HARNACK'S INEQUALITY FOR PARABOLIC DE GIORGI CLASSES IN METRIC SPACES

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ABSTRACT. In this paper we study problems related to parabolic partial differential equations in metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality. We give a definition of parabolic De Giorgi classes and compare this notion with that of parabolic quasiminimizers. The main result, after proving the local boundedness, is a scale and location invariant Harnack inequality for functions belonging to parabolic De Giorgi classes. In particular, the results hold true for parabolic quasiminimizers.

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1. INTRODUCTION

The purpose of this paper is to study problems related to the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

in metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality. We consider a notion of parabolic De Giorgi classes and parabolic quasiminimizers with quadratic structure conditions and study local regularity properties of functions belonging to these classes. More precisely, we show that functions in parabolic De Giorgi classes, satisfy a scale and location invariant Harnack inequality, see Theorem 5.7. Some consequences of the parabolic Harnack inequality are the local Hölder continuity and the strong maximum principle for the parabolic De Giorgi classes. Our assumptions on the metric space are rather standard to allow a reasonable first-order calculus; the reader should consult, e.g., Björn and Björn [3] and Heinonen [20], and the references therein.

Harnack type inequalities play an important role in the regularity theory of solutions to both elliptic and parabolic partial differential equations as it implies local Hölder continuity for the solutions. A parabolic Harnack inequality is logically stronger than an elliptic one since the reproduction at each time of the same harmonic function is a solution of the heat equation. There is, however, a well-known fundamental difference between elliptic and parabolic Harnack estimates. Roughly speaking, in the elliptic case the information of a positive solution on a ball is controlled by the infimum on the same ball. In the parabolic case a delay in time is needed: the information of a positive solution at a point and at instant t_0 is controlled by a ball centered at the same point but later time $t_0 + t_1$, where t_1 depends on the parabolic equation.

Elliptic quasiminimizers were introduced by Giaquinta–Giusti [14] and [15] as a tool for a unified treatment of variational integrals, elliptic equations and systems, and quasiregular mappings on $\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a Q-quasiminimizer, $Q \geq 1$, related to the power p in Ω if

$$\int_{\mathrm{supp}(\phi)} |\nabla u|^p \, dx \leq Q \int_{\mathrm{supp}(\phi)} |\nabla (u - \phi)|^p \, dx$$

for all $\phi \in W^{1,p}_0(\Omega)$. Giaquinta and Giusti realized that De Giorgi's method [7] could be extended to quasiminimizers, obtaining, in particular, local Hölder continuity. DiBenedetto and Trudinger [11] proved the Harnack inequality for quasiminimizers. These results were extended to metric spaces by Kinnunen and Shanmugalingam [24]. Elliptic quasiminimizers enable the study of elliptic problems, such as the p-Laplace equation and p-harmonic functions, in metric spaces under the doubling property of the measure and a Poincaré inequality. Compared with the theory of p-harmonic functions we have no partial differential equation, only the variational approach is available. There is also no comparison principle nor uniqueness for the Dirichlet problem for quasiminimizers. See, e.g., J. Björn [4], Kinnunen–Martio [23], Martio–Sbordone [28] and the references in these papers for more on elliptic quasiminimizers.

Following Giaquinta–Giusti, Wieser [36] generalized the notion of quasiminimizers to the parabolic setting in Euclidean spaces. A function $u : \Omega \times (0,T) \to \mathbf{R}$, $u \in L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(\Omega))$, is a parabolic Q-quasiminimizer, $Q \geq 1$, for the heat equation (thus related to the power 2) if

$$-\iint_{\mathrm{supp}(\phi)} u\frac{\partial \phi}{\partial t}\,dx\,dt + \iint_{\mathrm{supp}(\phi)} \frac{|\nabla u|^2}{2}\,dx\,dt \leq Q \iint_{\mathrm{supp}(\phi)} \frac{|\nabla (u-\phi)|^2}{2}\,dx\,dt$$

for every smooth compactly supported function ϕ in $\Omega \times (0,T)$. Parabolic quasiminimizers have also been studied by Zhou [37, 38], Gianazza–Vespri [13], Marchi [27], and Wang [35]. The literature for parabolic quasiminimizers is very small compared to the elliptic case. One of the goals of this work is to introduce parabolic quasiminimizers in metric metric spaces. This opens up a possibility to develop a systematic theory for parabolic problems in this generality.

The present paper is using the ideas of DiBenedetto [9] and is based on the lecture notes [12] of the course held by V. Vespri in Lecce. We would like to point out that the definition for the parabolic De Giorgi classes given by Gianazza and Vespri [13] is sligthly different from ours, and it seems that our class is larger. Naturally, our abstract setting causes new difficulties. For example, Lemma 2.5 plays a crucial role in the proof of Harnack's inequality. In Euclidean spaces this abstract lemma dates back to DiBenedetto–Gianazza-Vespri [10], but as the proof uses the linear sructure of the ambient space a new proof in the metric setting was needed.

Motivation for this work is to consider the Saloff-Coste–Grigor'yan theorem in metric measure spaces. Grigor'yan [17] and Saloff-Coste [29] observed independently that the doubling property for the measure and the Poincaré inequality are sufficient and necessary conditions for a scale invariant parabolic Harnack inequality for the heat equation on Riemannian manifolds. Sturm [32] generalized this result to the setting of local Dirichlet spaces; such approach works also in fractal geometries, but always when a Dirichlet form is defined. For references, see for instance Barlow–Bass–Kumagai [1] and also the forthcoming paper by Barlow–Grigor'yan–Kumagai [2].

In this paper we introduce a version of parabolic De Giorgi classes that include parabolic quasiminimizers and show the sufficiency of the Saloff-Coste–Grigor'yan theorem in metric measure spaces without using Dirichlet spaces nor the Cheeger differentiable structure [6]. Very recently a similar question has been studied for degenerate parabolic quasilinear partial differential equations in the subelliptic case by Caponga–Citti–Rea [5]. Their motivating example is a class of subelliptic operators associated to a family of Hörmander vector fields and their Carnot–Carathéodory distance. We show that the doubling property and the Poincaré inequality implies a scale and location invariant parabolic Harnack inequality for functions belonging to De Giorgi classes, and thus for parabolic quasiminimizers, in general metric measure spaces. It would be very interesting to know whether also necessity holds in this setting for De Giorgi classes or for parabolic quasiminimizers; using the results contained in Sturm [33, 34], it is possible to contruct a regular Dirichlet form, and then a diffusion process, on every locally compact metric spaces, and this, combined with Sturm [32] can be used to obtain the reverse implication. This is, however, a very abstract result and it is based on Γ-convergence of non-local Dirichlet forms, with no information on the limiting Dirichlet form. Such geometric characterization via the doubling property of the measure and a Poincaré inequality is not available for an elliptic Harnack inequality, see Delmotte [8].

The paper is organized as follows. In Section 2 we recall the definition of Newton–Sobolev spaces and prove some preliminary technical results; these results are general results on Sobolev functions and are of independent interest. In Section 3 we introduce the parabolic De Giorgi classes of order 2 and define parabolic quasiminimizers. In Section 4 we prove the‘ local boundedness of elements in the De Giorgi classes, and finally, in Section 5 we prove a Harnack-type inequality.

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2. Preliminaries

In this section we briefly recall the basic definitions and collect some results needed in the sequel. For a more detailed treatment we refer, for instance, to the forthcoming monograph by A. and J. Björn [3] and the references therein.

Standing assumptions in this paper are as follows. By the triplet (X, d, μ) we will denote a complete metric space X, where d is the metric and μ a Borel measure on X. The measure μ is supposed to be *doubling*, i.e., there exists a constant $c \geq 1$ such that

$$0 < \mu(B_{2r}(x)) \leq c\mu(B_r(x)) < \infty \tag{1}$$

for every $r > 0$ and $x \in X$. Here $B_r(x) = B(x, r) = \{y \in X : \ d(y, x) < r\}$ is the open ball centered at x with radius $r > 0$. We want to mention in passing that to require the measure of every ball in X to be positive and finite is anything but restrictive; it does not rule out any interesting measures. The *doubling constant* of μ is defined to be $c_d := \inf\{c \in (1, \infty) : \ (1) \text{ holds true}\}$. The doubling condition implies that for any $x \in X$, we have

$$\frac{\mu(B_R(x))}{\mu(B_r(x))} \leq c_d \left(\frac{R}{r}\right)^N = 2^N \left(\frac{R}{r}\right)^N, \tag{2}$$

for all $0 < r \leq R$ with $N := \log_2 c_d$. The exponent N serves as a counterpart of dimension related to the measure. Moreover, the product measure in the space $X \times (0, T)$, $T > 0$, is denoted by $\mu \otimes \mathcal{L}^1$, where $\mathcal{L}^1$ is the one dimensional Lebesgue measure.

We follow Heinonen and Koskela [21] in introducing upper gradients as follows. A Borel function $g : X \to [0, \infty]$ is said to be an *upper gradient* for an extended real-valued function u on X if for all rectifiable paths $\gamma : [0, l_\gamma] \to X$, we have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds. \tag{3}$$

If (3) holds for p–almost every curve, we say that g is a p–weak upper gradient of u; here by p–almost every curve we mean that (3) fails only for a curve family Γ with zero p–modulus. Recall, that the p–modulus of a curve family Γ is defined as

$$\text{Mod}_p\Gamma = \inf\left\{\int_X \varrho^p \, d\mu : \varrho \geq 0 \text{ is a Borel function}, \int_\gamma \varrho \geq 1 \text{ for all } \gamma \in \Gamma\right\}.$$

From the definition, it follows immediately that if g is a p-weak upper gradient for u, then g is a p-weak upper gradient also for $u - k$, and $|k|g$ for ku, for any $k \in \mathbf{R}$.

The p-weak upper gradients were introduced in Koskela–MacManus [25]. They also showed that if $g \in L^p(X)$ is a p–weak upper gradient of u, then, for any $\varepsilon > 0$, one can find an upper gradients g_ε of u such that $g_\varepsilon > g$ and $\|g_\varepsilon - g\|_{L^p(X)} < \varepsilon$. Hence for most practical purposes it is enough to consider upper gradients instead of p–weak upper gradients. If u has an upper gradient in $L^p(X)$, then it has a unique *minimal p-weak upper gradient* $g_u \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of u, $g_u \le g$ a.e., see Corollary 3.7 in Shanmugalingam [31] and Hajłasz [19] for the case $p = 1$.

Let Ω be an open subset of X and $1 \le p < \infty$. Following the definition of Shanmugalingam [30], we define for $u \in L^p(\Omega)$,

$$\|u\|^p_{N^{1,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + \inf \|g\|^p_{L^p(\Omega)},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* $N^{1,p}(\Omega)$ is the quotient space

$$N^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \|u\|_{N^{1,p}(\Omega)} < \infty\right\}/\!\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$. If $u, v \in N^{1,p}(X)$ and $v = u$ μ-almost everywhere, then $u \sim v$. Moreover, if $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ outside a set of zero p-capacity. The space $N^{1,p}(\Omega)$ is a Banach space (see Shanmugalingam [30, Theorem 3.7] and it is easily verified that it has a lattice structure. A function u belongs to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(V)$ for all bounded open sets V with $\overline{V} \subset \Omega$, the latter space being defined by considering V as a metric space with the metric d and the measure μ restricted to it.

