# THE BV-CAPACITY IN METRIC SPACES

HEIKKI HAKKARAINEN AND JUHA KINNUNEN

Abstract. We study basic properties of the BV-capacity and Sobolev capacity of order one in a complete metric space equipped with a doubling measure and supporting a weak Poincaré inequality. In particular, we show that the BV-capacity is a Choquet capacity and the Sobolev 1-capacity is not. However, these quantities are equivalent by two sided estimates and they have the same null sets as the Hausdorff measure of codimension one. The theory of functions of bounded variation plays an essential role in our arguments. The main tool is a modified version of the boxing inequality.

## 1. Introduction

The notion of capacity plays a crucial role in studying the pointwise behaviour of a Sobolev function, see [7, 9, 23, 27] for Euclidean and [18, 19, 4] for more general metric measure spaces. Let $1 \le p < \infty$ and according to [26], denote by $N^{1,p}(X)$ the first order Sobolev space on a metric measure space $X$. For the general theory of Sobolev functions on metric measure spaces we refer to a forthcoming monograph [3] by Björns. The Sobolev $p$-capacity of $E \subset X$ is defined as

$$\operatorname{cap}_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all admissible functions $u \in N^{1,p}(X)$ such that $0 \le u \le 1$ and $u = 1$ on a neighbourhood of $E$. The theory of Sobolev $p$-capacity in the setting of metric measure spaces, when $1 < p < \infty$, has been studied in papers [20, 21]. In particular, the Sobolev $p$-capacity is so called Choquet capacity when $1 < p < \infty$. This means that the capacity of a Borel set can be obtained by approximating with compact sets from inside and open sets from outside. In the Euclidean case with Lebesgue measure the Sobolev $p$-capacity is a Choquet capacity also when $p = 1$, but a slightly unexpected fact is that the Choquet property fails for $p = 1$ in the metric setting. We give an explicit example of this phenomenon inspired by an unpublised construction by Riikka Korte. During the past fifteen years, capacities in metric measure spaces have been studied, for example, in [11, 20, 21, 16]. However, little has been written about the case when $p = 1$.

2000 *Mathematics Subject Classification.* 28A12, 26A45.

In this paper we propose that the capacity defined in terms of the functions of bounded variation, see [9, 27], behaves better than the 1-Sobolev capacity in metric measure spaces. The BV-capacity of a set $E \subset X$ is

$$\operatorname{cap}_{\mathrm{BV}}(E) = \inf \big( \|u\|_{L^1(X)} + \|Du\|(X) \big),$$

where the infimum is taken over all $u \in \mathrm{BV}(X)$ such that $0 \le u \le 1$ and $u = 1$ on a neighbourhood of $E$. Here $\|Du\|$ is the total variation measure of $u$. In the metric setting a version of the BV-capacity, defined without the norm of the function, has been studied in [19]. However, the results in [19] apply for compact sets only and since we do not have the Choquet property, the passage to more general sets is not clear. One of the main advantages of using the BV-capacity in this work is that the results apply for more general sets as well. We show that the BV-capacity has many of the desired properties and it is, indeed, a Choquet capacity. In the Euclidean case with Lebesgue measure the BV-capacity equals to Sobolev 1-capacity, see [9] and [27], but in a complete metric space equipped with a doubling measure and supporting a weak Poincaré inequality, the BV-capacity is merely equivalent to the Sobolev 1-capacity by two sided estimates. For compact sets the BV-capacity and Sobolev 1-capacity coincide. We shall present examples which demonstrate that in general these two quantities are not equal.

The theory of BV-functions in metric measure spaces, see [24, 1, 2], with results like coarea formula and lower semicontinuity of the variation measure play an essential role in our approach. To prove the equivalence of capacities we use similar approach as in [9] and [19], where the boxing inequality, originally studied by Gustin in [12], plays an important role. However, we present a modified version of this inequality, since we are dealing with Sobolev capacities, where the norm of the function is also included. In [19] it is shown that the variational 1-capacity, which is defined without the norm of the function, is equivalent by two sided estimates to the Hausdorff content of codimension one. Here we show that the Sobolev 1-capacity and the BV-capacity have same null sets as the Hausdorff measure of codimesion one.

**Acknowledgements.** The research was supported by the Emil Aaltonen Foundation and the Finnish Academy of Science and Letters, the Vilho, Yrjö and Kalle Väsälä Foundation.

## 2. Preliminaries

Let $X = (X, d, \mu)$ denote a metric space equipped with a metric $d$ and a positive Borel regular outer measure $\mu$ such that $0 < \mu(B(x,r)) < \infty$ for all balls $B(x,r)$ of $X$. It is also assumed that $X$ contains at least two points. The measure $\mu$ is said to be *doubling* if there exists a constant $C_D \geq 1$, called the *doubling constant* of $\mu$, such that

$$\mu(B(x,2r)) \leq C_D \mu(B(x,r))$$

for all balls $B(x,r)$ of $X$. A *path* in $X$ is defined here as a rectifiable nonconstant continuous mapping from a compact interval to $X$. A path can be parameterized by arc length. In this paper, the definition of Sobolev spaces on metric measure space $X$ is based on the notion of $p$-weak upper gradients, see [26]. We will now recall the definition of the $p$-weak upper gradient.

**Definition 2.1.** A nonnegative Borel function $g$ on $X$ is an *upper gradient* of an extended real valued function $u$ on $X$ if for all paths $\gamma$ joining points $x$ and $y$ in $X$ we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds, \tag{2.2}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. Let $1 \leq p < \infty$. If $g$ is a nonnegative measurable function on $X$, and if the integral in (2.2) is well defined and the inequality holds for $p$-almost every path, then $g$ is a *p-weak upper gradient* of $u$.

The phrase that inequality (2.2) holds for *p-almost every path* with $1 \leq p < \infty$ means that it fails only for a path family with zero $p$-modulus, see for example [15]. Many usual rules of calculus hold true for upper gradients as well, see [3].

*Remark.* It is known that if $u$ has a $p$-weak upper gradient $g \in L^p_{\mathrm{loc}}(X)$, then there is a *minimal p-weak upper gradient* $g_u$ such that $g_u \leq g$ $\mu$-almost everywhere for every $p$-weak upper gradient of $u$, see [3].

The Sobolev spaces on $X$ are defined as follows.

**Definition 2.3.** Let $1 \leq p < \infty$. If $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \Big( \int_X |u|^p\, d\mu + \inf_g \int_X g^p\, d\mu \Big)^{1/p},$$

where the infimum is taken over all $p$-weak upper gradients of $u$. The *Newtonian space* on $X$ is the quotient space

$$N^{1,p}(X) = \big\{ u : \|u\|_{N^{1,p}(X)} < \infty \big\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

In order to obtain stronger connection between a function and its upper gradients and to develop first order calculus, one usually assumes that metric measure space supports some kind of Poincaré inequality.

**Definition 2.4.** The space $X$ supports a *weak (1,p)-Poincaré inequality* if there exists constants $C_P > 0$ and $\tau \geq 1$ such that for all balls

$B(x,r)$ of $X$, all locally integrable functions $u$ on $X$ and for all $p$-weak upper gradients $g$ of $u$

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le C_P r \Big( \fint_{B(x,\tau r)} g^p \, d\mu \Big)^{1/p},$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

It is known that $\mathrm{Lip}(X) \cap N^{1,p}(X)$ is dense in $N^{1,p}(X)$ if $\mu$ is doubling and $(1,p)$-Poincaré inequality is satisfied, see [26]. From this it easily follows that Lipschitz functions with compact support are dense in $N^{1,p}(X)$, if $X$ is also complete.

