A LOCAL ESTIMATE FOR NONLINEAR EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

Juha Kinnunen
Department of Mathematics
P.O.Box 4, FIN-00014 University of Helsinki, Finland

Shulin Zhou
Department of Mathematics
Peking University, Beijing 100871, P. R. China

1. Introduction

Let Ω be a domain in $\mathbf{R}^n$ and suppose that $1 < p < \infty$. We consider the weak solutions of the quasilinear equation

$$\operatorname{div}\big((ADu \cdot Du)^{(p-2)/2} ADu\big) = \operatorname{div}(|F|^{p-2}F), \tag{1.1}$$

where $A = (A_{ij}(x))_{n\times n}$ is a symmetric matrix with measurable coefficients satisfying the uniform ellipticity condition

$$\lambda\,|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda\,|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbf{R}^n$ and almost every $x \in \Omega$. Here λ and Λ are positive constants and $\eta \cdot \xi$ denotes the standard inner product of $\eta, \xi \in \mathbf{R}^n$. Suppose that $F \in L^p_{\mathrm{loc}}(\Omega)$. We recall that the function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution for equation (1.1) if

$$\int_\Omega (ADu \cdot Du)^{(p-2)/2} ADu \cdot D\varphi \, dx = \int_\Omega |F|^{p-2} F \cdot D\varphi \, dx \tag{1.3}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Equation (1.1) arises naturally in many different contexts. Just to mention few, we point out that it is the Euler equation for the variational integral

$$\int_\Omega \big((ADu \cdot Du)^{p/2} - p\,|F|^{p-2} F \cdot Du\big)\, dx.$$

In the case $p = n$ equation (1.1) (with $F = 0$) plays a key role in theory of quasiconformal mappings. If A is the identity matrix, then we have a non-homogeneous p-harmonic equation.

We are interested in studying how the regularity of F is reflected to the solutions under minimal assumptions on the coefficient matrix A. In particular, we are keen on having discontinuous coefficients. A natural weakening of the case with smooth coefficients is to assume that the coefficients of the matrix A are of vanishing mean oscillation. We recall that a locally integrable function f is of bounded mean oscillation, if

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, dy$$

is uniformly bounded as $B(x,r)$ ranges over all balls contained in Ω; here

$$f_{B(x,r)} = \fint_{B(x,r)} f(y)\, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

denotes the integral mean over the ball $B(x,r)$. If, in addition, we require that these averages tend to zero uniformly as r tends to zero, we say that

f is of vanishing mean oscillation and denote $f \in \mathrm{VMO}(\Omega)$, see [21]. Uniformly continuous functions are of vanishing mean oscillation, but in general functions of vanishing mean oscillation need not be continuous. Recently equations with coefficients of vanishing mean oscillation have obtained considerable attention, see [1], [2], [3], [4], [7], [8], [9], [10], [15] and [20]. Our main contribution is the following result.

1.4. Theorem. *Suppose that the coefficients of A are of vanishing mean oscillation and that $F \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > p$. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution for equation* (1.1). *Then for every $x_0 \in \Omega$, there exist $r > 0$ and $\gamma > 0$ such that $B(x_0, 6r) \subset \Omega$ and*

$$\int_{B(x_0,r)} |Du|^q \, dx \le \gamma \Big(\int_{B(x_0,6r)} |F|^q \, dx + \int_{B(x_0,6r)} |u|^q \, dx \Big). \tag{1.5}$$

Here r and γ depend only on n, p, q, λ, Λ, $\mathrm{dist}(x_0, \partial\Omega)$ and the VMO *data of A. In particular, this implies that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.*

Observe that the local estimate (1.5) holds above the natural exponent p; For $q = p$ it follows immediately by choosing the right test function.

The regularity theory for (1.1) has been studied extensively. We take the opportunity to briefly describe some developments related to our work.

There are two kinds of estimates in the literature. By local estimates we mean results similar to Theorem 1.4. On the other hand, if $u \in W^{1,p}(\mathbf{R}^n)$ is the weak solution of (1.1) with $F \in L^q(\mathbf{R}^n)$, then the question is whether there exists a constant $\gamma > 0$ such that

$$\int_{\mathbf{R}^n} |Du|^q \, dx \le \gamma \int_{\mathbf{R}^n} |F|^q \, dx.$$

We call this kind of results global estimates.

First suppose that we are in the linear case $p = 2$. Then (1.1) reduces to the equation

$$\mathrm{div}(ADu) = \mathrm{div}\, F. \tag{1.6}$$

If A is the unitary matrix, then global results follow from the classical L^p-theory for the Laplacian using the Calderón–Zygmund theory, see [12]. The case of bounded and uniformly continuous coefficients has been studied by Morrey et al, see [19]. Recently Di Fazio [7] proved a local result for (1.6) provided the coefficients are bounded functions of vanishing mean oscillation. His argument is based on representation formulas involving singular integral operators and commutators. A global result has been obtained by Iwaniec and Sbordone in [15].

Then we discuss the nonlinear case $p \neq 2$. If the matrix A is the unitary matrix, equation (1.1) reads

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F).$$

This is a non-homogeneous p-harmonic equation. In this case related results have been obtained by Iwaniec [14] and by DiBenedetto and Manfredi [6]. Their methods are based on maximal function inequalities and the regularity theory for the p-harmonic equation.

We generalize the local result of Di Fazio to a class of nonlinear equations. Even in the linear case our argument gives a new proof for the result of Di Fazio. Our approach is based on choosing the right test function, maximal function estimates and the regularity theory for the solutions with smooth coefficents. In particular, we do not have representation formulas for solutions available. Instead of using global maximal functions as in [6], we localize the problem and use maximal functions where the radii of balls are restricted. The drawback of our method is that it does not seem give the global estimate. On the other hand, our method can be modified to obtain a global estimate when Ω is a bounded $C^{1,1}$-domain using the boundary estimates of [17]. We hope to return to this question in a future paper.

Our paper is organized in the following way. In Section 2 we prove maximal function inequalities, which may be of independent interest. In Section 3 we establish an auxiliary local estimate, which is an essential tool in proving our main result. Finally in Section 4 we complete the proof of Theorem 1.4 using an approximation argument.

Our notation is standard. We use c to denote positive constants which may differ even on the same line. The dependence of the parameters is expressed, for example, by $c(n,p)$. We do not write down explicitly the dependence on the data. Throughout the paper we use many elementary inequalities without proofs. The proofs are scattered in the literature and difficult to locate, but some of our inequalities can be found, for example, in [14].