Newtonian spaces share many properties of the classical Sobolev spaces. For example, if $u, v \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then $g_u = g_v$ a.e. in $\{x \in \Omega : u(x) = v(x)\}$, in particular $g_{\min\{u,c\}} = g_u \chi_{\{u \neq c\}}$ for $c \in \mathbf{R}$.

We shall also need a Newtonian space with zero boundary values; for the detailed definition and main properties we refer to Shanmugalingam [31, Definition 4.1]. For a measurable set $E \subset X$, let

$$N^{1,p}_0(E) = \{f|_E : f \in N^{1,p}(X) \text{ and } f = 0 \; p\text{–a.e. on } X \setminus E\}.$$

The notion of p–a.e. is based on the notion of sets of null p–capacity; the p–capacity of a set E can be defined as

$$\mathrm{Cap}_p E = \inf \left\{ \|u\|^p_{N^{1,p}(X)} : u \in N^{1,p}(X), u \ge 1 \text{ on } E \right\}.$$

This space equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space.

We shall assume that X supports a *weak* $(1,2)$*-Poincaré inequality*, that is there exist constants $C_2 > 0$ and $\Lambda \ge 1$ such that for all balls $B_\rho \subset X$, all integrable functions u on X and all upper gradients g of u,

$$\fint_{B_\rho} |u - u_{B_\rho}|\, d\mu \le C_2 \rho \left(\fint_{B_{\Lambda\rho}} g^2 \, d\mu \right)^{1/2}, \tag{4}$$

where

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

It is noteworthy that by a result of Keith and Zhong [22] if a complete metric space is equipped with a doubling measure and supports a weak $(1,2)$-Poincaré inequality, then there exists $\varepsilon > 0$ such that the space admits a weak $(1,p)$-Poincaré inequality for each $p > 2 - \varepsilon$. We shall use this fact in the proof of Lemma 5.6 which is crucial for the proof of a parabolic Harnack inequality. For more detailed references of Poincaré inequality, see Heinonen–Koskela [21] and Hajłasz–Koskela [18]. In particular, in the latter it has been shown that if a weak $(1,2)$–Poincaré inequality is assumed, then the Sobolev embedding theorem holds true and so a weak $(q,2)$–Poincaré inequality holds for all $q \le 2^*$, where, for a fixed exponent p we have defined

$$p^* = \begin{cases} \dfrac{pN}{N-p}, & p < N, \\ \\ +\infty, & p \ge N. \end{cases} \tag{5}$$

In addition, we have that if $u \in N_0^{1,2}(B_\rho)$, $B_\rho \subset \Omega$, then the following Sobolev–type inequality is valid

$$\left(\fint_{B_\rho} |u|^q \, d\mu \right)^{1/q} \le c_* \, \rho \left(\fint_{B_\rho} g_u^2 \, d\mu \right)^{1/2}, \qquad 1 \le q \le 2^*; \tag{6}$$

for a proof of this fact we refer to Kinnunen–Shanmugalingam [24, Lemma 2.1]. The crucial fact here for us is that $2^* > 2$. We also point out that since $u \in N_0^{1,2}(B_\rho)$, then the balls in the previuous inequality have the same radius. The fact that a weak $(1,p)$–Poincaré inequality holds for $p > 2 - \varepsilon$ implies also the following Sobolev–type inequality

$$\left(\fint_{B_\rho} |u|^q \, d\mu \right)^{1/q} \le C_p \, \rho \left(\fint_{B_\rho} g^p \, d\mu \right)^{1/p}, \qquad 1 \le q \le p^*, \tag{7}$$

for any function u with zero boundary values and any g upper gradient of u. The constant c_* depends only on c_d and on the constants in the weak $(1,2)$–Poincaré inequality.

We also point out that requiring a Poincaré inequality implies in particular the existence of "enough" rectifiable curves; this also implies that the continuous embedding $N^{1,2} \to L^2$, given by the identity map, is not onto.

We now state and prove some results that are needed in the paper; these results are stated for functions in $N^{1,2}$, but can be easily generalized to any $N^{1,p}$, $1 \le p < +\infty$ if we assumed instead a weak $(1,p)$–Poincaré inequality.

Theorem 2.1. *Assume $u \in N_0^{1,2}(B_\rho)$, $0 < \rho < \operatorname{diam}(X)/3$ (any $\rho > 0$ in case X has infinite diameter); then there exist $\kappa > 1$ such that we have*

$$\fint_{B_\rho} |u|^{2\kappa} \, d\mu \le c_*^2 \rho^2 \left(\fint_{B_\rho} |u|^2 \, d\mu \right)^{\kappa - 1} \fint_{B_{\Lambda\rho}} g_u^2 \, d\mu.$$

Proof. Let $\kappa = 2 - 2/2^*$, where 2^* is as in the Sobolev inequality (6). By Hölder's inequality and (6), we obtain the claim

$$\begin{aligned}\fint_{B_\rho} |u|^{2\kappa} \, d\mu &\le \left(\fint_{B_\rho} |u|^2 \, d\mu \right)^{\kappa - 1} \left(\fint_{B_\rho} |u|^{2^*} \, d\mu \right)^{2/2^*} \\ &\le c_*^2 \rho^2 \left(\fint_{B_\rho} |u|^2 \, d\mu \right)^{\kappa - 1} \fint_{B_{\Lambda\rho}} g_u^2 \, d\mu.\end{aligned}$$

□

By integrating the previous inequality in time, we obtain a parabolic Sobolev inequality.

Proposition 2.2. *Assume $u \in C([s_1, s_2]; L^2(X)) \cap L^2(s_1, s_2; N_0^{1,2}(B_\rho))$. Then there exists $\kappa > 1$ such that*

$$\int_{s_1}^{s_2} \fint_{B_\rho} |u|^{2\kappa} \, d\mu \, dt \le c_*^2 \rho^2 \left(\sup_{t \in (s_1, s_2)} \fint_{B_\rho} |u(x,t)|^2 \, d\mu(x) \right)^{\kappa - 1} \int_{s_1}^{s_2} \fint_{B_\rho} g_u^2 \, d\mu \, dt.$$

We shall also need the following De Giorgi-type lemma.

Lemma 2.3. *Let $p > 2 - \varepsilon$ and $1 \le q \le p^*$; moreover let $k, l \in \mathbf{R}$ with $k < l$, and $u \in N^{1,2}(B_\rho)$. Then*

$$(l - k) \mu(\{u \le k\} \cap B_\rho)^{1/q} \mu(\{u > l\} \cap B_\rho)^{1/q} \le 2 C_p \rho \mu(B_\rho)^{2/q - 1/p} \left(\int_{\{k < u < l\} \cap B_{\Lambda\rho}} g_u^p \, d\mu \right)^{1/p}.$$

Remark 2.4. - The previous result holds in every open set $\Omega \subset X$, provided that (6) holds with Ω in place of B_ρ.

Proof. Denote $A = \{x \in B_\rho : u(x) \le k\}$; if $\mu(A) = 0$, the result is immediate, otherwise, if $\mu(A) > 0$, we define

$$v := \begin{cases} \min\{u, l\} - k, & \text{if } u > k, \\ 0, & \text{if } u \le k. \end{cases}$$

We have that

$$\int_{B_\rho} |v - v_{B_\rho}|^q \, d\mu = \int_{B_\rho \setminus A} |v - v_{B_\rho}|^q \, d\mu + \int_A |v_{B_\rho}|^q \, d\mu \ge |v_{B_\rho}|^q \mu(A)$$

and consequently

$$|v_{B_\rho}|^q \le \frac{1}{\mu(A)} \int_{B_\rho} |v - v_{B_\rho}|^q \, d\mu. \tag{8}$$

On the other hand, we see that

$$\begin{aligned} \int_{B_\rho} |v|^q \, d\mu &= \int_{\{u>l\}\cap B_\rho} (l-k)^q \, d\mu + \int_{\{k<u\le l\}\cap B_\rho} |v|^q \, d\mu \\ &\ge (l-k)^q \mu(\{u > l\} \cap B_\rho), \end{aligned} \tag{9}$$

and using (8), we obtain

$$\begin{aligned} \left(\int_{B_\rho} |v|^q \, d\mu \right)^{1/q} &\le \left(\int_{B_\rho} |v - v_{B_\rho}|^q \, d\mu \right)^{1/q} + \left(|v_{B_\rho}|^q \mu(B_\rho) \right)^{1/q} \\ &\le 2 \left(\frac{\mu(B_\rho)}{\mu(A)} \int_{B_\rho} |v - v_{B_\rho}|^q \, d\mu \right)^{1/q}. \end{aligned}$$

By (7) and the doubling property, we finally conclude that

$$(l-k)\mu(\{u > l\} \cap B_\rho)^{1/q} \le 2C_p \rho \frac{\mu(B_\rho)^{2/q - 1/p}}{\mu(A)^{1/q}} \left(\int_{B_{\Lambda\rho}} g_v^p \, d\mu \right)^{1/p},$$

which is the required inequality. □

The following measure-theoretic lemma is a generalization of a result obtained in [10] to the metric space setting. Roughly speaking, the lemma states that if the set where $u \in N^{1,1}_{\text{loc}}(X)$ is bounded away from zero occupies a good piece of the ball B, then there exists at least one point and a neighborhood about this point such that u remains large in a large portion of the neighborhood. In other words, the set where u is positive clusters about at least one point of the ball B.

Lemma 2.5. *Let $x_0 \in X$, $\rho_0 > \rho > 0$ with $\mu(\partial B_\rho(x_0)) = 0$ and $\alpha, \beta > 0$. Then, for every $\lambda, \delta \in (0,1)$ there exists $\eta \in (0,1)$ such that for every $u \in N^{1,2}_{\text{loc}}(X)$ satisfying*

$$\int_{B_{\rho_0}(x_0)} g_u^2 \, d\mu \le \beta \frac{\mu(B_{\rho_0}(x_0))}{\rho_0^2},$$

and

$$\mu(\{u > 1\} \cap B_\rho(x_0)) \ge \alpha \mu(B_\rho(x_0)),$$

there exists $x^* \in B_\rho(x_0)$ with $B_{\eta\rho}(x^*) \subset B_\rho(x_0)$ and

$$\mu(\{u > \lambda\} \cap B_{\eta\rho}(x^*)) > (1-\delta)\mu(B_{\eta\rho}(x^*)).$$

REMARK 2.6. - The assumption $\mu(\partial B_\rho(x_0)) = 0$ is not restrictive, since this property holds except for at most countably many radii $\rho > 0$ and we can choose the appropriate radius ρ as we like. We also point out that the two previous lemmas can also be stated for functions of bounded variation instead of Sobolev functions, once a weak $(1,1)$–Poincaré inequality is assumed; the proofs given here can be easily adapted to this case by using the notion of the perimeter.