Throughout the paper we assume that $X$ is a complete metric measure space endowed with a doubling measure and supporting a $(1,1)$-Poincaré inequality. The definition of functions of bounded variation on the metric space setting is based on [24] and [2]. For the classical case of $\mathbb{R}^n$, we refer to [10], [7], [23] and [27]. Notice that in [24] the functions of bounded variation are defined in terms of the lower pointwise dilation. However, we may use 1-weak upper gradients instead. For the proofs of the theorems in this section, we refer to [24, 1, 2].

**Definition 2.5.** Let $u \in L^1_{\mathrm{loc}}(X)$. For every open set $U \subset X$ we define

$$\|Du\|(U) = \inf \Big\{ \liminf_{i \to \infty} \int_U g_{u_i} \, d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(U), u_i \to u \text{ in } L^1_{\mathrm{loc}}(U) \Big\},$$

where $g_{u_i} \in L^1_{\mathrm{loc}}(U)$ is a 1-weak upper gradient of $u_i$. Function $u \in L^1(X)$ is of *bounded variation*, $u \in \mathrm{BV}(X)$, if $\|Du\|(X) < \infty$. A measurable set $E \subset X$ is said to have *finite perimeter* if $\|D\chi_E\|(X) < \infty$.

*Remark.* In the definition above, we may assume that $g_{u_i}$ is, indeed, the minimal 1-weak upper gradient of $u_i$. However, with the abuse of notation we denote the 1-weak upper gradient and the minimal 1-weak upper gradient of $u$ by $g_u$.

For the following result we refer to Theorem 3.4 in [24].

**Theorem 2.6.** *Let $u \in \mathrm{BV}(X)$. For every set $A \subset X$ we define*

$$\|Du\|(A) = \inf \big\{ \|Du\|(U) : A \subset U, \, U \subset X \text{ is open} \big\}.$$

*Then $\|Du\|(\cdot)$ is a finite Borel outer measure.*

The perimeter measure is also denoted by

$$P(E,A) = \|D\chi_E\|(A).$$

For any given $u \in \mathrm{BV}(X)$ there exists a sequence of functions $u_i \in \mathrm{Lip}_{\mathrm{loc}}(X)$, $i = 1, 2, \dots$, such that $u_i \to u$ in $L^1_{\mathrm{loc}}(X)$ and

$$\int_X g_{u_i} \, d\mu \to \|Du\|(X)$$

as $i \to \infty$. The following approximation result will be useful later.

**Lemma 2.7.** *Let $u \in \mathrm{BV}(X)$. Then there is a sequence of functions $u_i \in \mathrm{Lip}_c(X)$, $i = 1, 2, \ldots$, with upper gradients $g_{u_i}$ such that $u_i \to u$ in $L^1(X)$ and*

$$\int_X g_{u_i}\, d\mu \to \|Du\|(X),$$

*as $i \to \infty$.*

*Proof.* Let $i \in \mathbb{N}$ and fix $x \in X$. We choose $r > 0$ such that

$$\int_X |u|\, d\mu < \int_{B(x,r)} |u|\, d\mu + \frac{1}{i}.$$

Let $v_i \in \mathrm{Lip}_{\mathrm{loc}}(X)$ be such that

$$\int_{B(x,r+1)} |u - v_i|\, d\mu < \frac{1}{i}$$

and

$$\left| \|Du\|(X) - \int_X g_{v_i}\, d\mu \right| < \frac{1}{i}.$$

Let $\eta \in \mathrm{Lip}_c(X)$ be a 1-Lipschitz cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ in $B(x,r)$ and $\eta = 0$ in $X \setminus B(x, r+1)$. We define $u_i = v_i \eta \in \mathrm{Lip}_c(X)$ and obtain

$$\int_X |u - u_i|\, d\mu < \frac{2}{i}.$$

We notice that

$$g_{u_i} = |v_i| \chi_{B(x,r+1)\setminus B(x,r)} + g_{v_i}\eta$$

is a 1-weak upper gradient of $u_i$ and therefore

$$\begin{aligned}
\int_X g_{u_i}\, d\mu &= \int_{B(x,r+1)\setminus B(x,r)} |v_i|\, d\mu + \int_X g_{v_i}\eta\, d\mu \\
&\le \int_{B(x,r+1)} |u - v_i|\, d\mu + \int_{X\setminus B(x,r)} |u|\, d\mu + \int_X g_{v_i}\, d\mu \\
&< \|Du\|(X) + \frac{3}{i}.
\end{aligned}$$

Hence $u_i \to u$ in $L^1(X)$ as $i \to \infty$ and

$$\limsup_{i\to\infty} \int_X g_{u_i}\, d\mu \le \|Du\|(X).$$

By the definition of $\|Du\|(X)$ we have that

$$\|Du\|(X) \le \liminf_{i\to\infty} \int_X g_{u_i}\, d\mu$$

and thus

$$\int_X g_{u_i}\, d\mu \to \|Du\|(X)$$

as $i \to \infty$. $\square$

We now list some basic properties of the perimeter measure.

**Theorem 2.8.** *Let $u, v \in L^1_{\mathrm{loc}}(X)$, and $U \subset X$ be open set. Then*
(i) $\|D(\alpha u)\|(U) = |\alpha| \|Du\|(U)$ *for every* $\alpha \in \mathbb{R}$,
(ii) $\|D(u+v)\|(U) \le \|Du\|(U) + \|Dv\|(U)$ *and*
(iii) $\|D\max\{u,v\}\|(U) + \|D\min\{u,v\}\|(U) \le \|Du\|(U) + \|Dv\|(U)$.

*Proof.* We only give a proof for (iii). Without loss of generality we may assume that $\|Du\|(U) + \|Dv\|(U) < \infty$. Let $u_i, v_i \in \mathrm{Lip}_{\mathrm{loc}}(U)$, $i = 1, 2, \dots$, be such that $u_i \to u$, $v_i \to v$ in $L^1_{\mathrm{loc}}(U)$,

$$\int_U g_{u_i}\, d\mu \to \|Du\|(U)$$

and

$$\int_U g_{v_i}\, d\mu \to \|Dv\|(U)$$

as $i \to \infty$. Since $\max\{u_i, v_i\} \to \max\{u, v\}$ and $\min\{u_i, v_i\} \to \min\{u, v\}$ in $L^1_{\mathrm{loc}}(U)$ as $i \to \infty$, we obtain

$$\begin{aligned}
&\|D\max\{u,v\}\|(U) + \|D\min\{u,v\}\|(U) \\
&\le \liminf_{i\to\infty} \int_U g_{\max\{u_i,v_i\}}\, d\mu + \liminf_{i\to\infty} \int_U g_{\min\{u_i,v_i\}}\, d\mu \\
&\le \liminf_{i\to\infty} \int_U (g_{\max\{u_i,v_i\}} + g_{\min\{u_i,v_i\}})\, d\mu \\
&\le \lim_{i\to\infty} \int_U (g_{u_i} + g_{v_i})\, d\mu \\
&= \|Du\|(U) + \|Dv\|(U).
\end{aligned}$$

□

We obtain the metric space version of the relative isoperimetric inequality as a direct consequence of the weak $(1,1)$-Poincaré inequality.