2. Maximal function inequalities

The *Hardy-Littlewood maximal function* of a locally integrable function f is defined by

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy$$

and the *sharp maximal function* of f is defined by

$$f^{\#}(x) = \sup_{r>0} \fint_{B(x,r)} |f(y) - f_{B(x,r)}|\, dy.$$

In the definition of the *restricted sharp maximal function* $f^{\#}_{\rho}$ there is an additional requirement that the radii over which the supremum is taken must be less than or equal to a positive number ρ.

We recall the well-known estimates for the maximal operators.

2.1. Lemma. *Suppose that $f \in L^t(\mathbf{R}^n)$ with $t > 1$. Then there exists a constant $c = c(n,t)$ such that*

$$\|Mf\|_t \le c\, \|f\|_t \tag{2.2}$$

and

$$\|Mf\|_t \le c\, \|f^{\#}\|_t. \tag{2.3}$$

The first inequality is the maximal function theorem of Hardy, Littlewood and Wiener. The second inequality is due to Fefferman and Stein.

Observing that $|f| \leq Mf$ and $f^{\#} \leq 2\,Mf$, we see that the norms $\|f\|_t$, $\|Mf\|_t$ and $\|f^{\#}\|_t$ are equivalent. The corresponding result is not true for the restricted sharp maximal function. For example, for uniformly continuous functions we can make the restricted sharp maximal function arbtrarily small by taking the bound for the radii to be small enough. There are local versions of the Fefferman and Stein inequality, see for example Lemma 4 in [13], but we have not been able to make them to fit to our proof. However, the following local estimate will do for our purposes.

2.4. Lemma. *Suppose that $f \in L^t(\mathbf{R}^n)$ with $t > 1$ and $\operatorname{supp} f \subset B(x_0, R)$ for some $R > 0$. Then there exist constants $k = k(n,t) \geq 2$ and $c = c(n,t) > 0$ such that*

$$\int_{B(x_0,R)} |f(x)|^t\,dx \leq c \int_{B(x_0,kR)} f^{\#}_{kR}(x)^t\,dx. \tag{2.5}$$

Proof. Let $k \geq 2$ to be determined later.

First suppose that $x \in \mathbf{R}^n \setminus B(x_0, kR)$, then $|x - x_0| \geq kR$. If $B(x,r) \cap B(x_0,R) \neq \emptyset$, then $r \geq |x - x_0| - R \geq 1/2|x - x_0|$ and

$$\begin{aligned} f^{\#}(x) &\leq 2 \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap B(x_0,R)} |f(y)|\,dy \\ &\leq \frac{c}{|x-x_0|^n} \int_{B(x_0,R)} |f(y)|\,dy. \end{aligned}$$

Then suppose that $x \in B(x_0, kR)$. Clearly

$$\begin{aligned} f^{\#}(x) &\leq \sup_{r>kR} \fint_{B(x,r)} |f(y) - f_{B(x,r)}|\,dy + f^{\#}_{kR}(x) \\ &\leq \frac{2}{|B(x,kR)|} \int_{B(x_0,R)} |f(y)|\,dy + f^{\#}_{kR}(x). \end{aligned}$$

Using the above estimates, we have

$$
\begin{aligned}
\int_{\mathbf{R}^n} f^{\#}(x)^t\,dx =& \int_{\mathbf{R}^n\setminus B(x_0,kR)} f^{\#}(x)^t\,dx + \int_{B(x_0,kR)} f^{\#}(x)^t\,dx \\
\le& c\Big(\int_{B(x_0,R)} |f(y)|\,dy\Big)^t \int_{\mathbf{R}^n\setminus B(x_0,kR)} |x-x_0|^{-nt}\,dx \\
&+ c(kR)^{n(1-t)}\Big(\int_{B(x_0,R)} |f(y)|\,dy\Big)^t + c\int_{B(x_0,kR)} f^{\#}_{kR}(x)^t\,dx \\
=& I_1 + I_2 + I_3.
\end{aligned}
$$

An integration in the spherical coordinates and Hölder's inequality gives

$$
I_j \le ck^{n(1-t)} \int_{B(x_0,R)} |f(y)|^t\,dy, \qquad j = 1,2.
$$

Using the above estimates, we have

$$
\int_{\mathbf{R}^n} f^{\#}(x)^t\,dx \le c\Big(k^{n(1-t)} \int_{B(x_0,R)} |f(x)|^t\,dx + \int_{B(x_0,kR)} f^{\#}_{kR}(x)^t\,dx\Big). \tag{2.6}
$$

Finally applying Lemma 2.1 and estimate (2.6), we obtain

$$
\begin{aligned}
\int_{B(x_0,R)} |f(x)|^t\,dx &= \int_{\mathbf{R}^n} |f(x)|^t\,dx \le c\int_{\mathbf{R}^n} f^{\#}(x)^t\,dx \\
&\le c\Big(k^{n(1-t)} \int_{B(x_0,R)} |f(x)|^t\,dx + \int_{B(x_0,kR)} f^{\#}_{kR}(x)^t\,dx\Big)
\end{aligned}
$$

with $c = c(n,t)$. The claim follows by choosing $k = k(n,t) \ge 2$ large enough and absorbing the first term on the right side to the left side.

3. An auxiliary local estimate

In this section we prove an interior *a priori* estimate, which will be a crucial ingredient in the proof of Theorem 1.4. Our proof is based on the maximal function estimates and some known regularity results for the constant coefficient reference equation.

3.1. Proposition. *Let $1 < p < \infty$. Suppose that $F \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > p$ and that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is a weak solution of* (1.1). *Then for every $x_0 \in \Omega$ there exist $d > 0$ and $\gamma > 0$ such that $B(x_0, 3d) \subset \Omega$ and*

$$\int_{B(x_0,d/2)} |Du|^q \, dx \le \gamma \Big(\int_{B(x_0,3d)} |F|^q \, dx + \int_{B(x_0,3d)} |u|^q \, dx \Big). \tag{3.2}$$

Here d and γ depend only on n, p, q, λ, Λ, $\operatorname{dist}(x_0, \partial\Omega)$ *and the* VMO *data of A.*

Observe that we have the artificial assumption $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ in the statement of Proposition 3.1. This is a technicality to justify the absorption of some terms in the final part of the proof. Later we show that this assumption is redundant and that it follows from the assumptions of Theorem 1.4.