Proof. For every $\eta < (\rho_0 - \rho)/(2\Lambda\rho)$, we may consider a finite family of disjoint balls $\{B_{\eta\rho}(x_i)\}_{i\in I}$, $x_i \in B_\rho(x_0)$ for every $i \in I$, $B_{\eta\rho}(x_i) \subset B_\rho(x_0)$, such that

$$B_\rho(x_0) \subset \bigcup_{i\in I} B_{2\eta\rho}(x_i) \subset B_{\rho_0}(x_0).$$

Observe that $B_{2\Lambda\eta\rho}(x_i) \subset B_{\rho_0}(x_0)$ for every $i \in I$ and by the doubling property, the balls $B_{2\Lambda\eta\rho}(x_i)$ have bounded overlap with bound independent of η. We denote

$$I^+ = \left\{ i \in I : \mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) > \frac{\alpha}{2c_d}\mu(B_{2\eta\rho}(x_i)) \right\}$$

and

$$I^- = \left\{ i \in I : \mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) \leq \frac{\alpha}{2c_d}\mu(B_{2\eta\rho}(x_i)) \right\}.$$

By assumption, we get

$$\begin{aligned}
\alpha\mu(B_\rho(x_0)) &\leq \mu(\{u > 1\} \cap B_\rho(x_0)) \\
&\leq \sum_{i\in I^+} \mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) + \frac{\alpha}{2c_d} \sum_{i\in I^-} \mu(B_{2\eta\rho}(x_i)) \\
&\leq \sum_{i\in I^+} \mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) + \frac{\alpha}{2} \sum_{i\in I^-} \mu(B_{\eta\rho}(x_i)) \\
&\leq \sum_{i\in I^+} \mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) + \frac{\alpha}{2}\mu(B_{(1+\eta)\rho}(x_0))
\end{aligned}$$

and consequently

$$\frac{\alpha}{2}\left(\mu(B_\rho(x_0)) - \mu(B_{(1+\eta)\rho}(x_0) \setminus B_\rho(x_0))\right) \leq \sum_{i\in I^+} \mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)). \tag{10}$$

Assume by contradiction that

$$\mu(\{u > \lambda\} \cap B_{\eta\rho}(x_i)) \leq (1-\delta)\mu(B_{\eta\rho}(x_i)), \tag{11}$$

for every $i \in I^+$; this clearly implies that

$$\frac{\mu(\{u \leq \lambda\} \cap B_{\eta\rho}(x_i))}{\mu(B_{\eta\rho}(x_i))} \geq \delta.$$

The doubling condition on μ also implies that

$$\frac{\mu(\{u \leq \lambda\} \cap B_{2\eta\rho}(x_i))}{\mu(B_{2\eta\rho}(x_i))} \geq \frac{\delta}{c_d}.$$

By Lemma 2.3 with $q = 2$, $k = \lambda$ and $l = 1$, we obtain that

$$\begin{aligned}
\frac{\delta}{c_d}\mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) \leq& \frac{\mu(\{u \leq \lambda\} \cap B_{2\eta\rho}(x_i))}{\mu(B_{2\eta\rho}(x_i))}\mu(\{u > 1\} \cap B_{2\eta\rho}(x_i)) \\
\leq& \frac{16C_2^2\eta^2\rho^2}{(1-\lambda)^2} \int_{\{\lambda < u < 1\} \cap B_{2\Lambda\eta\rho}(x_i)} g_u^2 \, d\mu.
\end{aligned} \tag{12}$$

Summing up over I^+ and using the bounded overlapping property, from (10) we get

$$\begin{aligned}\frac{\alpha}{2}(1-\lambda)^2\frac{\delta}{c_d}\big(\mu(B_\rho(x_0))-\mu(B_{(1+\eta)\rho}(x_0)\setminus B_\rho(x_0)\big)\\ &\le 16C_2^2\eta^2\rho^2\sum_{i\in I^+}\int_{\{\lambda\le u<1\}\cap B_{2\Lambda\eta\rho}(x_i)} g_u^2\,d\mu\\ &\le c'\eta^2\rho^2\int_{B_{\rho_0}(x_0)} g_u^2\,d\mu\\ &\le c'\beta\mu(B_{\rho_0}(x_0))\eta^2,\end{aligned}$$

where the costant c' is given by $16C_2^2$ multiplied by the overlapping constant. The conclusion follows by passing to the limit with $\eta\to 0$, since the condition $\mu(\partial B_\rho(x_0))=0$ implies that the left hand side of the previous equation tends to

$$\frac{\alpha}{2}(1-\lambda)^2\frac{\delta}{c_d}\mu(B_\rho(x_0)).$$

□

We conclude with a result which will be needed later; for the proof we refer, for instance, to [16, Lemma 7.1].

Lemma 2.7. *Let $\{y_h\}_{h=0}^\infty$ be a sequence of positive real numbers such that*

$$y_{h+1}\le cb^h y_h^{1+\alpha},$$

where $c>0$, $b>1$ and $\alpha>0$. Then if $y_0\le c^{-1/\alpha}b^{-1/\alpha^2}$, we have

$$\lim_{h\to\infty} y_h=0.$$

3. Parabolic De Giorgi classes and quasiminimizers

We consider a variational approach related to the heat equation (see Definition 3.3)

$$\frac{\partial u}{\partial t}-\Delta u=0 \qquad \text{in } \Omega\times(0,T) \tag{13}$$

and provide a Harnack inequality for a class of functions in a metric measure space generalizing the known result for positive solutions of (13) in the Euclidean case. The following definition is essentially based on the approach of DiBenedetto–Gianazza–Vespri [10] and also of Wieser [36]; we refer also to the book of Lieberman [26] for a more detailed description.

Definition 3.1 (Parabolic De Giorgi classes of order 2)**.** *Let Ω be a non-empty open subset of X and $T>0$. A function $u:\Omega\times(0,T)\to\mathbf{R}$ belongs to the class $DG_+(\Omega,T,\gamma)$, if*

$$u\in C([0,T];L^2_{\text{loc}}(\Omega))\cap L^2_{\text{loc}}(0,T;N^{1,2}_{\text{loc}}(\Omega)),$$

and for all $k\in\mathbf{R}$ the following energy estimate holds

$$\begin{aligned}\sup_{t\in(\tau,s_2)}\int_{B_r(x_0)}(u-k)_+^2(x,t)\,d\mu+\int_\tau^{s_2}\!\!\int_{B_r(x_0)} g_{(u-k)_+}^2\,d\mu\,ds\le\alpha\int_{B_R(x_0)}(u-k)_+^2(x,s_1)\,d\mu(x)\\ +\gamma\left(1+\frac{1-\alpha}{\theta}\right)\frac{1}{(R-r)^2}\int_{s_1}^{s_2}\!\!\int_{B_R(x_0)}(u-k)_+^2\,d\mu\,ds,\end{aligned} \tag{14}$$

where $(x_0,t_0)\in\Omega\times(0,T)$, and $\theta>0$, $0<r<R$, $\alpha\in[0,1]$, $s_1,s_2\in(0,T)$, and $s_1<s_2$ are so that

$$\tau,t_0\in[s_1,s_2],\quad s_2-s_1=\theta R^2,\quad \tau-s_1=\theta(R-r)^2,$$

and $B_R(x_0) \times (t_0 - \theta R^2, t_0 + \theta R^2) \subset \Omega \times (0,T)$. *The function* u *belongs to* $DG_-(\Omega,T,\gamma)$ *if* (14) *holds with* $(u-k)_+$ *replaced by* $(u-k)_-$. *The function* u *is said to belong to the parabolic De Giorgi class of order 2, denoted* $DG(\Omega,T,\gamma)$, *if*

$$u \in DG_+(\Omega,T,\gamma) \cap DG_-(\Omega,T,\gamma).$$

In what follows, the estimate (14) given in Definition 3.1 is referred to as *energy estimate* or *Caccioppoli-type estimate*. We also point out that our definition of parabolic De Giorgi classes of order 2 is sligthly different from that given in the Euclidean case by Gianazza–Vespri [13]; our classes seem to be larger, but it is not known to us whether they are equivalent.

Denote $\mathcal{K}(\Omega \times (0,T)) = \{K \subset \Omega \times (0,T) : K \text{ compact}\}$ and consider the functional

$$E : L^2(0,T; N^{1,2}(\Omega)) \times \mathcal{K}(\Omega \times (0,T)) \to \mathbf{R}, \qquad E(w,K) = \frac{1}{2} \iint_K g_w^2 \, d\mu \, dt.$$

Definition 3.2 (Parabolic quasiminimizer)**.** *Let* Ω *be an open subset of* X. *A function*

$$u \in L^2_{\mathrm{loc}}(0,T; N^{1,2}_{\mathrm{loc}}(\Omega))$$

is said to be a parabolic Q*-quasiminimizer,* $Q \geq 1$, *related to the heat equation* (13) *if*

$$- \iint_{\mathrm{supp}(\phi)} u \frac{\partial \phi}{\partial t} \, d\mu \, dt + E(u, \mathrm{supp}(\phi)) \leq Q E(u - \phi, \mathrm{supp}(\phi)) \tag{15}$$

for every $\phi \in \mathrm{Lip}_c(\Omega \times (0,T)) = \{f \in \mathrm{Lip}(\Omega \times (0,T)) : \mathrm{supp}(f) \Subset \Omega \times (0,T)\}$.

In the Euclidean case with the Lebesgue measure it can be shown that u is a weak solution of (13) if and only if u is a 1-quasiminimizer for (13), see [36]. Hence 1-quasiminimizers can be seen as weak solutions of (13) in metric measure spaces. This motivates the following definition.

Definition 3.3. *A function* u *is a parabolic minimizer if* u *is a parabolic* Q*-quasiminimizer with* $Q = 1$.

We also point out that the class of Q-quasiminimizers is non-empty and non-trivial, since it contains the elliptic Q-quasiminimizers as defined in [14, 15] and as shown there, there exist many other examples as well.

Remark 3.4. - It is possible to prove, by using the Cheeger differentiable structure and the same proof contained in Wieser [36, Section 4], that a parabolic Q–quasiminimizer belongs to a suitable parabolic De Giorgi class. We are not able to prove this result directly without using the Cheeger differentiable structure; the main problem is that the map $u \mapsto g_u$ is only sublinear and not linear, and linearity is a main tool used in the argument.

