kappa_1}\Big)^\alpha d(x,y)^\alpha.$$

A similar estimate holds also for the second sum in (6.5) and hence we obtain

$$|f_t(x) - f_t(y)| \leq c d(x,y)^\alpha.$$

Then we consider the case when (6.4) does not hold. Then

$$
\begin{aligned}
|f_t(x) - f_t(y)| &\le \sum_{i=1}^{\infty} |\psi_i(x) - \psi_i(y)||f_{B(x_i,3r_i)} - f(x)| \\
&\le c \sum_{i=1}^{\infty} d(x,y) r_i^{\alpha-1} \qquad (6.6) \\
&\le c d(x,y) \Big(\frac{t}{c_2} \min\{\operatorname{dist}(x, X\setminus\Omega), \operatorname{dist}(y, X\setminus\Omega)\} \Big)^{\alpha-1}
\end{aligned}
$$

Since (6.4) does not hold, we have

$$
\begin{aligned}
&\min\{\operatorname{dist}(x, X\setminus\Omega), \operatorname{dist}(y, X\setminus\Omega)\} \\
&\ge \left(1 - \frac{1}{c_2}\right) \max\{\operatorname{dist}(x, X\setminus\Omega), \operatorname{dist}(y, X\setminus\Omega)\} \\
&\ge \left(1 - \frac{1}{c_2}\right) \frac{c_2}{t} d(x,y)
\end{aligned}
$$

which allows us to conclude that

$$
|f_t(x) - f_t(y)| \le c d(x,y) \Big(1 - \frac{1}{c_2}\Big)^{\alpha-1} d(x,y)^{\alpha-1} \le c d(x,y)^{\alpha}.
$$

Here we also used the fact that the number of non-zero terms in the sum (6.6) is uniformly bounded. Following the same reasoning as in the global case we obtain the claim. □

Remark 6.7. Similar arguments as above can be used to show that the discrete maximal operator preserves continuity, provided it is finite almost everywhere.

7. The discrete maximal function and BMO

A function $f \in L^1_{\mathrm{loc}}(X)$ belongs to $BMO(X)$ if

$$
\|f\|_{BMO(X)} = \sup_{B(x,r)} \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu < \infty.
$$

The functions of bounded mean oscillation in metric spaces have been studied, for example, in [6], [9], [11], [33], [35] and [36]. The fundamental property of functions in BMO is that they satisfy the following John-Nirenberg inequality: For any $f \in BMO(X)$ and any ball $B \subset X$ there exist constants c_1, c_2 depending only on the doubling constant of the measure μ such that

$$
\mu(\{x \in B : |f(x) - f_B| > \lambda\}) \le c_1 \mu(B) \mathrm{e}^{-c_2\lambda/\|f\|_{BMO(X)}} \qquad (7.1)
$$

for every $\lambda > 0$. The John-Nirenberg lemma implies that

$$
\fint_B \mathrm{e}^{\varepsilon(||f| - |f|_B|)} \, d\mu \le \frac{c_1}{1 - \varepsilon \|f\|_{BMO(X)}/c_2} \qquad (7.2)
$$

for every ball $B \subset X$ and $0 < \varepsilon < c_2/\|f\|_{BMO(X)}$. For the proof of the John-Nirenberg lemma in the context of metric measure spaces, we refer to [6], [25], [29]. See also [5].

The next theorem is a generalization of a Euclidean result in [3] and [4]. Our proof is a metric space version of the argument presented in [8]. See also [30].

Theorem 7.3. *If $f \in BMO(X)$, then $M^*f \in BMO(X)$ provided M^*f is not identically infinity.*

Proof. By choosing

$$\varepsilon = \frac{c_2}{2\|f\|_{BMO(X)}}$$

in (7.2) we obtain

$$\fint_B \exp(\varepsilon|f|)\,d\mu \le c\exp\Big(\varepsilon\fint_B |f|\,d\mu\Big).$$

Since c is independent of the ball B we have

$$M(\exp(\varepsilon|f|)) \le c\exp(\varepsilon Mf).$$

By Lemma 4.2 we conclude that

$$M^*(\exp(\varepsilon|f|)) \le c\exp(\varepsilon M^*f).$$

The reverse inequality holds as well. To see this we apply the following elementary inequality: If $a_1, a_2, \ldots, a_n$ are non-negative numbers whose sum is one and $b_1, b_2, \ldots, b_n$ are positive numbers, then

$$b_1^{a_1}b_2^{a_2}\ldots b_n^{a_n} \le a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

We apply the previous inequality together with Jensen's inequality to get

$$\mathrm{e}^{\varepsilon|f|_r} = \prod_{i=1}^{\infty}\left(\mathrm{e}^{\varepsilon|f|_{B(x_i,3r_i)}}\right)^{\psi_i} \le \sum_{i=1}^{\infty}\psi_i\mathrm{e}^{\varepsilon|f|_{B(x_i,r_i)}} \le \left(\mathrm{e}^{\varepsilon|f|}\right)_r.$$

Observe that the product and sum have only a bounded number of terms. By taking supremum on both sides we arrive at

$$\exp(\varepsilon M^*f) \le M^*(\exp(\varepsilon|f|)).$$

Hence, by Lemma 4.2, we have

$$c^{-1}\exp(\varepsilon M^*f) \le M(\exp(\varepsilon|f|)) \le c\exp(\varepsilon M^*f). \tag{7.4}$$

By Lemma 4.8, M^*f is finite almost everywhere and consequently

$$M(\exp(\varepsilon|f|)) < \infty$$

almost everywhere. By a theorem of Coifman-Rochberg, see [9] and [35], we conclude that

$$(M(\exp(\varepsilon|f|)))^{1/2}$$

is a Muckenhoupt A_1-weight. But then by (7.4) the function

$$\exp\big(\frac{\varepsilon}{2}M^*f\big)$$

is a Muckenhoupt A_1-weight. Since the logarithm of an A_1 weight is a function of bounded mean oscillation, the maximal function M^*f belongs to $BMO(X)$. □

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