Proof of Proposition 3.1. Set $t = q/p > 1$ and let $k = k(N, q/p) \ge 2$ be as in Lemma 2.4. Later we choose $h \ge 2$ and $d > 0$ appropriately so that $B(x_0, 2hkd) \subset \Omega$. Let $\zeta \in C_0^\infty(B(x_0, d))$ be a cut-off function such that $\zeta = 1$ in $B(x_0, d/2)$, $0 \le \zeta \le 1$ in $\mathbf{R}^n$ and $|D\zeta| \le c/d$. We set

$$w = u\zeta^{p'},$$

where $p' = p/(p-1)$. Then $w \in W^{1,q}(B(x,R))$ and, in particular, by Hölder's inequality $u \in W^{1,p}(B(x,R))$ for every R with $0 < R < \operatorname{dist}(x, \partial\Omega)$.

We begin with constructing a constant coefficient reference equation for which we have known regularity results. Later we compare the solutions of (1.1) to the solutions of the reference equation. To be more precise, for every $x \in B(x_0, kd)$ and every R, $0 < R \le hkd$, we have a unique weak solution $v \in W^{1,p}(B(x,R))$ for the equation

$$\operatorname{div}\big((A_B Dv \cdot Dv)^{(p-2)/2} A_B Dv\big) = 0 \tag{3.3}$$

with the boundary condition

$$v - w \in W_0^{1,p}(B(x,R)). \tag{3.4}$$

Here the matrix $A_B = A_{B(x,R)}$ is the integral mean (taken componentwise) of the matrix A over the ball $B = B(x,R)$. Obviously the averaged matrix A_B satisfies the ellipticity condition (1.2) with the same constants as A.

We need the following estimates for the solution of the constant coefficient Dirichlet problem. For the proofs of these inequalities we refer to [6] and [22]. Let $v \in W^{1,p}(B(x,R))$ be the unique weak solution of (3.3) with the boundary condition (3.4). Then there exists a constant $c = c(n,p,\lambda,\Lambda)$ such that

$$\operatorname*{ess\,sup}_{B(x,\rho)} |Dv| \le c \Big(\fint_{B(x,R)} |Dw|^p \, dy \Big)^{1/p} \tag{3.5}$$

and

$$\fint_{B(x,\rho)} |Dv - (Dv)_{B(x,\rho)}|^p \, dy \le c \Big(\frac{\rho}{R}\Big)^{\alpha} \fint_{B(x,R)} |Dw|^p \, dy \tag{3.6}$$

for every ρ, $0 < \rho \le R/2$, with $\alpha = \alpha(n,p,\lambda,\Lambda) > 0$. Observe that here $(Dv)_{B(x,\rho)}$ denotes the integral average of the vector taken componentwise.

The proof of Proposition 3.1 is based on the following technical result. We denote

$$\|A(y)\| = \max_{i,j} |A_{ij}(y)|$$

and

$$\|A\|_{*,R} = \sup \fint_{B(x,r)} \|A - A_{B(x,r)}\| \, dy,$$

where the supremum is taken over all balls $B(x,r)$ with $r \le R$.

3.7. Lemma. *Let $x \in \Omega$ and R, $0 < R \le hkd$, be such that $B(x,3R) \subset \Omega$ and denote $s = (p+q)/2$. Suppose that $v \in W^{1,p}(B(x,R))$ is the unique weak solution of* (3.3) *with the boundary condition* (3.4)*. Then for every ε, $0 < \varepsilon < 1$, we have*

$$\begin{aligned}\fint_{B(x,R)} |Dw - Dv|^p \, dy &\le c(\varepsilon) \|A\|_{*,R}^{1-p/s} \Big(\fint_{B(x,R)} |Dw|^s \, dy \Big)^{p/s} \\ &\quad + \varepsilon \fint_{B(x,R)} |Dw|^p \, dy + c(\varepsilon,h,d) \fint_{B(x,3R)} \big(|F|^p + |u|^p\big) \chi_{B(x_0,3d)} \, dy.\end{aligned} \tag{3.8}$$

The proof of (3.8) is not very difficult but it is lengthy. Basically we use Hölder's, Sobolev's and Young's inequalities successively. Observe, however, that the mean oscillation of the matrix A appears in the first term on the right side and this term can be made arbitrarily small if we choose $d > 0$ small enough. Note also that the right side of (3.8) is independent of v. We present the proof of Lemma 3.7 at the end of this section and now continue the proof of Proposition 3.1. So assume, for the moment, that we have proved Lemma 3.7.

For short, we set

$$G = (|F|^p + |u|^p)\chi_{B(x_0,3d)}.$$

Fix ρ, $0 < \rho \leq kd$, and let $R = h\rho$. Choose $\varepsilon = h^{-n-\alpha}$ in Lemma 3.7, where α is the same exponent as in (3.6). Since $h \geq 2$, we have $B(x,\rho) \subset B(x,R)$, and (3.8) implies that

$$\begin{aligned}\fint_{B(x,\rho)} |Dw - Dv|^p \, dy &\leq c(h)\|A\|_{*,R}^{1-p/s}\Big(\fint_{B(x,R)} |Dw|^s \, dy\Big)^{p/s} \\ &\quad + h^{-\alpha}\fint_{B(x,R)} |Dw|^p \, dy + c(h,d)\fint_{B(x,3R)} G \, dy.\end{aligned}$$

Suppose that $0 < \theta \leq 1$. Then

$$\big||\xi|^p - |\eta|^p\big| \leq c(p)\theta^{-p}|\xi - \eta|^p + \theta|\eta|^p$$

for every $\xi, \eta \in \mathbf{R}^n$. Using the previous elementary inequality we obtain

$$\begin{aligned}\fint_{B(x,\rho)} \big||Dw|^p - (|Dw|^p)_{B(x,\rho)}\big| \, dy &\leq c(\theta)\fint_{B(x,\rho)} |Dw - Dv|^p \, dy \\ &\quad + c(\theta)\fint_{B(x,\rho)} |Dv - (Dv)_{B(x,\rho)}|^p \, dy + \theta\fint_{B(x,\rho)} |Dv|^p \, dy.\end{aligned}$$

for every θ, $0 < \theta \leq 1$. Combining the above two estimates and applying

(3.5) and (3.6) we arrive at

$$\begin{aligned}
&\fint_{B(x,\rho)} \big||Dw|^p - (|Dw|^p)_{B(x,\rho)}\big| \, dy \\
&\qquad \le c(\theta,h)\|A\|_{*,hkd}^{1-p/s} \Big(\fint_{B(x,R)} |Dw|^s \, dy\Big)^{p/s} \\
&\qquad\quad + \big(c(\theta)h^{-\alpha} + c\theta\big) \fint_{B(x,R)} |Dw|^p \, dy + c(\theta,h,d) \fint_{B(x,3R)} G \, dy.
\end{aligned}$$