**Theorem 2.9.** *Let $E$ be a set of finite perimeter, then the following relative isoperimetric inequality holds*

$$\min\big\{\mu(B(x,r) \cap E), \mu(B(x,r) \setminus E)\big\} \le 2C_P r P(E, B(x, \tau r)),$$

*for every ball $B(x,r)$ of $X$.*

We need the following lower semicontinuity result.

**Theorem 2.10.** *Let $U \subset X$ be an open set and $u_i \in L^1_{\mathrm{loc}}(U)$ be a sequence such that $\|Du_i\|(U) < \infty$, for all $i = 1, 2 \dots$ and $u_i \to u$ in $L^1_{\mathrm{loc}}(U)$ as $i \to \infty$. Then*

$$\|Du\|(U) \le \liminf_{i\to\infty} \|Du_i\|(U).$$

Another useful result about functions of bounded variation is the *coarea formula*, see [24].

**Theorem 2.11.** *If $u \in \mathrm{BV}(X)$ and $U \subset X$ is an open set, then*

$$\|Du\|(U) = \int_{-\infty}^{\infty} P(\{u > t\}, U)\, dt.$$

## 3. BV-capacity

In the classical case the theory of capacity of order one relies strongly on theory of functions of bounded variation, see [9] and [27]. In the setting of metric measure spaces, we shall take the theory of functions of bounded variation presented in the second section as our starting point and define the capacity in terms of these functions. This approach on metric spaces has been used in [19].

**Definition 3.1.** Let $E \subset X$ and denote by $\mathcal{A}_{\mathrm{BV}}(E)$ the set of *admissible functions* $u \in \mathrm{BV}(X)$ such that $0 \le u \le 1$ and $u = 1$ on a neighbourhood of $E$. The BV-*capacity* of $E$ is

$$\mathrm{cap}_{\mathrm{BV}}(E) = \inf \Big( \int_X u\, d\mu + \|Du\|(X) \Big),$$

where the infimum is taken over all $u \in \mathcal{A}_{\mathrm{BV}}(E)$.

By the coarea formula we immediately obtain an equivalent definition.

**Lemma 3.2.** *If $E \subset X$, then*

$$\mathrm{cap}_{\mathrm{BV}}(E) = \inf \big( \mu(A) + P(A, X) \big),$$

*where the infimum is taken over all sets $A \subset X$ such that $E \subset \operatorname{int} A$.*

*Proof.* If $A \subset X$ with $E \subset \operatorname{int} A$ and $\mu(A) + P(A, X) < \infty$, then $\chi_A \in \mathcal{A}_{\mathrm{BV}}(E)$ and hence

$$\mathrm{cap}_{\mathrm{BV}}(E) \le \mu(A) + P(A, X).$$

By taking the infimum over all such sets $A$ we obtain

$$\mathrm{cap}_{\mathrm{BV}}(E) \le \inf \big( \mu(A) + P(A, X) \big).$$

In order to prove the opposite inequality, we may assume that $\mathrm{cap}_{\mathrm{BV}}(E) < \infty$. Let $\varepsilon > 0$ and $u \in \mathcal{A}_{\mathrm{BV}}(E)$ be such that

$$\int_X u\, d\mu + \|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

By the Cavalieri principle and the coarea formula we have

$$\int_X u\, d\mu + \|Du\|(X) = \int_0^1 \big( \mu(\{u > t\}) + P(\{u > t\}, X) \big) dt$$

and therefore there exists $0 < t_0 < 1$ such that

$$\mu(\{u > t_0\}) + P(\{u > t_0\}, X) < \mathrm{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

Since $u = 1$ in an open neighbourhood of $E$, we have $E \subset \operatorname{int}\{u > t_0\}$. The desired inequality now follows by letting $\varepsilon \to 0$. □

We will now prove basic properties of the BV-capacity.

**Theorem 3.3.** *The $BV$-capacity is an outer measure on $X$.*

*Proof.* Function $u \equiv 0$ is admissible for the empty set, therefore

$$\mathrm{cap}_{\mathrm{BV}}(\emptyset) = 0.$$

If $E_1 \subset E_2$ then $\mathcal{A}_{\mathrm{BV}}(E_2) \subset \mathcal{A}_{\mathrm{BV}}(E_1)$ and consequently

$$\mathrm{cap}_{\mathrm{BV}}(E_1) \le \mathrm{cap}_{\mathrm{BV}}(E_2).$$

To prove the countable subadditivity we may assume that

$$\sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) < \infty.$$

Let $\varepsilon > 0$ and choose functions $u_i \in \mathcal{A}_{\mathrm{BV}}(E_i)$ such that

$$\int_X u_i \, d\mu + \|Du_i\|(X) < \mathrm{cap}_{\mathrm{BV}}(E_i) + \varepsilon 2^{-i}$$

for $i = 1, 2, \ldots$ Let $u = \sup_{1 \le i < \infty} u_i$ and notice that

$$\int_X u \, d\mu \le \sum_{i=1}^{\infty} \int_X u_i \, d\mu \le \sum_{i=1}^{\infty} \big(\mathrm{cap}_{\mathrm{BV}}(E_i) + \varepsilon 2^{-i}\big) < \infty.$$

Hence $u \in L^1(X)$. For $i = 1, 2, \ldots$ we define

$$v_i = \max\{u_1, \ldots, u_i\}$$

and notice that $v_i \to u$ in $L^1(X)$ as $i \to \infty$. Therefore, by using Theorem 2.8 (iii) and Theorem 2.10 we obtain

$$\begin{aligned}\int_X u \, d\mu + \|Du\|(X) &\le \sum_{i=1}^{\infty} \int_X u_i \, d\mu + \liminf_{i \to \infty} \|Dv_i\|(X) \\ &\le \sum_{i=1}^{\infty} \int_X u_i \, d\mu + \sum_{i=1}^{\infty} \|Du_i\|(X) \\ &\le \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) + \varepsilon.\end{aligned}$$

Moreover, $u \in \mathcal{A}_{\mathrm{BV}}(\bigcup_{i=1}^{\infty} E_i)$. Hence by letting $\varepsilon \to 0$ we have

$$\mathrm{cap}_{\mathrm{BV}}\Big(\bigcup_{i=1}^{\infty} E_i\Big) \le \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}}(E_i).$$ □

The following theorem states that the BV-capacity behaves well with respect to limits of an increasing sequence of arbitrary sets.