4. Parabolic De Giorgi classes and local boundedness

We shall use the following notation;

$$Q^+_{\rho,\theta}(x_0,t_0) = B_\rho(x_0) \times [t_0, t_0 + \theta\rho^2),$$
$$Q^-_{\rho,\theta}(x_0,t_0) = B_\rho(x_0) \times (t_0 - \theta\rho^2, t_0],$$
$$Q_{\rho,\theta}(x_0,t_0) = B_\rho(x_0) \times (t_0 - \theta\rho^2, t_0 + \theta\rho^2).$$

When $\theta = 1$ we shall simplify the notation by writing $Q^+_\rho(x_0,t_0) = Q^+_{\rho,1}(x_0,t_0)$, $Q^-_\rho(x_0,t_0) = Q^-_{\rho,1}(x_0,t_0)$ and $Q_\rho(x_0,t_0) = Q_{\rho,1}(x_0,t_0)$.

We shall show that functions belonging to $DG(\Omega,T,\gamma)$ are locally bounded. Here we follow the analogous proof contained in [24] for the elliptic case. Consider $r, R > 0$ such that $R/2 < r < R$, $s_1, s_2 \in (0,T)$ with $2(s_2 - s_1) = R^2$ and $\sigma \in (s_1,s_2)$ such that $\sigma < (s_1+s_2)/2$, fix $x_0 \in X$. We define level sets at scale $\rho > 0$ as follows

$$A(k;\rho;t_1,t_2) := \{(x,t) \in B_\rho(x_0) \times (t_1,t_2) : u(x,t) > k\}.$$

Let $\tilde r := (R+r)/2$, i.e., $R/2 < r < \tilde r < R$, and $\eta \in \mathrm{Lip}_c(B_{\tilde r})$ such that $0 \le \eta \le 1$, $\eta = 1$ on B_r, and $g_\eta \le 2/(R-r)$. Then $v = (u-k)_+\eta \in N_0^{1,2}(B_{\tilde r})$ and $g_v \le g_{(u-k)_+} + 2(u-k)_+/(R-r)$. We have

$$\begin{aligned}
&\fint\!\!\fint_{B_r\times(\sigma,s_2)} (u-k)_+^2\,d\mu\,dt \le 2^N \fint\!\!\fint_{B_{\tilde r}\times(\sigma,s_2)} (u-k)_+^2\eta^2\,d\mu\,dt\\
&\le \frac{2^N}{\mu\otimes\mathcal{L}^1(B_{\tilde r}\times(\sigma,s_2))}\left(\iint_{B_{\tilde r}\times(\sigma,s_2)} (u-k)_+^q\eta^q\,d\mu\,dt\right)^{2/q} \left(\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2)\right)^{(q-2)/q}\\
&\le 2^N\left(\frac{\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2))}{\mu\otimes\mathcal{L}^1(B_{\tilde r}\times(\sigma,s_2))}\right)^{(q-2)/q}\left(\fint\!\!\fint_{B_{\tilde r}\times(\sigma,s_2)} (u-k)_+^q\eta^q\,d\mu\,dt\right)^{2/q}.
\end{aligned}$$

We now use Proposition 2.2 taking $q = 2\kappa$. We get

$$\begin{aligned}
\fint\!\!\fint_{B_r\times(\sigma,s_2)} (u-k)_+^2\,d\mu\,dt &\le 2^N 2^{2/\kappa} c_*^{2/\kappa} r^{2/\kappa}\left(\frac{\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2))}{\mu\otimes\mathcal{L}^1(B_{\tilde r}\times(\sigma,s_2))}\right)^{(\kappa-1)/\kappa}\\
&\times\left(\sup_{t\in(\sigma,s_2)}\fint_{B_{\tilde r}} (u-k)_+^2\eta^2\,d\mu\right)^{(\kappa-1)/\kappa}\left(\fint\!\!\fint_{B_{\tilde r}\times(\sigma,s_2)} g^2_{(u-k)_+\eta}\,d\mu\,dt\right)^{1/\kappa}.
\end{aligned}$$

By applying (14) with $\tau = \sigma$, $\alpha = 0$, and $\theta = 1/2$, since $\kappa > 1$, we arrive at

$$\begin{aligned}
\fint\!\!\fint_{B_r\times(\sigma,s_2)} (u-k)_+^2\,d\mu\,dt &\le 2^{N+2}\frac{c_*^{2/\kappa} r^{2/\kappa}}{(s_2-\sigma)^{1/\kappa}}\left(\frac{\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2))}{\mu\otimes\mathcal{L}^1(B_{\tilde r}\times(\sigma,s_2))}\right)^{(\kappa-1)/\kappa}\\
&\times\left(\sup_{t\in(\sigma,s_2)}\fint_{B_{\tilde r}} (u-k)_+^2(x,t)\,d\mu\right)^{(\kappa-1)/\kappa}\left(2\int_\sigma^{s_2}\fint_{B_{\tilde r}} g^2_{(u-k)_+}\,d\mu\,dt\right.\\
&\left.+\frac{8}{(R-r)^2}\int_\sigma^{s_2}\fint_{B_{\tilde r}} (u-k)_+^2\,d\mu\,dt\right)^{1/\kappa}\\
&\le 2^{N+2}\frac{c_*^{2/\kappa} r^{2/\kappa}}{(s_2-\sigma)^{1/\kappa}}\left(\frac{\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2))}{\mu\otimes\mathcal{L}^1(B_{\tilde r}\times(\sigma,s_2))}\right)^{(\kappa-1)/\kappa}\frac{\mu(B_R)}{\mu(B_{\tilde r})}\\
&\times\frac{6\gamma+2^{2N+3}}{(R-r)^2}\int_{s_1}^{s_2}\fint_{B_R} (u-k)_+^2\,d\mu\,dt.
\end{aligned}$$

By the the choice of σ, we see that $(s_2-\sigma)^{-1} < 2(s_2-s_1)^{-1}$, and consequently

$$\begin{aligned}
\fint\!\!\fint_{B_r\times(\sigma,s_2)} (u-k)_+^2\,d\mu\,dt &\le 2^{N+4}(3\gamma+2^{2N+2}) c_*^{2/\kappa} r^{2/\kappa}\left(\frac{\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2))}{\mu\otimes\mathcal{L}^1(B_{\tilde r}\times(\sigma,s_2))}\right)^{(\kappa-1)/\kappa}\\
&\times\frac{\mu(B_R)}{\mu(B_{\tilde r})}(s_2-\sigma)^{(\kappa-1)/\kappa}\frac{1}{(R-r)^2}\fint\!\!\fint_{B_R\times(s_1,s_2)} (u-k)_+^2\,d\mu\,dt
\end{aligned} \tag{16}$$

Consider $h < k$. Then

$$\begin{aligned}
(k-h)^2\left(\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2))\right) &\le \iint_{A(k;\tilde r;\sigma,s_2)} (u-h)_+^2\,d\mu\,dt\\
&\le \iint_{A(h;\tilde r;\sigma,s_2)} (u-h)_+^2\,d\mu\,dt = \int_\sigma^{s_2}\int_{B_{\tilde r}} (u-h)_+^2\,d\mu\,dt,
\end{aligned}$$

from which, using the doubling property (2), it follows that

$$\begin{aligned}\mu\otimes\mathcal{L}^1(A(k;\tilde r;\sigma,s_2)) &\le \frac{1}{(k-h)^2}\left(\mu\otimes\mathcal{L}^1\left(B_{\tilde r}\times(\sigma,s_2)\right)\right)u(h;\tilde r;\sigma,s_2)^2\\ &\le \frac{2^{N+1}}{(k-h)^2}\left(\mu\otimes\mathcal{L}^1\left(B_{\tilde r}\times(\sigma,s_2)\right)\right)u(h;R;s_1,s_2)^2,,\end{aligned} \tag{17}$$

where

$$u(l;\rho;t_1,t_2):=\left(\fint\!\!\fint_{B_\rho\times(t_1,t_2)}(u-l)_+^2\,d\,\mu\,dt\right)^{1/2}.$$

By plugging (17) into (16) and arranging terms we arrive at

$$u(k;r;\sigma,s_2)\le \bar c\frac{r^{1/\kappa}(s_2-\sigma)^{(\kappa-1)/2\kappa}}{(k-h)^{(\kappa-1)/\kappa}(R-r)}u(k;R;s_1,s_2)u(h;R;s_1,s_2)^{(\kappa-1)/\kappa}, \tag{18}$$

with $\bar c = 2^{N+2+(N+1)(\kappa-1)/(2\kappa)}(3\gamma+2^{2N+2})^{1/2}c_*^{1/\kappa}$.

Let us consider the following sequences: for $n\in\mathbf{N}$, $k_0\in\mathbf{R}$ and fixed d we define

$$\begin{aligned}k_n &:= k_0+d\left(1-\frac{1}{2^n}\right)\nearrow k_0+d,\\ r_n &:= \frac{R}{2}+\frac{R}{2^{n+1}}\searrow\frac{R}{2},\quad\text{and}\\ \sigma_n &:= \frac{s_1+s_2}{2}-\frac{R^2}{4^{n+1}}\nearrow\frac{s_1+s_2}{2}.\end{aligned}$$

This is possible since $2(s_2-s_1)=R^2$. The following technical result will be useful for us.

Lemma 4.1. *Let* $u_0:=u(k_0;R;s_1,s_2)$, $u_n:=u(k_n;r_n;\sigma_n,s_2)$,

$$\theta:=\frac{\kappa-1}{\kappa},\quad a:=\frac{1+\theta}{\theta}=1+\frac{\kappa}{\kappa-1},$$

and

$$d^\theta=\bar c\,2^{1+\theta/2+a(1+\theta)}u_0^\theta,$$

where $\bar c$ *is the constant in* (18). *Then*

$$u_n\le\frac{u_0}{2^{an}}. \tag{19}$$

Proof. We prove the lemma by induction. First notice that (19) is true for $n=0$. Then assume that (19) is true for fixed $n\in\mathbf{N}$. In (18), we first estimate $r^{1/\kappa}(s_2-\sigma)^{(\kappa-1)/2\kappa}$ by $R^{1/\kappa}(s_2-s_1)^{(\kappa-1)/2\kappa}$. Then we replace r with r_{n+1}, R with r_n, σ with σ_{n+1}, s_1 with σ_n, h with k_n, and k with k_{n+1}. With these replacements we arrive at

$$u_{n+1}\le\frac{\bar c\,R^{1/\kappa}(s_2-s_1)^{(\kappa-1)/(2\kappa)}}{(k_{n+1}-k_n)^{(\kappa-1)/\kappa}(r_n-r_{n+1})}u_n^{1+(\kappa-1)/\kappa}.$$

Denote $c':=\bar c\,R^{1/\kappa}(s_2-s_1)^{(\kappa-1)/(2\kappa)}$ so that we have

$$u_{n+1}\le\frac{c'u_n^{1+\theta}}{(k_{n+1}-k_n)^\theta(r_n-r_{n+1})}.$$

Since $r_n-r_{n+1}=2^{-(n+2)}R$ and $k_{n+1}-k_n=2^{-(n+1)}d$, we obtain

$$u_{n+1}\le c'\frac{2^{(n+1)\theta+n+2}}{d^\theta R}u_n^{1+\theta}=2c'\frac{2^{(n+1)(1+\theta)}}{d^\theta R}u_n^{1+\theta}\le 2c'\frac{2^{(n+1)(1+\theta)}}{d^\theta R}\left(\frac{u_0}{2^{an}}\right)^{1+\theta}.$$

As $2(s_2 - s_1) = R^2$, we have

$$c'' := \frac{2c'}{R} = 2\bar{c}\, R^{1/\kappa}(s_2 - s_1)^{(\kappa-1)/(2\kappa)}\frac{1}{R} = 2^{1+\theta/2}\bar{c}.$$

Point being that the constant c'' is independent of R, s_1, and s_2. Finally, since $(1-a)(1+\theta) = -a$ we arrive at

$$u_{n+1} \le c''\frac{2^{(n+1)(1+\theta)}}{d^\theta}\left(\frac{u_0}{2^{an}}\right)^{1+\theta} = 2^{-a(n+1)}u_0.$$

This completes the proof. □

Before proving the main result of this section, we need the following proposition.