Then we interpret the obtained inequality in terms of maximal functions. We observe that the previous estimate is independent of v and that it holds for every $x \in B(x_0, kd)$ and ρ with $0 < \rho \le kd$. Taking the supremum over radii yields

$$\begin{aligned}
(|Dw|^p)^{\#}_{kd}(x) \le & c(\theta,h)\|A\|_{*,hkd}^{1-p/s} \big(M(|Dw|^s)(x)\big)^{p/s} \\
& + \big(c(\theta)h^{-\alpha} + c\theta\big) M(|Dw|^p)(x) + c(\theta,h,d)\, MG(x)
\end{aligned}$$

for every $x \in B(x_0, kd)$. Since $\operatorname{supp} w \subset B(x_0, d)$, we may apply Lemma 2.4 with $f = |Dw|^p$ and $t = q/p > 1$. This implies that

$$\begin{aligned}
\int_{B(x_0,d)} |Dw|^q \, dx \le & c \int_{B(x_0,kd)} \big|(|Dw|^p)^{\#}_{kd}\big|^{q/p} \, dx \\
\le & c(\theta,h)\|A\|_{*,hkd}^{q/p-q/s} \int_{\mathbf{R}^n} \big(M(|Dw|^s)\big)^{q/s} \, dx \\
& + \big(c(\theta)h^{-\alpha} + c\theta\big)^{q/p} \int_{\mathbf{R}^n} \big(M(|Dw|^p)\big)^{q/p} \, dx \\
& + c(\theta,h,d) \int_{\mathbf{R}^n} (MG)^{q/p} \, dx \\
= & I_1 + I_2 + I_3.
\end{aligned}$$

We estimate the obtained integrals by the Hardy-Littlewood-Wiener maximal function theorem, see Lemma 2.1. Recalling that $p < s < q$, we may apply (2.2) and obtain

$$\begin{aligned}
I_1 &\le c(\theta,h)\|A\|_{*,hkd}^{q/p-q/s} \int_{B(x_0,d)} |Dw|^q \, dx, \\
I_2 &\le \big(c(\theta)h^{-\alpha} + c\theta\big)^{q/p} \int_{B(x_0,d)} |Dw|^q \, dx, \\
I_3 &\le c(\theta,h,d) \int_{B(x_0,3d)} G^{q/p} \, dx.
\end{aligned}$$

Now we choose θ small enough and then h large enough such that

$$\big(c(\theta)h^{-\alpha} + c\theta\big)^{q/p} < \frac{1}{4}.$$

Combining the estimates above we arrive at

$$\begin{aligned}\int_{B(x_0,d)} |Dw|^q \, dx \leq &\Big(c\|A\|_{*,hkd}^{q/p-q/s} + \frac{1}{4}\Big) \int_{B(x_0,d)} |Dw|^q \, dx \\ &+ c(d) \int_{B(x_0,3d)} \big(|F|^q + |u|^q\big) \, dx.\end{aligned} \tag{3.9}$$

We observe that the first term on the right side can be absorbed to the left side by choosing $d > 0$ small enough such that $2khd \leq \operatorname{dist}(x_0, \partial\Omega)$ and

$$c\,\|A\|_{*,hkd}^{q/p-q/s} \leq \frac{1}{4}.$$

This completes the proof of Proposition 3.1.

We still have to prove Lemma 3.7.

Proof of Lemma 3.7. Denote

$$w = u\zeta^{p'} \quad \text{and} \quad g = -uD(\zeta^{p'}).$$

Let $v \in W^{1,p}(B(x,R))$ be the unique weak solution of (3.3) with $v - w \in W_0^{1,p}(B(x,R))$. A direct calculation using (3.3) shows that

$$\begin{aligned}&\operatorname{div}\big((A_B Dw \cdot Dw)^{(p-2)/2} A_B Dw - (A_B Dv \cdot Dv)^{(p-2)/2} A_B Dv\big) \\ &\quad = \operatorname{div}\big((A_B Dw \cdot Dw)^{(p-2)/2} A_B Dw - (ADw \cdot Dw)^{(p-2)/2} ADw\big) \\ &\qquad + \operatorname{div}\big((ADw \cdot Dw)^{(p-2)/2} ADw\big),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}&\operatorname{div}\big((ADw \cdot Dw)^{(p-2)/2} ADw\big) \\ &\quad = \operatorname{div}\big((ADu \cdot Du)^{(p-2)/2} ADu\zeta^p\big) + \operatorname{div}\big((ADw \cdot Dw)^{(p-2)/2} ADw \\ &\qquad - (A(Dw+g) \cdot (Dw+g))^{(p-2)/2} A(Dw+g)\big).\end{aligned} \tag{3.11}$$

Using the fact that u is a solution of (1.1) we conclude that

$$\begin{aligned}
&\operatorname{div}\big((ADu \cdot Du)^{(p-2)/2} ADu\zeta^p\big) \\
&\quad = \operatorname{div}(|F|^{p-2}F)\zeta^p + (ADu \cdot Du)^{(p-2)/2} ADu \cdot D(\zeta^p).
\end{aligned} \tag{3.12}$$

As usual, equations (3.10), (3.11) and (3.12) have to be understood in the weak sense. Collecting (3.10), (3.11) and (3.12) and choosing the test function $w - v \in W_0^{1,p}(B(x,R))$ we obtain

$$\begin{aligned}
&\fint_{B(x,R)} \big((A_B Dw \cdot Dw)^{(p-2)/2} A_B Dw \\
&\qquad - (A_B Dv \cdot Dv)^{(p-2)/2} A_B Dv\big) \cdot (Dw - Dv)\, dy \\
&\quad = I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
I_1 =& \fint_{B(x,R)} \big((A_B Dw \cdot Dw)^{(p-2)/2} A_B Dw \\
&- (ADw \cdot Dw)^{(p-2)/2} ADw\big) \cdot (Dw - Dv)\, dy, \\
I_2 =& \fint_{B(x,R)} \big((ADw \cdot Dw)^{(p-2)/2} ADw \\
&- (A(Dw+g) \cdot (Dw+g)^{(p-2)/2} A(Dw+g)\big) \cdot (Dw - Dv)\, dy, \\
I_3 =& \fint_{B(x,R)} |F|^{p-2} F \cdot D\big((w-v)\zeta^p\big)\, dy, \\
I_4 =& -\fint_{B(x,R)} (ADu \cdot Du)^{(p-2)/2} ADu \cdot D(\zeta^p)(w-v)\, dy.
\end{aligned}$$

We denote the left side of (3.13) by I_0.