**Theorem 3.4.** *If $E_1 \subset \ldots \subset E_i \subset E_{i+1} \subset \ldots \subset X$, then*

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) = \mathrm{cap}_{\mathrm{BV}}\Big(\bigcup_{i=1}^{\infty} E_i\Big).$$

*Proof.* Clearly

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) \le \mathrm{cap}_{\mathrm{BV}}\Big(\bigcup_{i=1}^{\infty} E_i\Big)$$

and the equality holds if

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) = \infty.$$

Let $\varepsilon > 0$ and assume that

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) < \infty.$$

For every index $i = 1, 2, \ldots$, we may choose $u_i \in \mathcal{A}_{\mathrm{BV}}(E_i)$ such that

$$\int_X u_i \, d\mu + \|Du_i\|(X) \le \mathrm{cap}_{\mathrm{BV}}(E_i) + \varepsilon 2^{-i}.$$

For $i = 1, 2, \ldots$ we define functions

$$v_i = \max\{u_1, \ldots, u_i\} = \max\{v_{i-1}, u_i\}$$

and

$$w_i = \min\{v_{i-1}, u_i\}.$$

Here we set $v_0 \equiv 0$ and $E_0 = \emptyset$. Notice that $v_i$, $w_i \in \mathrm{BV}(X)$ and $E_{i-1} \subset \mathrm{int}\{w_i = 1\}$ for every $i = 1, 2, \ldots$ By Theorem 2.8 (iii) we obtain

$$\begin{aligned}
&\int_X v_i \, d\mu + \|Dv_i\|(X) + \mathrm{cap}_{\mathrm{BV}}(E_{i-1}) \\
&\quad \le \int_X v_i \, d\mu + \|Dv_i\|(X) + \int_X w_i \, d\mu + \|Dw_i\|(X) \\
&\quad \le \int_X v_{i-1} \, d\mu + \|Dv_{i-1}\|(X) + \int_X u_i \, d\mu + \|Du_i\|(X) \\
&\quad \le \int_X v_{i-1} \, d\mu + \|Dv_{i-1}\|(X) + \mathrm{cap}_{\mathrm{BV}}(E_i) + \varepsilon 2^{-i}
\end{aligned}$$

for every index $i = 1, 2 \ldots$ It then follows by adding that

$$\int_X v_i \, d\mu + \|Dv_i\|(X) \le \mathrm{cap}_{\mathrm{BV}}(E_i) + \sum_{j=1}^{i} \varepsilon 2^{-j}.$$

Let $v = \lim_{i\to\infty} v_i$. By the monotone convergence theorem we obtain

$$\int_X v \, d\mu = \lim_{i\to\infty} \int_X v_i \, d\mu \le \lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(E_i) + \varepsilon < \infty,$$

whereas Theorem 2.10 implies that

$$\|Dv\|(X) \le \liminf_{i\to\infty} \|Dv_i\|(X) < \infty.$$

Thus, $v \in \mathcal{A}_{\mathrm{BV}}(\bigcup\limits_{i=1}^{\infty} E_i)$ and

$$
\begin{aligned}
\operatorname{cap}_{\mathrm{BV}}\Big(\bigcup_{i=1}^{\infty} E_i\Big) &\le \int_X v\,d\mu + \|Dv\|(X) \\
&\le \liminf_{i\to\infty} \int_X v_i\,d\mu + \liminf_{i\to\infty} \|Dv_i\|(X) \\
&\le \liminf_{i\to\infty} \Big( \int_X v_i\,d\mu + \|Dv_i\|(X) \Big) \\
&\le \lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(E_i) + \varepsilon.
\end{aligned}
$$

The claim follows by letting $\varepsilon \to 0$. □

The next two results follow directly from the definition. The first theorem states that the BV-capacity is an outer capacity.

**Theorem 3.5.** *For every $E \subset X$ we have*

$$\operatorname{cap}_{\mathrm{BV}}(E) = \inf\{\operatorname{cap}_{\mathrm{BV}}(U) : U \supset E,\, U \text{ is open}\}.$$

*Proof.* By monotonicity

$$\operatorname{cap}_{\mathrm{BV}}(E) \le \inf\{\operatorname{cap}_{\mathrm{BV}}(U) : U \supset E,\, U \text{ is open}\}.$$

To prove the opposite inequality, we may assume that $\operatorname{cap}_{\mathrm{BV}}(E) < \infty$. Let $\varepsilon > 0$ and take $u \in \mathcal{A}_{\mathrm{BV}}(E)$ such that

$$\int_X u\,d\mu + \|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

Since $u \in \mathcal{A}_{\mathrm{BV}}(E)$ there is an open set $U \supset E$ such that $u = 1$ on $U$, from which it follows that

$$\operatorname{cap}_{\mathrm{BV}}(U) \le \int_X u\,d\mu + \|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

Hence

$$\inf\{\operatorname{cap}_{\mathrm{BV}}(U) : U \supset E,\, U \text{ is open}\} \le \operatorname{cap}_{\mathrm{BV}}(E).$$ □

The next result states that the BV-capacity behaves well with respect to limits of a decreasing sequence of compact sets.

**Theorem 3.6.** *If $K_1 \supset \ldots \supset K_i \supset K_{i+1} \supset \ldots$ are compact subsets of $X$, then*

$$\operatorname{cap}_{\mathrm{BV}}\Big(\bigcap_{i=1}^{\infty} K_i\Big) = \lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(K_i).$$

*Proof.* By monotonicity

$$\lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(K_i) \ge \operatorname{cap}_{\mathrm{BV}}(K),$$

where $K = \bigcap_{i=1}^{\infty} K_i$. Let $U$ be an open set containing $K$. Now by the compactness of $K$, $K_i \subset U$ for all sufficiently large $i$, and therefore

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(K_i) \le \mathrm{cap}_{\mathrm{BV}}(U)$$

and since the BV-capacity is an outer capacity, we obtain the claim by taking infimum over all open sets $U$ containing $K$. □

It turns out that the BV-capacity satisfies the following strong subadditivity property.

**Theorem 3.7.** *If $E_1, E_2 \subset X$, then*

$$\mathrm{cap}_{\mathrm{BV}}(E_1 \cup E_2) + \mathrm{cap}_{\mathrm{BV}}(E_1 \cap E_2) \le \mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2).$$

*Proof.* We may assume that $\mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2) < \infty$. Let $\varepsilon > 0$ and $u_1 \in \mathcal{A}_{\mathrm{BV}}(E_1)$ and $u_2 \in \mathcal{A}_{\mathrm{BV}}(E_2)$ be such that

$$\int_X u_1\, d\mu + \|Du_1\|(X) < \mathrm{cap}_{\mathrm{BV}}(E_1) + \frac{\varepsilon}{2}$$

and

$$\int_X u_2\, d\mu + \|Du_2\|(X) < \mathrm{cap}_{\mathrm{BV}}(E_2) + \frac{\varepsilon}{2}.$$

Clearly $\max\{u_1, u_2\} \in \mathcal{A}_{\mathrm{BV}}(E_1 \cup E_2)$ and $\min\{u_1, u_2\} \in \mathcal{A}_{\mathrm{BV}}(E_1 \cap E_2)$. By Theorem 2.8 (iii) we obtain

$$\begin{aligned}
&\mathrm{cap}_{\mathrm{BV}}(E_1 \cup E_2) + \mathrm{cap}_{\mathrm{BV}}(E_1 \cap E_2) \\
&\qquad\le \int_X \max\{u_1, u_2\}\, d\mu + \int_X \min\{u_1, u_2\}\, d\mu \\
&\qquad\quad + \|D\max\{u_1, u_2\}\|(X) + \|D\min\{u_1, u_2\}\|(X) \\
&\qquad\le \int_X (u_1 + u_2)\, d\mu + \|Du_1\|(X) + \|Du_2\|(X) \\
&\qquad= \int_X u_1\, d\mu + \|Du_1\|(X) + \int_X u_2\, d\mu + \|Du_2\|(X) \\
&\qquad< \mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the claim. □

Theorems 3.3, 3.4, 3.5 and 3.6, together with the general theory of capacities in [6], imply that BV-capacity is a *Choquet capacity*, and therefore we have the following result.