Proposition 4.2. *For every number $k_0 \in \mathbf{R}$ we have*

$$u(k_0 + d; R/2; (s_1 + s_2)/2, s_2) = 0,$$

where d is defined as in Lemma 4.1.

Proof. Since $k_n \le k_0 + d$, $R/2 \le r_n \le R$, $s_1 \le \sigma_n \le (s_1 + s_2)/2$, the doubling property implies that

$$0 \le u(k_0 + d; R/2; (s_1 + s_2)/2, s_2) \le 2^{N+1}u(k_n; r_n; \sigma_n, s_2) = u_n.$$

By Lemma 4.1, we have $\lim_{n\to\infty} u_n = 0$ and the claim follows. □

We close this section by proving local boundedness for functions in the De Giorgi class.

Theorem 4.3. *Suppose $u \in DG(\Omega, T, \gamma)$. Then there is a constant c_∞ depending only on c_d, γ, and the constants in the weak $(1,2)$–Poincaré inequality, such that for all $B_R \times (s_1, s_2) \subset \Omega \times (0,T)$, we have*

$$\operatorname*{ess\,sup}_{B_{R/2}\times((s_1+s_2)/2,s_2)} |u| \le c_\infty \left(⨏\!\!⨏_{B_R\times(s_1,s_2)} |u|^2 d\mu\, dt\right)^{1/2}.$$

Proof. The Proposition 4.2 implies that

$$\operatorname*{ess\,sup}_{B_{R/2}\times((s_1+s_2)/2,s_2)} u \le k_0 + d,$$

where d is defined in Lemma 4.1. Then

$$\operatorname*{ess\,sup}_{B_{R/2}\times((s_1+s_2)/2,s_2)} u \le k_0 + c_\infty \left(⨏\!\!⨏_{B_R\times(s_1,s_2)} (u-k_0)_+^2 d\mu\, dt\right)^{1/2},$$

with $c_\infty = \bar{c}^{1/\theta}\, 2^{1+\theta/2+a(1+\theta)}$, $\bar{c}$ the constant in (18). The previous inequality with $k_0 = 0$ can be written as follows

$$\operatorname*{ess\,sup}_{B_{R/2}\times((s_1+s_2)/2,s_2)} u \le c_\infty \left(⨏\!\!⨏_{B_R\times(s_1,s_2)} u_+^2 d\mu\, dt\right)^{1/2} \le c_\infty \left(⨏\!\!⨏_{B_R\times(s_1,s_2)} |u|^2 d\mu\, dt\right)^{1/2}.$$

Since also $-u \in DG(\Omega, T, \gamma)$ the analogous argument applied to $-u$ gives the claim. □

5. Parabolic De Giorgi classes and Harnack inequality

In this section we shall prove a scale-invariant parabolic Harnack inequality for functions in the De Giorgi class of order 2 and, in particular, for parabolic quasiminimizers.

Proposition 5.1. *Let ρ, $\theta > 0$ be chosen such that the cylinder $Q^-_{\rho,\theta}(y,s) \subset \Omega \times (0,T)$. Then for each choice of $a, \sigma \in (0,1)$ and $\bar\theta \in (0,\theta)$, there is ν_+, depending only on $N, \gamma, c_*, a, \theta, \bar\theta$, such that for every $u \in DG_+(\Omega,T,\gamma)$ and m_+ and ω for which*

$$m_+ \geq \operatorname*{ess\,sup}_{Q^-_{\rho,\theta}(y,s)} u \quad \textit{and} \quad \omega \geq \operatorname*{osc}_{Q^-_{\rho,\theta}(y,s)} u,$$

the following claim holds true: if

$$\mu \otimes \mathcal{L}^1 \left(\{(x,t) \in Q^-_{\rho,\theta}(y,s) : u(x,t) > m_+ - \sigma\omega\} \right) \leq \nu_+ \mu \otimes \mathcal{L}^1 \left(Q^-_{\rho,\theta}(y,s) \right),$$

then

$$u(x,t) \leq m_+ - a\sigma\omega \qquad \mu \otimes \mathcal{L}^1\text{-a.e. in } B_{\rho/2}(y) \times (s - \bar\theta\rho^2, s].$$

Proof. Define the following sequences, $h \in \mathbf{N}$,

$$\begin{aligned}
&\rho_h := \frac{\rho}{2} + \frac{\rho}{2^{h+1}} \searrow \frac{\rho}{2}, \quad \theta_h = \bar\theta + \frac{1}{2^h}(\theta - \bar\theta) \searrow \bar\theta, \\
&B_h := B_{\rho_h}(y), \quad s_h := s - \theta_h\rho^2 \nearrow s - \bar\theta\rho^2, \quad Q^-_h := B_h \times (s_h, s], \\
&\sigma_h := a\sigma + \frac{1-a}{2^h}\sigma \searrow a\sigma, \quad \text{and} \quad k_h = m_+ - \sigma_h\omega \nearrow m_+ - a\sigma\omega.
\end{aligned}$$

Consider a sequence of Lipschitz continuous functions ζ_h, $h \in \mathbf{N}$, satisfying the following:

$$\begin{aligned}
&\zeta_h \equiv 1 \text{ in } Q^-_{h+1}, \qquad \zeta_h \equiv 0 \text{ in } Q^-_{\rho,\theta}(y,s) \setminus Q^-_h \\
&g_{\zeta_h} \leq \frac{1}{\rho_h - \rho_{h+1}} = \frac{2^{h+2}}{\rho}, \qquad 0 \leq (\zeta_h)_t \leq \frac{2^{h+1}}{\theta - \bar\theta}\frac{1}{\rho^2}.
\end{aligned}$$

Denote $A_h := \{(x,t) \in Q^-_h : u(x,t) > k_h\}$. We have

$$\begin{aligned}
\iint_{Q^-_h} (u-k_h)^2_+ \zeta_h^2 \, d\mu \, dt &\geq \iint_{Q^-_{h+1}} (u-k_h)^2_+ \, d\mu \, dt \geq \iint_{A_{h+1}} (u-k_h)^2_+ \, d\mu \, dt \\
&\geq \iint_{A_{h+1}} (k_{h+1}-k_h)^2 \, d\mu \, dt = \frac{((1-a)\sigma\omega)^2}{2^{2h+2}} \mu \otimes \mathcal{L}^1(A_{h+1}),
\end{aligned}$$

and consequently

$$\int_{s_h}^s \fint_{B_h} (u-k_h)^2_+ \zeta_h^2 \, d\mu \, dt \geq \frac{((1-a)\sigma\omega)^2}{2^{2h+2}} \frac{\mu \otimes \mathcal{L}^1(A_{h+1})}{\mu(B_{h+1})}. \tag{20}$$

On the other hand, if we use first Hölder's inequality and then Proposition 2.2, we obtain the following estimate

$$
\begin{aligned}
\int_{s_h}^{s} \fint_{B_h} (u-k_h)_+^2 \zeta_h^2 \, d\mu \, dt &\le \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{(\kappa-1)/\kappa} \left(\int_{s_h}^{s} \fint_{B_h} (u-k_h)_+^{2\kappa} \zeta_h^{2\kappa} \, d\mu \, dt \right)^{1/\kappa} \\
&\le c_*^{2/\kappa} \rho^{2/\kappa} \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{(\kappa-1)/\kappa} \left(\sup_{t \in (s_h, s)} \fint_{B_h} (u-k_h)_+^2 \zeta_h^2 \, d\mu \right)^{(\kappa-1)/\kappa} \times \\
&\quad \times \left(\int_{s_h}^{s} \fint_{B_h} \left(2\, \zeta_h^2 \, g^2_{(u-k_h)_+} + 2 g^2_{\zeta_h} (u-k_h)_+^2 \right) d\mu \, dt \right)^{1/\kappa} \\
&\le c_*^{2/\kappa} \rho^{2/\kappa} \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{(\kappa-1)/\kappa} \frac{1}{\mu(B_h)} \left(\sup_{t \in (s_h, s)} \int_{B_h} (u-k_h)_+^2 \, d\mu \right)^{(\kappa-1)/\kappa} \times \\
&\quad \times \left(2 \int_{s_h}^{s} \int_{B_h} g^2_{(u-k_h)_+} \, d\mu \, dt + \frac{2^{2h+5}}{\rho^2} \int_{s_h}^{s} \int_{B_h} (u-k_h)_+^2 \, d\mu \, dt \right)^{1/\kappa} \\
&\le 2^{1/\kappa} c_*^{2/\kappa} \rho^{2/\kappa} \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{(\kappa-1)/\kappa} \frac{1}{\mu(B_h)} \left(\sup_{t \in (s_h, s)} \int_{B_h} (u-k_h)_+^2 \, d\mu + \right. \\
&\quad \left. + \int_{s_h}^{s} \int_{B_h} g^2_{(u-k_h)_+} \, d\mu \, dt + \frac{2^{2h+4}}{\rho^2} \int_{s_h}^{s} \int_{B_h} (u-k_h)_+^2 \, d\mu \, dt \right).
\end{aligned}
$$