Let $\sigma > 0$ and $\varepsilon > 0$ be parameters to be fixed later.

We estimate the integrals I_i, $i = 0, 1, 2, 3, 4$. The crucial term for us is I_1, since here we need the assumption that the coefficients of A are of vanishing mean oscillation. The estimates for the other terms are rather standard. The proof is divided into two cases.

Case 1. $p \geq 2$.

We begin with estimating I_0. If $p \geq 2$, then for every $\xi, \eta \in \mathbf{R}^n$ we have

$$\big((A\xi \cdot \xi)^{(p-2)/2} A\xi - (A\eta \cdot \eta)^{(p-2)/2} A\eta\big) \cdot (\xi - \eta) \geq c\,|\xi - \eta|^p \qquad (3.14)$$

with $c = c(p, \lambda, \Lambda)$. Inequality (3.14) implies that

$$I_0 \geq c \fint_{B(x,R)} |Dw - Dv|^p \, dy. \qquad (3.15)$$

This is our estimate for I_0.

Then we consider I_1. Recall that $s = (p+q)/2$ and let

$$t = \frac{ps}{(p-1)(s-p)} > 1.$$

For every $\xi, \eta \in \mathbf{R}^n$ we have

$$\big|(A\xi \cdot \xi)^{(p-2)/2} A\xi - (A_B\xi \cdot \xi)^{(p-2)/2} A_B\xi\big| \leq c\,\|A - A_B\| |\xi|^{p-1} \qquad (3.16)$$

with $c = c(p, \lambda, \Lambda)$. By (3.16), Hölder's and Young's inequalities, we have

$$\begin{aligned} I_1 \leq & \fint_{B(x,R)} \|A - A_B\| |Dw|^{p-1} |Dw - Dv| \, dy \\ \leq & c(\sigma) \Big(\fint_{B(x,R)} \|A - A_B\|^t \, dy \Big)^{1-p/s} \Big(\fint_{B(x,R)} |Dw|^s \, dy \Big)^{p/s} \\ & + \sigma \fint_{B(x,R)} |Dw - Dv|^p \, dy. \end{aligned}$$

The mean oscillation of the coefficients of A can be estimated by the ellipticity assumption, which implies, in particular, that the coefficients of A are bounded. Hence we obtain

$$\fint_{B(x,R)} \|A - A_B\|^t \, dy \leq c\,\|A\|_{*,R}$$

with constant c depending on the data. Consequently, we have

$$I_1 \leq c(\sigma) \|A\|_{*,R}^{1-p/s} \Big(\fint_{B(x,R)} |Dw|^s \, dy \Big)^{p/s} + \sigma \fint_{B(x,R)} |Dw - Dv|^p \, dy. \quad (3.17)$$

This is the desired estimate for I_1.

To bound I_2 we use the inequality

$$\left|(A\xi\cdot\xi)^{(p-2)/2}A\xi-(A\eta\cdot\eta)^{(p-2)/2}A\eta\right|\le c\left(|\xi|+|\eta|\right)^{p-2}|\xi-\eta|,$$

which holds for every $\xi,\eta\in\mathbf{R}^n$ with $c=c(p,\lambda,\Lambda)$. We obtain

$$\begin{aligned}I_2\le&c\fint_{B(x,R)}\left(|Dw|^{p-2}+|Dw+g|^{p-2}\right)|g||Dw-Dv|\,dy\\ \le&\sigma\fint_{B(x,R)}|Dw-Dv|^p\,dy+\varepsilon\fint_{B(x,R)}|Dw|^p\,dy\\ &+c(\sigma,\varepsilon)\fint_{B(x,R)}|g|^p\,dy.\end{aligned}\tag{3.18}$$

This estimate takes care of I_2.

To estimate I_3 we recall that $w-v\in W_0^{1,p}(B(x,R))$. We use Hölder's, Sobolev's and Young's inequalities in this order and conclude that

$$\begin{aligned}I_3\le&\fint_{B(x,R)}|F|^{p-1}\zeta^p|D(w-v)|\,dy+c\fint_{B(x,R)}|F|^{p-1}|w-v||D\zeta|\,dy\\ \le&\sigma\fint_{B(x,R)}|Dw-Dv|^p\,dy+c(\sigma,h)\fint_{B(x,R)}|F|^p\chi_{B(x_0,d)}\,dy.\end{aligned}\tag{3.19}$$

Finally we deal with I_4. An easy calculation shows that

$$I_4\le c(d)\fint_{B(x,R)}|Dw+g|^{p-2}|Du|\chi_{B(x_0,d)}|w-v|\,dy.$$

We use Hölder's then Sobolev's and finally Young's inequality to obtain

$$\begin{aligned}I_4\le&\sigma\fint_{B(x,R)}|Dw-Dv|^p\,dy+\varepsilon\fint_{B(x,R)}|Dw|^p\,dy\\ &+c(\sigma,\varepsilon,h,d)R^p\fint_{B(x,R)}|Du|^p\chi_{B(x_0,d)}\,dy\\ &+c(\sigma,\varepsilon,h,d)\fint_{B(x,R)}|g|^p\,dy.\end{aligned}\tag{3.20}$$

By geometry we can find a ball $B(z,r)$ where the center z lies on the line joining x_0 and x and the radius $r=\min(d,R)$ is such that $B(x,R)\cap$

$B(x_0, d) \subset B(z, r)$. Let $\eta \in C_0^\infty(B(z, 2r))$ be a cut-off function with $\eta = 1$ in $B(z, r)$, $0 \le \eta \le 1$ in $\mathbf{R}^n$ and $|D\eta| \le c/r$. Choosing the test function $\eta^p u$ in (1.3), after standard calculations, we arrive at the Caccioppoli type inequality

$$r^p \int_{B(z,r)} |Du|^p \, dy \le c \int_{B(z,2r)} \big(r^p |F|^p + |u|^p\big) \, dy.$$