**Corollary 3.8.** *All Suslin sets $E \subset X$ are capacitable, this is,*

$$\begin{aligned}
\mathrm{cap}_{\mathrm{BV}}(E) &= \inf\{\mathrm{cap}_{\mathrm{BV}}(U) : E \subset U,\, U \text{ is open}\} \\
&= \sup\{\mathrm{cap}_{\mathrm{BV}}(K) : K \subset E,\, K \text{ is compact}\}.
\end{aligned}$$

*In particular, all Borel sets are capacitable.*

The next theorem states that for compact sets, we only need to consider compactly supported Lipschitz functions in the definition of the BV-capacity.

**Theorem 3.9.** *Let $K$ be a compact subset of $X$. Then*

$$\operatorname{cap}_{\mathrm{BV}}(K) = \inf\Big(\int_X u\,d\mu + \|Du\|(X)\Big),$$

*where infimum is taken over all functions $u \in \operatorname{Lip}_c(X)$ such that $0 \le u \le 1$ and $u = 1$ on a neighbourhood of $K$.*

*Proof.* Clearly

$$\operatorname{cap}_{\mathrm{BV}}(K) \le \inf\Big(\int_X u\,d\mu + \|Du\|(X)\Big),$$

where infimum is taken over all functions $u \in \operatorname{Lip}_c(X)$, $0 \le u \le 1$ such that $u = 1$ on a neighbourhood of $K$. In order to prove the inequality in the opposite direction, let $u \in \mathcal{A}_{\mathrm{BV}}(K)$ be such that

$$\int_X u\,d\mu + \|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(K) + \varepsilon,$$

and $u = 1$ in open set $U \supset K$ with $\mu(U) < \infty$. Since $K$ is compact and $X \setminus U$ is a closed set, we can find an open set $U' \subset U$ such that $K \subset U' \subset\subset U$ and $\operatorname{dist}(U', X \setminus U) > 0$. By Lemma 2.7 there is a sequence of functions $u_i \in \operatorname{Lip}_c(X)$, $i = 1, 2, \dots$, with $0 \le u_i \le 1$, such that $u_i \to u$ in $L^1(X)$ and

$$\int_X g_{u_i}\,d\mu \to \|Du\|(X),$$

as $i \to \infty$. Let $\eta \in \operatorname{Lip}_c(X)$ be a cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ in $U'$ and $\eta = 0$ in $X \setminus U$. For every $i = 1, 2, \dots$ we define functions

$$v_i = (1-\eta)u_i + \eta,$$

and notice that $v_i \in \operatorname{Lip}_c(X)$. Also $v_i = 1$ in $U'$ for every index $i$ and $(1-u_i)g_\eta + (1-\eta)g_{u_i}$ is a 1-weak upper gradient of $v_i$. To see this, we use the fact that $v_i$ is absolutely continuous on paths as in Lemma 3.1 of [22]. See also [3]. Clearly we can assume that $g_\eta$ is bounded and $g_\eta = 0$ in $X \setminus U$. Since $u = (1-\eta)u + \eta$, we obtain

$$\begin{aligned}
&\limsup_{i\to\infty}\Big(\int_X v_i\,d\mu + \|Dv_i\|(X)\Big)\\
&\le \limsup_{i\to\infty}\int_X v_i\,d\mu + \limsup_{i\to\infty}\|Dv_i\|(X)\\
&\le \int_X u\,d\mu + \limsup_{i\to\infty}\int_X (1-u_i)g_\eta\,d\mu + \limsup_{i\to\infty}\int_X g_{u_i}\,d\mu\\
&\le \int_X u\,d\mu + \|g_\eta\|_\infty \limsup_{i\to\infty}\int_U |u-u_i|\,d\mu + \|Du\|(X)\\
&= \int_X u\,d\mu + \|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(K) + \varepsilon.
\end{aligned}$$

Therefore, for every $\varepsilon > 0$ we can find a compactly supported admissible Lipschitz function $v$ such that

$$\int_X v\,d\mu + \|Dv\|(X) < \operatorname{cap}_{\mathrm{BV}}(K) + \varepsilon.$$

The desired inequality follows by taking infimum over such functions. □

## 4. Equivalence of the capacities

In this section we prove that the Sobolev 1-capacity is equivalent to the BV-capacity by two sided estimates. The equivalence of capacities follows from a modification of a metric space version of Gustin's boxing inequality. We will also give examples, which demonstrate that the Sobolev 1-capacity is not necessarily a Choquet capacity.

**Definition 4.1.** Let $E \subset X$. Denote by $\mathcal{A}_1(E)$ the set of *admissible functions* $u \in N^{1,1}(X)$ such that $0 \le u \le 1$ and $u = 1$ on a neighbourhood of $E$. The *Sobolev 1-capacity* of $E$ is

$$\operatorname{cap}_1(E) = \inf \|u\|_{N^{1,1}(X)},$$

where the infimum is taken over all functions $u \in \mathcal{A}_1(E)$.

*Remark.* The functions in $N^{1,p}(X)$ with $1 \le p < \infty$ are necessarily *$p$-quasicontinuous* (see [5] and [3]) and thus the above definition of the capacity agrees with the definition of the Sobolev 1-capacity where the functions are required to satisfy $u = 1$ only in $E$.

It is well known that many of the results presented in the third section are also true for the Sobolev 1-capacity. Indeed,

(i) $\operatorname{cap}_1(\cdot)$ is an outer measure,
(ii) $\operatorname{cap}_1(\cdot)$ is an outer capacity,
(iii) If $K_1 \supset \ldots \supset K_i \supset K_{i+1} \supset \ldots$ are compact subsets of $X$, then

$$\operatorname{cap}_1\Big(\bigcap_{i=1}^{\infty} K_i\Big) = \lim_{i\to\infty} \operatorname{cap}_1(K_i),$$

(iv) $\operatorname{cap}_1(\cdot)$ satisfies the strong subadditivity property.

However, as we will see, the Sobolev 1-capacity fails to be a Choquet capacity. Our next goal is to prove that the BV-capacity and the Sobolev 1-capacity are equivalent. To this end, we need the following modified version of the *boxing inequality*, see [12], [8] and [19].