We continue by applying the energy estimate (14) with $r = \rho_h$, $R = \rho_{h-1}$, $\alpha = 0$, $s_2 = s$, $\tau = s_h$, $s_1 = s_{h-1}$ and get

$$
\begin{aligned}
&\int_{s_h}^{s} \fint_{B_h} (u-k_h)_+^2 \zeta_h^2 \, d\mu \, dt \\
&\qquad \le 2^{1/\kappa} c_*^{2/\kappa} \rho^{2/\kappa} \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{(\kappa-1)/\kappa} \frac{1}{\mu(B_h)} \left(\frac{2^{2h+4}}{\rho^2} \int_{s_h}^{s} \int_{B_h} (u-k_h)_+^2 \, d\mu \, dt + \right. \\
&\qquad\quad \left. + \gamma \left(1 + \frac{2^h}{\theta - \bar{\theta}} \right) \frac{2^{2h+2}}{\rho^2} \int_{s_{h-1}}^{s} \int_{B_{h-1}} (u-k_h)_+^2 \, d\mu \, dt \right) \\
&\qquad \le C_1 \, \rho^{2/\kappa} \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{(\kappa-1)/\kappa} \frac{1}{\mu(B_h)} \frac{2^{3h+4}}{\rho^2} \iint_{Q^-_{h-1}} (u-k_h)_+^2 \, d\mu \, dt
\end{aligned}
$$

where $C_1 = 2^{1/\kappa} c_*^{2/\kappa} (1 + \gamma + \gamma/(\theta - \bar{\theta}))$. We also have that $u - k_h \le m_+ - k_h = \sigma_h \omega$ and then

$$
\iint_{Q^-_{h-1}} (u-k_h)_+^2 \, d\mu dt \le \mu \otimes \mathcal{L}^1(A_{h-1}) (\sigma_h \omega)^2 \le \mu \otimes \mathcal{L}^1(A_{h-1}) (\sigma \omega)^2 .
$$

This implies that

$$
\begin{aligned}
\int_{s_h}^{s} \fint_{B_h} (u-k_h)_+^2 \zeta_h^2 \, d\mu \, dt &\le \; 2^{2h+4} \, C_1 (\sigma\omega)^2 \, \frac{1}{\rho^{2 \frac{\kappa-1}{\kappa}}} \left(\frac{\mu \otimes \mathcal{L}^1(A_h)}{\mu(B_h)} \right)^{\frac{\kappa-1}{\kappa}} \frac{\mu \otimes \mathcal{L}^1(A_{h-1})}{\mu(B_h)} \\
&\le \; 2^{2h+4} \, C_1 (\sigma\omega)^2 \, \theta^{\frac{\kappa-1}{\kappa}} \, 2^{N(1 + \frac{\kappa-1}{\kappa})} \left(\frac{\mu \otimes \mathcal{L}^1(A_{h-1})}{\mu \otimes \mathcal{L}^1(B_{h-1})} \right)^{\frac{\kappa-1}{\kappa}} \frac{\mu \otimes \mathcal{L}^1(A_{h-1})}{\mu(B_{h-1})}
\end{aligned}
$$

where we have estimated $\mu(B_{h-1})/\mu(B_h) \le 2^N$. By the last inequality and (20), if we call C_2 the constant $C_1\,\theta^{\frac{\kappa-1}{\kappa}} 2^{N(1+\frac{\kappa-1}{\kappa})+6}(1-a)^{-2}$, we obtain

$$\frac{\mu\otimes\mathcal{L}^1(A_{h+1})}{\mu(B_{h+1})} \le C_2\,2^{4h}\left(\frac{\mu\otimes\mathcal{L}^1(A_{h-1})}{\mu\otimes\mathcal{L}^1(B_{h-1})}\right)^{\frac{\kappa-1}{\kappa}}\frac{\mu\otimes\mathcal{L}^1(A_{h-1})}{\mu(B_{h-1})};$$

finally, dividing by $s-s_{h+1}$ and since $(s-s_{h-1})/(s-s_{h+1}) \le \theta/\bar{\theta}$, we can summarize what we have obtain by writing

$$y_{h+1} \le C_3\,2^{4h}\,y_{h-1}^{1+(\kappa-1)/\kappa} \tag{21}$$

where we have defined

$$y_h := \frac{\mu\otimes\mathcal{L}^1(A_h)}{\mu\otimes\mathcal{L}^1(Q_h^-)} \qquad \text{and} \qquad C_3 = C_2\frac{\theta}{\bar{\theta}}$$

i.e.

$$C_3 = 2^{1/\kappa}\,c_*^{2/\kappa}\left(1+\gamma+\frac{\gamma}{\theta-\bar{\theta}}\right)\theta^{\frac{\kappa-1}{\kappa}}2^{N(1+\frac{\kappa-1}{\kappa})+6}\frac{1}{(1-a)^2}\frac{\theta}{\bar{\theta}}\,.$$

We observe that the hypotheses of Lemma 2.7 are satisfied with $c = C_3$, $b = 2^4$ and $\alpha = (\kappa-1)/\kappa$. Then if

$$y_0 \le c^{-1/\alpha}b^{-1/\alpha^2}$$

we would be able to conclude, since $\{y_h\}_h$ is a decreasing sequence, that

$$\lim_{h\to\infty} y_h = 0.$$

Since $y_0 = \mu\otimes\mathcal{L}^1(A_0)/\mu\otimes\mathcal{L}^1(Q_0^-)$, where

$$Q_0^- = B_\rho(y)\times(s-\theta\rho^2, s], \quad \text{and} \quad A_0 = \{(x,t)\in Q_0^- : u(x,t) > m_+ - \sigma\omega\}.$$

To do this it is sufficient to choose ν_+ to be

$$\nu_+ = C_3^{-\kappa/(\kappa-1)}16^{-\kappa^2/(\kappa-1)^2}\,.$$

By definition of y_h and A_h we see that

$$u \le m_+ - a\sigma\omega \qquad \mu\otimes\mathcal{L}^1\text{-a.e. in } B_{\rho/2}(y)\times\left(s-\bar{\theta}\rho^2, s\right],$$

which completes the proof. □

An analogous argument proves the following claim.

Proposition 5.2. *Let ρ, $\theta > 0$ be chosen such that the cylinder $Q^-_{\rho,\theta}(y,s) \subset \Omega\times(0,T)$. Then for each choice of $a,\sigma\in(0,1)$ and $\bar{\theta}\in(0,\theta)$, there is ν_-, depending only on $N,\gamma,c_*,a,\theta,\bar{\theta}$, such that for every $u\in DG_-(\Omega,T,\gamma)$ and m_+ and ω for which*

$$m_- \le \operatorname*{ess\,inf}_{Q^-_{\rho,\theta}(y,s)} u \quad \text{and} \quad \omega \ge \operatorname*{osc}_{Q^-_{\rho,\theta}(y,s)} u,$$

the following claim holds true: if

$$\mu\otimes\mathcal{L}^1\left(\{(x,t)\in Q^-_{\rho,\theta}(y,s) : u(x,t) < m_- + \sigma\omega\}\right) \le \nu_-\mu\otimes\mathcal{L}^1\left(Q^-_{\rho,\theta}(y,s)\right),$$

then

$$u(x,t) \ge m_- + a\sigma\omega \qquad \mu\otimes\mathcal{L}^1\text{-a.e. in } B_{\rho/2}(y)\times(s-\bar{\theta}\rho^2, s].$$

Proof. It is sufficient to argue as in the proof of Proposition 5.1 considering $(u-\hat{k}_h)_-$ in place of $(u-k_h)_+$, where $\hat{k}_h = m_- + \sigma_h\omega$. □

The next result is the so called *expansion of positivity.* Following the approach of DiBenedetto [9] we show that pointwise information in a ball B_ρ implies pointwise information in the expanded ball $B_{2\rho}$ at a further time level.

Proposition 5.3. *Let $(x^*,t^*) \in \Omega \times (0,T)$ and $\rho > 0$ with $B_{5\Lambda\rho}(x^*) \times [t^* - \rho^2, t^* + \rho^2] \subset \Omega \times (0,T)$. Then there exists $\tilde\theta \in (0,1)$, depending only on γ, such that for every $\hat\theta \in (0,\tilde\theta)$ there exists $\lambda \in (0,1)$, depending on $\tilde\theta$ and $\hat\theta$, such that for every $h > 0$ and for every $u \in DG(\Omega,T,\gamma)$ the following is valid. If*

$$u(x,t^*) \geq h \qquad \mu - a.e. \text{ in } B_\rho(x^*),$$

then

$$u(x,t) \geq \lambda h \qquad \mu - a.e. \text{ in } B_{2\rho}(x^*), \qquad \text{for every } t \in [t^* + \hat\theta\rho^2, t^* + \tilde\theta\rho^2].$$

From now on, let us denote

$$A_{h,\rho}(x^*,t^*) := \{x \in B_\rho(x^*) : u(x,t^*) < h\}.$$

Remark 5.4. - Let $(x^*,t^*) \in \Omega \times (0,T)$ and $h > 0$ be fixed. Then if $u(x,t^*) \geq h$ for μ-a.e. $x \in B_\rho(x^*)$ we have that

$$A_{h,4\rho}(x^*,t^*) \subset B_{4\rho}(x^*) \setminus B_\rho(x^*).$$

The doubling property implies

$$\mu(A_{h,4\rho}(x^*,t^*)) \leq \left(1 - \frac{1}{4^N}\right)\mu(B_{4\rho}(x^*)).$$

The proof of Proposition 5.3 requires some preliminary lemmas.

Lemma 5.5. *Given (x^*,t^*) for which $B_{4\rho}(x^*) \times [t^*, t^* + \theta\,\rho^2] \subset \Omega \times (0,T)$, there exist $\eta \in (0,1)$ and $\tilde\theta \in (0,\theta)$ such that, given $h > 0$ and $u \geq 0$ in $DG(\Omega,T,\gamma)$ for which the following holds*

$$u(x,t^*) \geq h \qquad \mu - a.e. \text{ in } B_\rho(x^*)$$

then

$$\mu(A_{\eta h,4\rho}(x^*,t)) < \left(1 - \frac{1}{4^{N+1}}\right)\mu(B_{4\rho}(x^*))$$

for every $t \in [t^*, t^* + \tilde\theta\rho^2]$.