Observing that $B(z, 2r) \subset B(x_0, 3d) \cap B(x, 3R)$, we have

$$\begin{aligned} R^p \fint_{B(x,R)} &|Du|^p \chi_{B(x_0,d)} \, dy \\ &\le c(h, d) \fint_{B(x,3R)} \big(|F|^p + |u|^p\big) \chi_{B(x_0,3d)} \, dy. \end{aligned} \tag{3.21}$$

Combining estimates (3.15), (3.17), (3.18), (3.19), (3.20) and (3.21), we obtain

$$\begin{aligned} \fint_{B(x,R)} |Dw - Dv|^p \, dy \le{}& c(\sigma) \|A\|_{*,R}^{1-p/s} \Big(\fint_{B(x,R)} |Dw|^s \, dy \Big)^{p/s} \\ &+ 4\sigma \fint_{B(x,R)} |Dw - Dv|^p \, dy + 2\varepsilon \fint_{B(x,R)} |Dw|^p \, dy \\ &+ c(\sigma, \varepsilon, h, d) \fint_{B(x,3R)} \big(|F|^p + |u|^p\big) \chi_{B(x_0,3d)} \, dy. \end{aligned} \tag{3.22}$$

Now choosing σ small enough we see that the second term on the right side can be absorbed to the left side. This completes the proof of (3.8) in the case $p \ge 2$.

Case 2. $1 < p < 2$.

For every ε, $0 < \varepsilon < 1$, there is $c = c(p, \lambda, \Lambda)$ such that

$$\begin{aligned} |\xi - \eta|^p \le{}& c\varepsilon^{(p-2)/p} \big((A\xi \cdot \xi)^{(p-2)/2} A\xi \\ &- (A\eta \cdot \eta)^{(p-2)/2} A\eta \big) \cdot (\xi - \eta) + \varepsilon |\eta|^p \end{aligned} \tag{3.23}$$

for every $\xi, \eta \in \mathbf{R}^n$. It follows from (3.23) that

$$\begin{aligned}
\fint_{B(x,R)} |Dw - Dv|^p \, dy \leq c(\varepsilon) \fint_{B(x,R)} & \big((A_B Dw \cdot Dw)^{(p-2)/2} A_B Dw \\
& - (A_B Dv \cdot Dv)^{(p-2)/2} A_B Dv\big) \cdot (Dw - Dv) \, dy \\
& + \varepsilon \fint_{B(x,R)} |Dw|^p \, dy \\
\leq & c(\varepsilon)(I_1 + I_2 + I_3 + I_4) + \varepsilon \fint_{B(x,R)} |Dw|^p \, dy
\end{aligned}$$

where I_i, $i = 1, 2, 3, 4$, are the same as before. The integrals I_1 and I_3 may be estimated as above.

For I_2 we use the inequality

$$\big|(A\xi \cdot \xi)^{(p-2)/2} A\xi - (A\eta \cdot \eta)^{(p-2)/2} A\eta\big| \leq c\, |\xi - \eta|^{p-1},$$

which holds for every $\xi, \eta \in \mathbf{R}^n$ with constant $c = c(p, \lambda, \Lambda)$, together with Hölder's and Young's inequalities and obtain

$$\begin{aligned}
I_2 \leq & c \fint_{B(x,R)} |g|^{p-1} |Dw - Dv| \, dy \\
\leq & \sigma \fint_{B(x,R)} |Dw - Dv|^p \, dy + c(\sigma) \fint_{B(x,R)} |g|^p \, dy.
\end{aligned}$$

For I_4 we use Hölder's, Sobolev's and Young's inequalities to obtain

$$\begin{aligned}
I_4 \leq & c(d) \fint_{B(x,R)} |Du|^{p-1} \chi_{B(x_0,d)} |w - v| \, dy \\
\leq & \sigma \fint_{B(x,R)} |Dw - Dv|^p \, dy + c(\sigma, d) R^p \fint_{B(x,R)} |Du|^p \chi_{B(x_0,d)} \, dy.
\end{aligned}$$

The last term may be estimated as in (3.21). Therefore

$$\begin{aligned}
I_4 \leq & \sigma \fint_{B(x,R)} |Dw - Dv|^p \, dy \\
& + c(\sigma, h, d) \fint_{B(x,3R)} \big(|F|^p + |u|^p\big) \chi_{B(x_0,3d)} \, dy.
\end{aligned}$$

We obtain estimate (3.8) for $1 < p < 2$ by combining the estimates for I_i, $i = 0, 1, 2, 3, 4$, exactly in the same way as in (3.22) and then choosing $\sigma = \sigma(\varepsilon)$ small enough and absorbing terms. This completes the proof of (3.8) in the case $1 < p < 2$ and Lemma 3.7 follows.

4. The proof of the main result

In this section we complete the proof of Theorem 1.4 using an approximation argument and Proposition 3.1. In particular, we show that if $F \in L^q_{\mathrm{loc}}(\Omega)$ with $q > p$, then the weak solutions of (1.1) belong to $W^{1,q}_{\mathrm{loc}}(\Omega)$. Hence the regularity is reflected to the solutions. Our strategy is to build up a sequence of equations with smooth coefficients, use the fact that the solutions of the regularized problems are locally Lipschitz and then show that they converge to the solution of the original problem.

It seems to be crucial for us that the weak solutions of (1.1) belong to a higher Sobolev space. This is a well-known result, but we restate it here for the sake of completeness.

4.1. Proposition. *Suppose that $F \in L^q_{\mathrm{loc}}(\Omega)$ with $q > p$ and let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of* (1.1)*. Then there exist s, $p < s < q$, and $c \geq 1$ such that*

$$\begin{aligned}\Big(\fint_{B(x_0,R)} |Du|^s\,dx\Big)^{1/s} &\leq c\Big(\fint_{B(x_0,2R)} |Du|^p\,dx\Big)^{1/p} \\ &\quad + c\Big(\fint_{B(x_0,2R)} |F|^q\,dx\Big)^{1/q}\end{aligned} \tag{4.2}$$

for every $B(x_0,2R) \subset \Omega$. Here s and c depend only on n, p, q, λ and Λ.