**Lemma 4.2.** *Let $E \subset X$ be a $\mu$-measurable set such that*

$$\lim_{r\to 0} \frac{\mu\big(E \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1$$

*for every $x \in E$. Then for every $0 < R \le 1$ there exists a collection of disjoint balls $B(x_i, \tau r_i)$, $i = 1, 2, \dots$, such that $0 < r_i \le R$ and*

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i)$$

*and disjoint sets of indices $I_1, I_2$, $I_1 \cup I_2 = \mathbb{N}$, such that $r_i \ge R/2$ for every $i \in I_1$ and*

$$\sum_{i \in I_1} \mu\big(B(x_i, 5\tau r_i)\big) + \sum_{i \in I_2} \frac{\mu\big(B(x_i, 5\tau r_i)\big)}{5\tau r_i} \le c\big(\mu(E) + P(E, X)\big).$$

*Here $\tau$ is the dilation constant in the weak Poincaré inequality and the constant $c$ depends only on the doubling constant and the constants in the weak $(1,1)$-Poincaré inequality.*

*Proof.* We may assume that $\mu(E) + P(E, X) < \infty$. Let $x \in E$ and denote

$$\widetilde{r}_x = \sup\Big\{0 < r \le R : \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \le 1 - \frac{3}{4C_D}\Big\}. \tag{4.3}$$

We choose $r_x$, which satisfies $r_x < \widetilde{r}_x < 2r_x$, such that the inequality in (4.3) holds for $r_x$. We apply a covering argument to obtain pairwise disjoint balls $B(x_i, \tau r_i)$, $i = 1, 2, \dots$, such that

$$\bigcup_{x \in E} B(x, \tau r_x) \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i).$$

It follows that

$$\mu(B(x_i, r_i) \cap E) \ge \frac{3}{4C_D} \mu(B(x_i, r_i)),$$

Denote by $I_2$ the indices for which

$$\mu(B(x_i, r_i) \setminus E) > \frac{1}{4} \mu(B(x_i, r_i)).$$

By the relative isoperimetric inequality (Theorem 2.9), we obtain

$$\begin{aligned}\frac{\mu(B(x_i, r_i))}{r_i} &\le \Big(4 + \frac{4C_D}{3}\Big) \frac{\min\{\mu(B(x_i, r_i) \cap E), \mu(B(x_i, r_i) \setminus E)\}}{r_i} \\ &\le cP(E, B(x_i, \tau r_i))\end{aligned}$$

for every index $i \in I_2$.

Let $I_1 = \mathbb{N} \setminus I_2$. Then

$$\mu(B(x_i, r_i) \cap E) \ge \frac{3}{4} \mu(B(x_i, r_i))$$

for every $i \in I_1$. We claim that $r_i \geq R/2$ for every $i \in I_1$. Indeed, we have

$$\begin{aligned}\mu(B(x_i, 2r_i) \setminus E) &= \mu(B(x_i, 2r_i)) - \mu(B(x_i, 2r_i) \cap E)\\ &\leq \mu(B(x_i, 2r_i)) - \mu(B(x_i, r_i) \cap E)\\ &\leq \mu(B(x_i, 2r_i)) - \frac{3}{4}\mu(B(x_i, r_i))\\ &\leq \mu(B(x_i, 2r_i)) - \frac{3}{4C_D}\mu(B(x_i, 2r_i)).\end{aligned}$$

Hence

$$\frac{\mu(B(x_i, 2r_i) \setminus E)}{\mu(B(x_i, 2r_i))} \leq 1 - \frac{3}{4C_D},$$

and if $2r_i < R$, then this contradicts with the choice of $r_i$. Therefore $r_i \geq R/2$ for every $i \in I_1$.

By the doubling property of the measure $\mu$ we obtain

$$\begin{aligned}&\sum_{i\in I_1}\mu(B(x_i, 5\tau r_i)) + \sum_{i\in I_2}\frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i}\\ &\quad\leq c\Big(\sum_{i\in I_1}\mu(B(x_i, r_i)) + \sum_{i\in I_2}\frac{\mu(B(x_i, r_i))}{r_i}\Big)\\ &\quad\leq c\Big(\sum_{i\in I_1}\mu(B(x_i, r_i)\cap E) + \sum_{i\in I_2}P(E, B(x_i, \tau r_i))\Big)\\ &\quad\leq c\Big(\mu\big(\bigcup_{i\in I_1}B(x_i, r_i)\cap E\big) + P\big(E, \bigcup_{i\in I_2}B(x_i, \tau r_i)\big)\Big)\\ &\quad\leq c(\mu(E) + P(E, X)).\end{aligned}$$

Here we also used the facts that the balls are disjoint and that $P(E, \cdot)$ is a Borel measure by Theorem 2.6. $\square$

Now we are ready to prove the main result of this section.

**Theorem 4.4.** *For any set $E \subset X$, we have*

$$\operatorname{cap}_{\mathrm{BV}}(E) \leq \operatorname{cap}_1(E) \leq c \operatorname{cap}_{\mathrm{BV}}(E),$$

*where the constant $c$ depends only on the doubling constant and the constants in the weak $(1,1)$-Poincaré inequality.*

*Proof.* Clearly $\operatorname{cap}_{\mathrm{BV}}(E) \leq \operatorname{cap}_1(E)$. To prove the second inequality, we may assume that $\operatorname{cap}_{\mathrm{BV}}(E) < \infty$. Let $\varepsilon > 0$ and choose a function $u \in \mathcal{A}_{\mathrm{BV}}(E)$ such that

$$\int_X u\, d\mu + \|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

By the coarea formula and the Cavalieri principle

$$\int_X u\,d\mu + \|Du\|(X) = \int_0^1 \big(\mu(\{u>t\}) + P(\{u>t\},X)\big)dt,$$

and thus

$$\mu(\{u>t_0\}) + P(\{u>t_0\},X) < \mathrm{cap}_{\mathrm{BV}}(E) + \varepsilon$$

for some $0 < t_0 < 1$. We denote $E_{t_0} = \{u > t_0\}$ and

$$E^*_{t_0} = \Big\{x \in E_{t_0} : \lim_{r\to 0} \frac{\mu(E_{t_0} \cap B(x,r))}{\mu(B(x,r))} = 1\Big\}.$$

From the Lebesgue's differentiation theorem, see [15], it follows that $\mu(E^*_{t_0}) = \mu(E_{t_0}) < \infty$ and hence $P(E^*_{t_0}, X) = P(E_{t_0}, X) < \infty$. Furthermore,

$$E \subset \mathrm{int}\{u = 1\} \subset E^*_{t_0}.$$

We apply Lemma 4.2 with $R = 1$ to obtain a covering

$$E^*_{t_0} \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i)$$

such that

$$\sum_{i\in I_1} \mu(B(x_i,5\tau r_i)) + \sum_{i\in I_2} \frac{\mu(B(x_i,5\tau r_i))}{5\tau r_i} \le c\big(\mu(E_{t_0}) + P(E_{t_0},X)\big).$$

It follows that

$$\begin{aligned}\mathrm{cap}_1(E) \le \mathrm{cap}_1(E^*_{t_0}) &\le \sum_{i=1}^{\infty} \mathrm{cap}_1(B(x_i,5\tau r_i))\\ &\le \sum_{i\in I_1} \mathrm{cap}_1(B(x_i,5\tau r_i)) + \sum_{i\in I_2} \mathrm{cap}_1(B(x_i,5\tau r_i)).\end{aligned}$$

By applying the admissible function

$$u_i(x) = \Big(1 - \frac{\mathrm{dist}(x, B(x_i,5\tau r_i))}{5\tau r_i}\Big)^+$$

for every index $i = 1, 2, \dots$, we observe that

$$\mathrm{cap}_1(B(x_i,5\tau r_i)) \le c\Big(\mu(B(x_i,5\tau r_i)) + \frac{\mu(B(x_i,5\tau r_i))}{5\tau r_i}\Big).$$

Since $r_i \ge 1/2$ for every $i \in I_1$ we have

$$\begin{aligned}\sum_{i\in I_1} \mathrm{cap}_1(B(x_i,5\tau r_i) &\le c\sum_{i\in I_1} \Big(\mu(B(x_i,5\tau r_i)) + \frac{\mu(B(x_i,5\tau r_i))}{5\tau r_i}\Big)\\ &\le c\sum_{i\in I_1} \mu(B(x_i,5\tau r_i)).\end{aligned}$$