Proof. We may assume that $h = 1$, otherwise we consider the scaled function u/h. We apply the energy estimate of Definition 3.1 with $R = 4\rho$, $r = 4\rho(1-\sigma)$, $s_1 = t^*$, $s_2 = t^* + \tilde\theta\rho^2$ with $\tilde\theta$ to be chosen, $\tau = t^*$, $\sigma \in (0,1)$, and $\alpha = 1$. This gives us

$$\begin{aligned}\sup_{t^*<t<t^*+\tilde\theta\rho^2} \int_{B_{4\rho(1-\sigma)}(x^*)} (u-1)_-^2(x,t)d\mu(x) + \int_{t^*}^{t^*+\tilde\theta\rho^2} \int_{B_{4\rho(1-\sigma)}(x^*)} g_{(u-1)_-}^2 \, d\mu\, dt \\ \leq \int_{B_{4\rho}(x^*)} (u-1)_-^2(x,t^*)d\mu(x) + \frac{\gamma}{16\sigma^2\rho^2} \int_{t^*}^{t^*+\tilde\theta\rho^2} \int_{B_{4\rho}(x^*)} (u-1)_-^2 \, d\mu\, dt.\end{aligned}$$

Since $u \geq 1$ in $B_\rho(x^*)$, we deduce from Remark 5.4 that

$$\mu(\{x \in B_{4\rho}(x^*) : u(x,t^*) < 1\}) < \left(1 - \frac{1}{4^N}\right)\mu(B_{4\rho}(x^*)).$$

Notice that $(u-1)_- \le 1$; thus we have in particular

$$
\begin{aligned}
\sup_{t^*<t<t^*+\tilde\theta\rho^2} & \int_{B_{4\rho(1-\sigma)}(x^*)} (u-1)_-^2(x,t)\,d\mu(x) \\
&\le \int_{B_{4\rho}(x^*)} (u-1)_-^2(x,t^*)\,d\mu(x) + \frac{\gamma}{16\sigma^2\rho^2}\int_{t^*}^{t^*+\tilde\theta\rho^2}\int_{B_{4\rho}(x^*)} (u-1)_-^2\,d\mu\,dt \\
&\le \left(1-\frac{1}{4^N}\right)\mu(B_{4\rho}(x^*)) + \frac{\gamma}{16\sigma^2\rho^2}\int_{t^*}^{t^*+\tilde\theta\rho^2}\int_{B_{4\rho}(x^*)} (u-1)_-^2\,d\mu\,dt \\
&\le \left(1-\frac{1}{4^N}\right)\mu(B_{4\rho}(x^*)) + \frac{\gamma\tilde\theta}{16\sigma^2}\mu(B_{4\rho}(x^*))
\end{aligned}
$$

Writing $A_{h,\rho}(t)$ in place of $A_{h,\rho}(x^*,t)$, decomposing

$$A_{\eta,4\rho}(t) = A_{\eta,4\rho(1-\sigma)}(t) \cup \{x \in B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*) : u(x,t) < \eta\},$$

and using the doubling property we have

$$\mu(A_{\eta,4\rho}(t)) \le \mu(A_{\eta,4\rho(1-\sigma)}(t)) + \mu\big(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*)\big).$$

On the other hand,

$$\int_{B_{4\rho(1-\sigma)}(x^*)} (u-1)_-^2(x,t)\,d\mu(x) \ge \int_{A_{\eta,4\rho(1-\sigma)}(t)} (u-1)_-^2(x,t)\,d\mu(x) \ge (1-\eta)^2\mu(A_{\eta,4\rho(1-\sigma)}(t)).$$

Finally, we obtain

$$
\begin{aligned}
\mu(A_{\eta,4\rho}(t)) &\le \mu(A_{\eta,4\rho(1-\sigma)}(t)) + \mu\left(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*)\right) \\
&\le (1-\eta)^{-2}\int_{B_{4\rho(1-\sigma)}(x^*)} (u-1)_-^2(x,t)d\mu + \mu\left(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*)\right) \\
&\le (1-\eta)^{-2}\left(1-\frac{1}{4^N}+\frac{\gamma\tilde\theta}{16\sigma^2}\right)\mu(B_{4\rho}(x^*)) + \mu\left(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*)\right).
\end{aligned}
\tag{22}
$$

If the claim of the lemma was false, then for every $\tilde\theta, \eta \in (0,1)$ there exists $\bar t \in [t^*, t^*+\tilde\theta\rho^2]$ for which

$$\mu(A_{\eta,4\rho}(\bar t)) \ge \left(1-\frac{1}{4^{N+1}}\right)\mu\left(B_{4\rho}(x^*)\right).$$

Applying this last estimate, then (22) for $t = \bar t$ and dividing by $\mu\left(B_{4\rho}(x^*)\right)$ we would have

$$\left(1-\frac{1}{4^{N+1}}\right) \le (1-\eta)^{-2}\left(1-\frac{1}{4^N}+\frac{\gamma\tilde\theta}{16\sigma^2}\right) + \frac{\mu(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*))}{\mu(B_{4\rho}(x^*))}.$$

Choosing, for instance, $\tilde\theta = \sigma^3$ and letting η and σ go to zero we would have the contradiction $1-4^{-N-1} \le 1-4^{-N}$. □

Lemma 5.6. *Assume $u \in DG(\Omega,T,\gamma)$, $u \ge 0$. Let $\tilde\theta$ be as in Lemma 5.5 and $h > 0$. Consider (x^*,t^*) in such a way that $B_{5\Lambda\rho}(x^*) \times [t^*-\tilde\theta\rho^2, t^*+\tilde\theta\rho^2] \subset \Omega \times (0,T)$ and assume that*

$$u(x,t^*) \ge h, \qquad \mu - a.e.\ x \in B_\rho(x^*).$$

Then for every $\epsilon > 0$ there exists $\eta_1 \in (0,1)$, depending on ϵ, c_d, γ, $\tilde\theta$, and the constant in the weak Poincaré inequality, such that

$$\mu \otimes \mathcal{L}^1\left(\{(x,t) \in B_{4\rho}(x^*) \times [t^*, t^*+\tilde\theta\rho^2] : u(x,t) < \eta_1 h\}\right) < \epsilon\mu \otimes \mathcal{L}^1\left(B_{4\rho}(x^*) \times [t^*, t^*+\tilde\theta\rho^2]\right).$$

Proof. Apply the energy estimate (14) in $B_{5\Lambda\rho}(x^*) \times (t^* - 2\tilde{\theta}\rho^2, t^*)$ with

$$R = 5\Lambda\rho,\ r = 4\Lambda\rho,\ s_2 = t^* + \tilde{\theta}\rho^2,\ s_1 = t^* - \tilde{\theta}\rho^2,\ \tau = t^*,\ \text{ and }\ \alpha = 0,$$

at the level $k = \eta h 2^{-m}$, where $\eta > 0$ and $m \in \mathbf{N}$. We obtain

$$\begin{aligned}
(23)\qquad & \int_{t^*}^{t^*+\tilde{\theta}\rho^2} \int_{B_{4\Lambda\rho}(x^*)} g^2_{(u-\frac{\eta h}{2^m})_-}\, d\mu\, dt \\
&\le \gamma\left(1 + \frac{1}{2\tilde{\theta}}\right) \frac{1}{(\Lambda\rho)^2} \int_{t^*-\tilde{\theta}\rho^2}^{t^*+\tilde{\theta}\rho^2} \int_{B_{5\Lambda\rho}(x^*)} \left(u - \frac{\eta h}{2^m}\right)_-^2 d\mu\, dt \\
&\le \gamma\left(1 + \frac{1}{2\tilde{\theta}}\right) \frac{1}{(\Lambda\rho)^2} \frac{\eta^2 h^2}{2^{2m}} 2\tilde{\theta}\rho^2 \mu\left(B_{5\Lambda\rho}(x^*)\right) \\
&\le \gamma(2\tilde{\theta} + 1) \frac{\eta^2 h^2}{2^{2m}} \mu\left(B_{5\Lambda\rho}(x^*)\right).
\end{aligned}$$

To simplify notation, let us write $A_{h,\rho}(t)$ instead of $A_{h,\rho}(x^*, t)$. Lemma 2.3 with parameters $k = \eta h/2^m$, $l = \eta h/2^{m-1}$, $q = 1$ and $2 - \varepsilon < p < 2$, implies

$$\begin{aligned}
(24)\qquad \int_{B_{4\rho}(x^*)} \left(u - \frac{\eta h}{2^m}\right)_- (x,t)\, d\mu &\le \frac{\eta h}{2^m} \mu(A_{\eta h 2^{-m}, 4\rho}(t)) \\
&\le \frac{8 C_p \rho \mu(B_{4\rho}(x^*))^{2-1/p}}{\mu(B_{4\rho}(x^*) \setminus A_{\eta h 2^{-m+1}, 4\rho}(\tau))} \left(\int_{\tilde{A}(t)} g^p_{(u-\frac{\eta h}{2^m})_-}(x,t)\, d\mu \right)^{1/p},
\end{aligned}$$

for every $t \in [t^*, t^* + \tilde{\theta}\rho^2]$, where $\tilde{A}(t) := A_{\eta h 2^{-m+1}, 4\Lambda\rho}(t) \setminus A_{\eta h 2^{-m}, 4\Lambda\rho}(t)$. Clearly,

$$B_{4\rho}(x^*) \setminus A_{\eta h 2^{-m+1}, 4\rho}(t) \supseteq B_{4\rho}(x^*) \setminus A_{\eta h, 4\rho}(t).$$

If we choose η so that it satisfies the hypothesis of Lemma 5.5 and write

$$\mu(B_{4\rho}(x^*) \setminus A_{\eta h, 4\rho}(t)) + \mu(A_{\eta h, 4\rho}(t)) = \mu(B_{4\rho}(x^*)),$$

then we deduce that

$$\mu(B_{4\rho}(x^*) \setminus A_{\eta h, 4\rho}(t)) > 4^{-N-1} \mu(B_{4\rho}(x^*))$$

for every $t \in [t^*, t^* + \tilde{\theta}\rho^2]$. We finally arrive at

$$\int_{B_{4\rho}(x^*)} \left(u - \frac{\eta h}{2^m}\right)_- (x,t)\, d\mu \le 8 C_p 4^{N+1} \mu(B_{4\rho}(x^*))^{1-1/p} \rho \left(\int_{\tilde{A}(t)} g^p_{(u-\frac{\eta h}{2^m})_-}(x,t)\, d\mu \right)^{1/p}.$$

Integrating this with respect to t and defining the decreasing sequence $\{a_{m,\rho}\}_{m=0}^{\infty}$ as

$$\begin{aligned}
a_{m,\rho} :&= \int_{t^*}^{t^*+\tilde{\theta}\rho^2} \mu\left(A_{\eta h 2^{-m}, \rho}(t)\right) dt \\
&= \mu \otimes \mathcal{L}^1 \left(\left\{ (x,t) \in B_\rho(x^*) \times [t^* - \tilde{\theta}\rho^2, t^*] : u(x,t) < \frac{\eta h}{2^m} \right\} \right),
\end{aligned}$$

we get by the Hölder inequality

$$\begin{aligned}(25)\quad &\int_{t^*}^{t^*+\tilde\theta\rho^2}\int_{B_{4\rho}(x^*)}\Big(u-\frac{\eta h}{2^m}\Big)_-(x,t)\,d\mu\,dt\\ &\le 8C_p4^{N+1}\mu(B_{4\rho}(x^*))^{1-1/p}\rho\Big(\int_{t^*}^{t^*+\tilde\theta\rho^2}\int_{\tilde A(t)} g^p_{(u-\frac{\eta h}{2^m})_-}(x,t)\,d\mu\,dt\Big)^{1/p}\\ &\le 8C_p4^{N+1}\mu(B_{4\rho}(x^*))^{1-1/p}\rho\Big(\int_{t^*}^{t^*+\tilde\theta\rho^2}\int_{B_{4\Lambda\rho}(x^*)} g^2_{(u-\frac{\eta h}{2^m})_-}\,d\mu\,dt\Big)^{1/2}(a_{m-1,4\Lambda\rho}-a_{m,4\Lambda\rho})^{(2-p)/2}.\end{aligned}$$