Proof. Let $\zeta \in C^\infty_0(B(x_0,2R))$ be a cut-off function such that $\zeta = 1$ in $B(x_0,R)$, $0 \leq \zeta \leq 1$ and $|D\zeta| \leq c/R$. We test (1.3) with $\varphi = \zeta^p(u - u_{B(x_0,2R)}) \in W^{1,p}_0(\Omega)$. Using the ellipticity assumption we have

$$\begin{aligned}\int_{B(x_0,2R)} |Du|^p\zeta^p\,dx &\leq c\Big(\int_{B(x_0,2R)} |Du|^{p-1}|D(\zeta^p)||u - u_{B(x_0,2R)}|\,dx \\ &\quad + \int_{B(x_0,2R)} |F|^{p-1}|Du||\zeta^p|\,dx \\ &\quad + \int_{B(x_0,2R)} |F|^{p-1}|u - u_{B(x_0,2R)}||D(\zeta^p)|\,dx\Big).\end{aligned}$$

By Hölder's and Young's inequalities we obtain

$$\int_{B(x_0,R)} |Du|^p\,dx \le c\Big(R^{-p}\int_{B(x_0,2R)} |u-u_{B(x_0,2R)}|^p\,dx + \int_{B(x_0,2R)} |F|^p\,dx\Big)$$

and finally we use Sobolev's inequality to conclude that

$$\begin{aligned}\Big(\fint_{B(x_0,R)} |Du|^p\,dx\Big)^{1/p} &\le c\Big(\fint_{B(x_0,2R)} |Du|^{p_*}\,dx\Big)^{1/p_*}\\ &+c\Big(\fint_{B(x_0,2R)} |F|^p\,dx\Big)^{1/p},\end{aligned}$$

where $p_* = \max\{1, np/(n+p)\}$. In other words, the gradient of a solution satisfies a reverse Hölder inequality. Then we use the fact that reverse Hölder inequalities improve themselves. The version of the result needed here was first proved by Meyers and Elcrat [18] using the idea of Gehring, see also Giaquinta's book [11]. This completes the proof.

Then we construct a sequence of approximating equations with smooth coefficients. To this end, fix a ball $B(x_0, 2R) \subset \Omega$. Then there are matrices $A_m \in C^\infty(B(x_0, 2R))$, $m = 1, 2, \dots$, satisfying the ellipticity condition (1.2) such that $A_m \to A$ in $L^1(B(x_0, 2R))$ as $m \to \infty$ and

$$\|A_m\|_{*,r} \le \|A\|_{*,r}, \qquad m = 1, 2, \dots,$$

for every r with $0 < r \le R$. This is easy to see by convolving the coefficients of A by a smooth approximate identity as in [S].

Let $F_m \in C_0^\infty(B(x_0, R))$, $m = 1, 2, \dots$, be functions such that $F_m \to F$ in $L^q(B(x_0, R))$ as $m \to \infty$ and

$$\int_{B(x_0,R)} |F_m|^q\,dx \le \int_{B(x_0,R)} |F|^q\,dx, \qquad m = 1, 2, \dots$$

Suppose that $u \in W^{1,p}(B(x_0, R))$. For every $m = 1, 2, \dots$, there is a unique solution $u_m \in W^{1,p}(B(x_0, R))$ for the Dirichlet problem

$$\operatorname{div}\big((A_m Du_m \cdot Du_m)^{(p-2)/2} A_m Du_m\big) = \operatorname{div}(|F_m|^{p-2} F_m) \qquad (4.3)$$

with

$$u_m - u \in W_0^{1,p}(B(x_0, R)). \tag{4.4}$$

It seems natural to expect that the solutions to the approximating problems converge to the solution of the original equation in some suitable sense. This is the content of the following proposition.

4.5. Proposition. *Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of* (1.1)*. Fix a ball $B(x_0, 2R) \subset \Omega$. Let $u_m \in W^{1,p}(B(x_0, R))$, $m = 1, 2, \ldots$, be the unique solution of* (4.3) *with the boundary condition* (4.4)*. Then $u_m \to u$ in $W^{1,p}(B(x_0, R))$ as $m \to \infty$.*

Proof. Fix m. Using (4.3) and (1.1) we have

$$\begin{aligned}\int_{B(x_0,R)} &\big((A_m Du_m \cdot Du_m)^{(p-2)/2} A_m Du_m \\ &\quad - (A_m Du \cdot Du)^{(p-2)/2} A_m Du\big) \cdot (Du_m - Du)\, dx \\ =& I_1 + I_2,\end{aligned} \tag{4.6}$$

where

$$\begin{aligned} I_1 =& \int_{B(x_0,R)} \big((ADu \cdot Du)^{(p-2)/2} ADu \\ &- (A_m Du \cdot Du)^{(p-2)/2} A_m Du\big) \cdot (Du_m - Du)\, dx, \\ I_2 =& \int_{B(x_0,R)} (|F_m|^{p-2} F_m - |F|^{p-2} F) \cdot (Du_m - Du)\, dx. \end{aligned}$$

The left side of equality (4.6) is denoted by I_0. Again, we divide the estimates for (4.6) into two cases.

Case 1. $p \geq 2$.

Using the elementary inequality (3.14) we see that

$$I_0 \geq c \int_{B(x_0,R)} |Du_m - Du|^p\, dx.$$

The two terms on the right side of (4.6) are estimated as below.

For I_1 we use (3.16) and obtain

$$
\begin{aligned}
I_1 \le & c \int_{B(x_0,R)} \|A - A_m\| |Du|^{p-1} |Du_m - Du| \, dx \\
\le & \theta \int_{B(x_0,R)} |Du_m - Du|^p \, dx + c(\theta) \int_{B(x_0,R)} \|A - A_m\|^{p'} |Du|^p \, dx,
\end{aligned}
\tag{4.7}
$$

where $0 < \theta < 1$. The last term of (4.7) is controlled by the reverse Hölder inequality. Using (4.2) we obtain

$$
\begin{aligned}
& \int_{B(x_0,R)} \|A - A_m\|^{p'} |Du|^p \, dx \\
& \qquad \le \Big(\int_{B(x_0,R)} |Du|^s \, dx \Big)^{p/s} \Big(\int_{B(x_0,R)} \|A - A_m\|^{p's/(s-p)} \, dx \Big)^{(s-p)/s} \\
& \qquad \le \gamma \Big(\int_{B(x_0,R)} \|A - A_m\| \, dx \Big)^{(s-p)/s},
\end{aligned}
$$

where γ is independent of m.