On the other hand, since $0 < r_i \leq 1$ for every index $i$ we obtain

$$\begin{aligned}\sum_{i \in I_2} \mathrm{cap}_1(B(x_i, 5\tau r_i)) &\leq c \sum_{i \in I_2} \Big( \mu\big(B(x_i, 5\tau r_i)\big) + \frac{\mu\big(B(x_i, 5\tau r_i)\big)}{5\tau r_i} \Big) \\ &\leq c \sum_{i \in I_2} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i}.\end{aligned}$$

Consequently

$$\begin{aligned}\mathrm{cap}_1(E) &\leq c \sum_{i \in I_1} \mu(B(x_i, 5\tau r_i)) + c \sum_{i \in I_2} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \\ &\leq c(\mu(E_{t_0}) + P(E_{t_0}, X)) \\ &< c\big(\mathrm{cap}_{\mathrm{BV}}(E) + \varepsilon\big).\end{aligned}$$

The claim follows letting $\varepsilon \to 0$. $\square$

By Theorem 4.4 the Sobolev 1-capacity is equivalent to the BV-capacity and we have the following immediate consequence.

**Corollary 4.5.** *The Sobolev* 1*-capacity satisfies the following properties:*

(i) *There is a constant $c$ such that for any incresing sequence of sets $E_1 \subset E_2 \subset \ldots \subset E_i \subset \ldots \subset X$ we have*

$$\lim_{i \to \infty} \mathrm{cap}_1(E_i) \leq \mathrm{cap}_1\big(\bigcup_{i=1}^{\infty} E_i\big) \leq c \lim_{i \to \infty} \mathrm{cap}_1(E_i),$$

(ii)

$$\mathrm{cap}_1(E) \leq c \sup\{\mathrm{cap}_1(K) : K \subset E,\ K \text{ is compact}\}$$

From Theorem 3.9 it follows that the BV-capacity and the Sobolev 1-capacity are equal for compact sets. However, the following example demonstrates that the BV-capacity and the Sobolev 1-capacity are not ncessarily equal for noncompact sets. Therefore, the equivalence results like Theorem 4.4 and Corollary 4.5 cannot be improved and the Sobolev 1-capacity is not a Choquet capacity. Originally, this kind of example has been constructed by Riikka Korte, see also [3].

**Example 4.6.** Let $m$ denote the ordinary Lebesgue measure in $\mathbb{R}^2$ and $\gamma = \mathrm{cap}_1(B(0,1))$ the usual Sobolev 1-capacity of the unit ball in $\big(\mathbb{R}^2, |\cdot|, m\big)$. Clearly $\gamma > \pi$. We define $d\mu = w\, dm$, where

$$w(x) = \begin{cases} (\gamma - \pi)/4\pi, & x \in B(0,1), \\ 1, & x \in \mathbb{R}^2 \setminus B(0,1) \end{cases}$$

and obtain a weighted measure $\mu$. The space $X = (\mathbb{R}^2, |\cdot|, \mu)$ is a metric measure space equipped with a doubling measure and supporting the weak $(1,1)$-Poincaré inequality. For every index $i = 2,3,4,\dots$ let $B_i = B(0, 1-i^{-1})$ and

$$u_i(x) = \min\big\{1, \max\{0, -2i|x| + 2i - 1\}\big\}$$

and notice that $u_i \in \mathcal{A}_1(B_i)$. Hence, for every index $i$ we can estimate the 1-capacity of the set $B_i$ in $X$ by

$$\begin{aligned}
\operatorname{cap}_1(B_i) &\le \int_{B(0,1)} 1\,d\mu + \int_{\mathbb{R}^2} |Du_i|\,d\mu \\
&= \frac{\gamma-\pi}{4} + \frac{\gamma-\pi}{4\pi}\int_{B_{2i}\setminus B_i} 2i\,dm \\
&= \frac{\gamma-\pi}{4} + \frac{\gamma-\pi}{2}\Big(1 - \frac{3}{4i}\Big) < \frac{3(\gamma-\pi)}{4}.
\end{aligned}$$

It is clear that in $\big(\mathbb{R}^2, |\cdot|, \mu\big)$

$$\operatorname{cap}_1\big(\bigcup_{i=2}^{\infty} B_i\big) = \operatorname{cap}_1(B(0,1)) = \gamma - \Big(1 - \frac{\gamma-\pi}{4\pi}\Big)\pi = \frac{5(\gamma-\pi)}{4}.$$

Thus

$$\lim_{i\to\infty} \operatorname{cap}_1(B_i) \le \frac{3(\gamma-\pi)}{4} < \operatorname{cap}_1(B(0,1)).$$

A modification of the previous example shows that the Sobolev 1-capacity is not necessarily a Choquet capacity.

**Example 4.7.** Let $\gamma, \mu, B_i$ and $u_i$ be as in the Example 4.6. Then for any compact set $K \subset B(0,1) = B$, we have that $\operatorname{dist}(K, X\setminus B) > 0$ and therefore $K \subset B_i$ for some index $i$. Thus, by the previous example

$$\operatorname{cap}_1(K) \le \operatorname{cap}_1(B_i) < \frac{3(\gamma-\pi)}{4}.$$

Hence

$$\sup\{\operatorname{cap}_1(K) : K \subset B,\ K \text{ is compact}\} < \operatorname{cap}_1(B),$$

and the Sobolev 1-capacity is not a Choquet capacity.

## 5. Connections to Hausdorff measure

The *restricted spherical Hausdorff content of codimension one* of $E$ is defined as

$$\mathcal{H}_R(E) = \inf\Big\{\sum_{i=1}^{\infty} \frac{\mu\big(B(x_i,r_i)\big)}{r_i} : E \subset \bigcup_{i=1}^{\infty} B(x_i,r_i), r_i \le R\Big\},$$

where $0 < R < \infty$. The Hausdorff measure of codimension one is obtained as a limit

$$\mathcal{H}(E) = \lim_{R\to 0} \mathcal{H}_R(E).$$

Next theorem shows that the BV-capacity and the Hausdorff measure of codimension one have the same null sets.

**Theorem 5.1.** *Let $E \subset X$. Then $\operatorname{cap}_{\mathrm{BV}}(E) = 0$ if and only if $\mathcal{H}(E) = 0$.*

*Proof.* Let us first assume that $\mathcal{H}(E) = 0$. Let $\varepsilon > 0$ and $B(x_i, r_i)$, $i = 1, 2, \dots$, be a covering of $E$ such that $r_i \le 1$ for every $i = 1, 2 \dots$ and

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} < \varepsilon.$$

Let

$$u_i(x) = \Big(1 - \frac{\operatorname{dist}(x, B(x_i, r_i))}{r_i}\Big)^+$$

and observe that

$$\begin{aligned}\operatorname{cap}_{\mathrm{BV}}(B(x_i, r_i)) &\le C_D\Big(\mu(B(x_i, r_i)) + \frac{\mu(B(x_i, r_i))}{r_i}\Big)\\ &\le 2C_D \frac{\mu(B(x_i, r_i))}{r_i}.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{cap}_{\mathrm{BV}}(E) &\le \sum_{i=1}^{\infty} \operatorname{cap}_{\mathrm{BV}}(B(x_i, r_i))\\ &\le 2C_D \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \le 2C_D\varepsilon.\end{aligned}$$

By taking $\varepsilon \longrightarrow 0$ it follows that $\operatorname{cap}_{\mathrm{BV}}(E) = 0$.

Then assume that $\operatorname{cap}_{\mathrm{BV}}(E) = 0$. Then for every index $i = 1, 2, \dots$ we can choose function $u_i \in \mathcal{A}_{\mathrm{BV}}(E)$ such that

$$\int_X u_i \, d\mu + \|Du_i\|(X) < \frac{1}{i^2}.$$

By the Cavalieri principle and the coarea formula, as in the proof of Theorem 4.4, for every $i = 1, 2, \dots$ we obtain $0 < t_i < 1$ such that

$$\mu(\{u_i > t_i\}) + P(\{u_i > t_i\}, X) < \frac{1}{i^2}.$$

We denote $E_{t_i} = \{u_i > t_i\}$ and

$$E_{t_i}^* = \Big\{x \in E_{t_i} : \lim_{r \to 0} \frac{\mu(E_{t_i} \cap B(x, r))}{\mu(B(x, r))} = 1\Big\}.$$

From the Lebesgue's differentiation theorem it follows that $\mu(E_{t_i}^*) = \mu(E_{t_i}) < \infty$, and consequently, we have $P(E_{t_i}^*, X) = P(E_{t_i}, X) < \infty$. As in the proof of Theorem 4.4, we have

$$E \subset \operatorname{int}\{u = 1\} \subset E_{t_i}^*.$$

For every $i = 1, 2, \dots$ we apply Lemma 4.2 for $E^*_{t_i}$ with $R = 1/(10\tau i)$ to obtain a covering

$$E^*_{t_i} \subset \bigcup_{j=1}^{\infty} B(x^i_j, 5\tau r^i_j)$$

such that

$$\sum_{j \in I^i_1} \mu(B(x^i_j, 5\tau r^i_j)) + \sum_{j \in I^i_2} \frac{\mu(B(x^i_j, 5\tau r^i_j))}{5\tau r^i_j} \le c\big(\mu(E_{t_i}) + P(E_{t_i}, X)\big).$$

For every $i = 1, 2, \dots$ we have following estimate

$$\begin{aligned}
\mathcal{H}_{1/i}(E^*_{t_i}) &\le \sum_{j=1}^{\infty} \frac{\mu(B(x^i_j, 5\tau r^i_j))}{5\tau r^i_j} \\
&\le \sum_{j \in I^i_1} \frac{1}{5\tau r^i_j} \mu(B(x^i_j, 5\tau r^i_j)) + \sum_{j \in I^i_2} \frac{\mu(B(x^i_j, 5\tau r^i_j))}{5\tau r^i_j} \\
&\le 4i\Big( \sum_{j \in I^i_1} \mu(B(x^i_j, 5\tau r^i_j)) + \sum_{j \in I^i_2} \frac{\mu(B(x^i_j, 5\tau r^i_j))}{5\tau r^i_j} \Big) \\
&< \frac{4ci}{i^2} = \frac{c}{i}.
\end{aligned}$$

Hence

$$\mathcal{H}(E) = \lim_{i \to \infty} \mathcal{H}_{1/i}(E) \le \limsup_{i \to \infty} \mathcal{H}_{1/i}(E^*_{t_i}) \le \limsup_{i \to \infty} \frac{c}{i} = 0.$$

□

*Remark.* By Theorem 4.4 we obtain that the Sobolev 1-capacity and te Hausdorff measure of codimension one have the same null sets.

## References


[1] L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. Math. **159** (2001), no. 1, 51–67.

[2] L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces. Calculus of variations, nonsmooth analysis and related topics*, Set-Valued Anal. **10** (2002), no. 2–3, 111–128.

[3] A. Björn, J. Björn, *Nonlinear potential theory in metric spaces*, to appear.

[4] A. Björn, J. Björn and M. Parviainen, *Lebesgue points and the fundamental convergence theorem for superharmonic functions*, to appear in Rev. Mat. Iberoamericana.

[5] A. Björn, J. Björn, and N. Shanmugalingam, *Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces*, Houston Math. J. **34** (2008), no. 4, 1197–1211.

[6] G. Choquet, *Forme abstraite du téorème de capacitabilité*, (French) Ann. Inst. Fourier. (Grenoble) **9** (1959), 83–89.

[7] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[8] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York, 1969.

[9] H. Federer, W.P. Ziemer, *The Lebesgue set of a function whose distribution derivatives are p-th power summable*, Indiana Univ. Math. J. **22** (1972/73), 139–158.
[10] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.
[11] V. Gol'dshtein, M. Troyanov, *Capacities in metric spaces*, Integral Equations Operator Theory **44** (2002), no. 2, 212–242.
[12] W. Gustin, *Boxing inequalities*, J. Math. Mech. **9** (1960) 229-239.
[13] P. Hajłasz, *Sobolev spaces on metric-measure spaces. (Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002))*, 173-218, Contemp. Math., **338**, Amer. Math. Soc., Providence, RI, 2003.
[14] P. Hajłasz, P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688.
[15] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.
[16] S. Kallunki, N. Shanmugalingam, *Modulus and continuous capacity*, Ann. Acad. Sci. Fenn. Math. **26** (2001), no. 2, 455–464.
[17] T. Kilpeläinen, J. Kinnunen, O. Martio, *Sobolev spaces with zero boundary values on metric spaces*, Potential Anal. **12** (2000), no. 3, 233–247.
[18] J. Kinnunen and V. Latvala, Lebesgue points for Sobolev functions on metric spaces, Rev. Mat. Iberoamericana **18** (2002), 685–700.
[19] J. Kinnunen, R. Korte, N. Shanmugalingam, H. Tuominen, *Lebesgue points and capacities via the boxing inequality in metric spaces*, Indiana Univ. Math. J. **57** (2008), no. 1, 401–430.
[20] J. Kinnunen, O. Martio, *The Sobolev capacity on metric spaces*, Ann. Acad. Sci. Fenn. Math. **21** (1996), no. 2, 367–382.
[21] J. Kinnunen, O. Martio, *Choquet property for the Sobolev capacity in metric spaces*, Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), 285-290, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.
[22] J. Kinnunen and O. Martio, *Sobolev space properties of superharmonic functions on metric spaces*, Results Math. **44** (2003), no. 1-2, 114–129.
[23] V.G. Maz'ja, *Sobolev spaces. Translated from the Russian by T. O. Shaposhnikova*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.
[24] M. Miranda, Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004.
[25] T. Mäkäläinen, *Adams inequality on metric measure spaces*, Rev. Mat. Iberoamericana **25** (2) (2009), 533–558.
[26] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279.
[27] W.P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120, Springer-Verlag, New York, 1989.

Addresses:
H.H: Department of Mathematical Sciences, P.O. Box 3000, FI-90014 University of Oulu, Finland.
E-mail: `heikki.hakkarainen@oulu.fi`

J.K.: Department of Mathematics, P.O. Box 1100, FI-02015 Helsinki University of Technology, Finland.

E-mail: juha.kinnunen@tkk.fi