To estimate I_2 we apply the inequality

$$
\big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big| \le c(p) \big(|\xi| + |\eta| \big)^{p-2} |\xi - \eta|,
$$

where $\xi, \eta \in \mathbf{R}^n$. We conclude that

$$
\begin{aligned}
I_2 \le & c \int_{B(x_0,R)} (|F_m| + |F|)^{p-2} |F_m - F| |Du_m - Du| \, dx \\
\le & \theta \int_{B(x_0,R)} |Du_m - Du|^p \, dx + c(\theta) \Big(\int_{B(x_0,R)} |F_m - F|^p \, dx \Big)^{p'/p} \\
& \cdot \Bigg(\Big(\int_{B(x_0,R)} |F_m|^p \, dx \Big)^{1/p} + \Big(\int_{B(x_0,R)} |F|^p \, dx \Big)^{1/p} \Bigg)^{p'(p-2)},
\end{aligned}
$$

where $0 < \theta < 1$.

Choosing θ small enough, combining the above estimates and absorbing terms we arrive at

$$
\begin{aligned}
\int_{B(x_0,R)} |Du_m - Du|^p \, dx \le & \gamma \Big(\int_{B(x_0,R)} \|A - A_m\| \, dx \Big)^{(s-p)/s} \\
& + \gamma \Big(\int_{B(x_0,R)} |F_m - F|^q \, dx \Big)^{p'/q},
\end{aligned}
$$

where γ is independent of m. Proposition 4.5 follows in the case $p \geq 2$ letting $m \to \infty$.

Case 2. $1 < p < 2$.

It follows from (3.23) that, for every ε with $0 < \varepsilon \leq 1$, we have

$$\begin{aligned}\int_{B(x_0,R)} |Du_m - Du|^p \, dx \leq c(\varepsilon) \int_{B(x_0,R)} &\big((A_m Du_m \cdot Du_m)^{(p-2)/2} A_m Du_m \\ &- (A_m Du \cdot Du)^{(p-2)/2} A_m Du\big) \cdot (Du_m - Du) \, dx \\ &+ \varepsilon \int_{B(x_0,R)} |Du|^p \, dx \\ \leq & c(\varepsilon)(I_1 + I_2) + \varepsilon \int_{B(x_0,R)} |Du|^p \, dx.\end{aligned}$$

Here I_1 and I_2 denote the same integrals as in (4.6). The integral I_1 can be estimated exactly in the same way as before.

To estimate I_2 we apply the inequality

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \leq c(p)|\xi - \eta|^{p-1},$$

where $\xi, \eta \in \mathbf{R}^n$. We obtain

$$\begin{aligned} I_2 \leq & c \int_{B(x_0,R)} |F_m - F|^{p-1} |Du_m - Du| \, dx \\ \leq & \theta \int_{B(x_0,R)} |Du_m - Du|^p \, dx + c(\theta) \int_{B(x_0,R)} |F_m - F|^p \, dx, \end{aligned}$$

where $0 < \theta < 1$.

Choosing $\theta = \theta(\varepsilon)$ small enough and absorbing terms we conclude that

$$\begin{aligned}\int_{B(x_0,R)} |Du_m - Du|^p \, dx \leq \varepsilon \int_{B(x_0,R)} |Du|^p \, dx \\ + \gamma\Big(\int_{B(x_0,R)} \|A_m - A\| \, dx\Big)^{(s-p)/s} + \gamma\Big(\int_{B(x_0,R)} |F_m - F|^q \, dx\Big)^{p/q},\end{aligned}$$

where γ does not depend on m. Letting $m \to \infty$ we have

$$\limsup_{m\to\infty} \int_{B(x_0,R)} |Du_m - Du|^p \, dx \leq \varepsilon \int_{B(x_0,R)} |Du|^p \, dx.$$

Since $\varepsilon > 0$ is arbitrary, the above inequality implies that

$$\limsup_{m\to\infty} \int_{B(x_0,R)} |Du_m - Du|^p \, dx = 0.$$

This proves Proposition 4.5 also in the case $1 < p < 2$ and we are done.

Proof of Theorem 1.4. Let $B(x_0, R) \subset \Omega$. First we assume that $q \le p^* = \min\{np/(n-p)_+, 2p\}$. Suppose that $u_m \in W^{1,p}(B(x_0, R))$, $m = 1, 2, \dots$, are weak solutions of (4.3) with the boundary condition (4.4). By the regularity theory the gradients of u_m are locally Hölder continuous, see [5], [16] and [22]. In particular, this implies that $Du_m \in L^\infty_{\mathrm{loc}}(B(x_0, R))$ and hence $u_m \in W^{1,q}_{\mathrm{loc}}(B(x_0, R))$ for every $m = 1, 2, \dots$ This means that the functions u_m, $m = 1, 2, \dots$, satisfy the assumptions of Proposition 3.1. Hence by (3.2) there is $d > 0$ such that $B(x_0, 3d) \subset B(x_0, R)$ and

$$\int_{B(x_0,d/2)} |Du_m|^q \, dx \le \gamma \Big(\int_{B(x_0,3d)} |F_m|^q \, dx + \int_{B(x_0,3d)} |u_m|^q \, dx \Big), \qquad (4.8)$$

where d and γ do not depend on m.

Proposition 4.5 and Sobolev's inequality give us a subsequence of (u_m) (denoted by u_m) such that $Du_m \to Du$ almost everywhere in $B(x_0, 3d)$ and $u_m \to u$ in $L^q(B(x_0, 3d))$ as $m \to \infty$. It follows from (4.8) using Fatou's lemma that

$$\begin{aligned}\int_{B(x_0,d/2)} |Du|^q \, dx &\le \limsup_{m\to\infty} \int_{B(x_0,d/2)} |Du_m|^q \, dx \\ &\le \gamma \Big(\int_{B(x_0,3d)} |F|^q \, dx + \int_{B(x_0,3d)} |u|^q \, dx \Big).\end{aligned}$$

In particular, this implies that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.

If $q \ge p^*$, we iterate the above procedure. First we see that $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ since $F \in L^q_{\mathrm{loc}}(\Omega) \subset L^{p^*}_{\mathrm{loc}}(\Omega)$. In the first step we improved the integrability of the gradient of u by a fixed amount. Then we use the same reasoning again. After finitely many steps we obtain that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and we may use Proposition 3.1 to complete the proof of Theorem 1.4.

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