On the other hand,

$$\int_{t^*}^{t^*+\tilde\theta\rho^2}\int_{B_{4\rho}(x^*)}\Big(u-\frac{\eta h}{2^m}\Big)_-(x,t)\,d\mu\,dt\ge\frac{\eta h}{2^{m+1}}a_{m+1,4\rho}$$

from which, using first (25) and then (23), we obtain

$$a_{m+1,4\rho}^{2/(2-p)}\le c(a_{m-1,4\Lambda\rho}-a_{m,4\Lambda\rho}),$$

where $c=(C_p^2 16^{N+3}\gamma(2\tilde\theta+1)\mu(B_{4\rho}(x^*))^{2(1-1/p)}\mu(B_{5\Lambda\rho}(x^*))\rho^2)^{1/(2-p)}$. Hence for every $m_*\in\mathbf{N}$ we have

$$\sum_{m=1}^{m_*}a_{m+1,4\rho}^{2/(2-p)}\le c(a_{0,4\Lambda\rho}-a_{m_*,4\Lambda\rho}).$$

Since $\{a_{m,\rho}\}_{m=0}^\infty$ is decreasing the sum $\sum_{m=1}^\infty a_{m+1,4\rho}^{2/(2-p)}$ converges, and consequently

$$\lim_{m\to\infty}a_{m,4\rho}=0.$$

This completes the proof. □

Proof of Proposition 5.3. The proof is a direct consequence of Proposition 5.2 used with the right parameters. Fix $\theta=1$ and let $\tilde\theta$ be as in Lemma 5.5; choose also $\hat\theta\in(0,\tilde\theta)$ and let ν_- be the constant in Proposition 5.2 determined by these parameters and $a=1/2$. Apply Lemma 5.6 with $\epsilon=\nu_-$ and obtain the constant η_1 for which the assumptions of Proposition 5.2 are satisfied with

$$y=x^*,\ s=t^*+\tilde\theta\rho^2,\bar\theta:=\tilde\theta-\hat\theta,m_-=0\text{ and }\sigma=\frac{\eta_1 h}{\omega}.$$

This concludes the proof with $\lambda=\frac12\eta_1$. □

The following is the main result of this paper.

Theorem 5.7 (Parabolic Harnack)**.** *Assume $u\in DG(\Omega,T,\gamma)$, $u\ge0$. For any constant $c_2\in(0,1]$, there exists $c_1>0$, depending on c_d, γ, and the constants in the weak $(1,2)$–Poincaré inequality, such that for any Lebesgue point $(x_0,t_0)\in\Omega\times(0,T)$ with $B_{5\Lambda\rho}(x_0)\times(t_0-\rho^2,t_0+5\rho^2)\subset\Omega\times(0,T)$ we have*

$$u(x_0,t_0)\le c_1\operatorname*{ess\,inf}_{B_\rho(x_0)}u(x,t_0+c_2\rho^2).$$

As a consequence, u is locally α-Hölder continuous with $\alpha=-\log_2\frac{1-\gamma}{\gamma}$ and satisfies the strong maximum principle.

Proof. Suppose $t_0=0$; up to rescaling, we may write $u(x_0,0)=\rho^{-\xi}$ for some $\xi>0$ to be fixed later. Define the functions

$$\mathcal{M}(s)=\sup_{Q_s^-(x_0,0)}u,\qquad\mathcal{N}(s)=(\rho-s)^{-\xi},\qquad s\in[0,\rho).$$

Let us denote by $s_0 \in [0, \rho)$ the largest solution of $\mathcal{M}(s) = \mathcal{N}(s)$. Define

$$M := \mathcal{N}(s_0) = (\rho - s_0)^{-\xi},$$

and fix $(y_0, \tau_0) \in Q^-_{s_0}(x_0, 0)$ in such a way that

$$\frac{3M}{4} < \sup_{Q^-_{\rho_0/4}(y_0,\tau_0)} u \leq M, \tag{26}$$

where $\rho_0 = (\rho - s_0)/2$; this implies that $Q^-_{\rho_0}(y_0, \tau_0) \subset Q^-_{(\rho+s_0)/2}(x_0, 0)$, as well as that

$$\sup_{Q^-_{\rho_0}(y_0,\tau_0)} u \leq \sup_{Q^-_{(\rho+s_0)/2}(x_0,0)} u < \mathcal{N}\left(\frac{\rho + s_0}{2}\right) = 2^\xi M.$$

Let us divide the proof into five steps.

Step 1. We assert that

$$\mu \otimes \mathcal{L}^1 \left(\left\{(x, t) \in Q^-_{\rho_0/2}(y_0, \tau_0) : u(x, t) > \frac{M}{2}\right\}\right) > \nu_+ \mu \otimes \mathcal{L}^1 \left(Q^-_{\rho_0/2}(y_0, \tau_0)\right), \tag{27}$$

where ν_+ is the constant in Proposition 5.1. To see this, assume on the contrary that equation (27) is not true. Then set $k = 2^\xi M$ and

$$m_+ = \omega = k,\ \theta = 1,\ \rho = \frac{\rho_0}{2},\ \sigma = 1 - 2^{-\xi-1}, \text{ and } a = \sigma^{-1}\left(1 - \frac{3}{2^{\xi+2}}\right).$$

We obtain from Proposition 5.1 that

$$u \leq \frac{3M}{4} \quad \text{in } Q^-_{\frac{\rho_0}{4}}(y_0, \tau_0),$$

which contradicts (26).

Step 2. We show that there exists

$$\bar{t} \in \left(\tau_0 - \frac{\rho_0^2}{4}, \tau_0 - \frac{\nu_+}{8}\frac{\rho_0^2}{4}\right]$$

such that

$$\mu\left(\left\{x \in B_{\rho_0/2}(y_0) : u(x, \bar{t}) \geq \frac{M}{2}\right\}\right) > \frac{\nu_+}{2}\mu(B_{\rho_0/2}(y_0)), \tag{28}$$

and

$$\int_{B_{\rho_0/2}(y_0)} g_u^2(x, \bar{t})\, d\mu(x) \leq \alpha\frac{\mu(B_{\rho_0}(y_0))}{\rho_0^2}k^2, \tag{29}$$

for some sufficiently large $\alpha > 0$. For this, we define the sets $A(t)$, I, and J_α as follows

$$A(t) := \left\{x \in B_{\rho_0/2}(y_0) : u(x, t) \geq \frac{M}{2}\right\},$$

$$I := \left\{t \in (\tau_0 - \frac{\rho_0^2}{4}, \tau_0] : \mu(A(t)) > \frac{\nu_+}{2}\mu(B_{\rho_0/2}(y_0))\right\},$$

and

$$J_\alpha := \left\{t \in (\tau_0 - \frac{\rho_0^2}{4}, \tau_0] : \int_{B_{\rho_0/2}(y_0)} g_u^2(x, t) d\mu(x) \leq \alpha\frac{\mu(B_{\rho_0}(y_0))}{\rho_0^2}k^2\right\}.$$

From (27) we have that

$$\begin{aligned}\nu_+\mu\otimes\mathcal{L}^1(Q^-_{\rho_0/2}(y_0,\tau_0)) &< \int_{\tau_0-\rho_0^2/4}^{\tau_0}\mu(A(t))\,dt\\ &= \int_I \mu(A(t))dt + \int_{(\tau_0-\rho_0^2/4]\setminus I}\mu(A(t))\,dt\\ &\le \mu(B_{\rho_0/2}(y_0))|I| + \frac{\nu_+}{2}\mu\otimes\mathcal{L}^1(Q^-_{\rho_0/2}(y_0,\tau_0))\\ &= \mu\otimes\mathcal{L}^1(Q^-_{\rho_0/2}(y_0,\tau_0))\left(|I|\left(\frac{4}{\rho_0^2}\right)+\frac{\nu_+}{2}\right).\end{aligned}$$

This implies the following lower bound

$$|I| \ge \frac{\nu_+\rho_0^2}{8}.$$

On the other hand, if we apply (14) with $R=\rho_0$, $r=\rho_0/2$, $\alpha=0$, and $\theta=1$, we obtain

$$\begin{aligned}\int_{Q^-_{\rho_0/2}(y_0,\tau_0)} g_u^2\,d\mu\,dt &= \int_{Q^-_{\rho_0/2}(y_0,\tau_0)} g^2_{(u-k)_-}\,d\mu\,dt\\ &\le \frac{8\gamma}{\rho_0^2}\int_{Q^-_{\rho_0}(y_0,\tau_0)}(u-k)^2_-\,d\mu\,dt \le \frac{8\gamma k^2}{\rho_0^2}\mu\otimes\mathcal{L}^1(Q^-_{\rho_0}(y_0,\tau_0))\\ &= 8\gamma k^2\mu(B_{\rho_0}(y_0)).\end{aligned}\tag{30}$$

This estimate implies

$$\begin{aligned}4\gamma k^2\mu(B_{\rho_0}(y_0)) &\ge \int_{(\tau_0-\rho_0^2/4,\tau_0]}dt\int_{B_{\rho_0/2}(y_0)} g_u^2(x,t)\,d\mu\\ &\ge \alpha\frac{\mu(B_{\rho_0}(y_0))}{\rho_0^2}k^2\left(\frac{\rho_0^2}{4}-|J_\alpha|\right),\end{aligned}$$

which in turn gives us

$$|J_\alpha| \ge \frac{\rho_0^2}{4}\left(1-\frac{16\gamma}{\alpha}\right).$$

Choosing $\alpha = 64\gamma/\nu_+$, we obtain

$$|I\cap J_\alpha| = |I| + |J_\alpha| - |I\cup J_\alpha| \ge \frac{\nu_+\rho_0^2}{16}.$$

Then if we set

$$T = \left(\tau_0-\frac{\rho_0^2}{4}, \tau_0-\frac{\nu_+}{8}\frac{\rho_0^2}{4}\right],$$

we get

$$|I\cap J_\alpha\cap T| = |I\cap J_\alpha| + |T| - |(I\cap J_\alpha)\cup T| \ge \frac{\rho_0^2}{4}\frac{\nu_+}{8},$$

and in particular $I\cap J_\alpha\cap T\neq\emptyset$.

Step 3. We fix $\bar t\in T$; by Lemma 2.5 we have that for any $\delta\in(0,1)$, there exist $x^*\in B_{\rho_0/2}(y_0)$ and $\eta\in(0,1)$ such that

$$\mu\left(\left\{u(\cdot,\bar t) > \frac{M}{4}\right\}\cap B_{\eta\rho_0/2}(x^*)\right) > (1-\delta)\mu(B_{\eta\rho_0/2}(x^*)).\tag{31}$$

Step 4. We show that for $\varepsilon > 0$ to be fixed, there exists $\bar{x}$ such that $Q^+_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{t}) \subset Q^-_{\rho_0}(y_0,\tau_0)$ and

$$\mu \otimes \mathcal{L}^1\left(\left\{u \le \frac{M}{8}\right\} \cap Q^+_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{t})\right) \le 4^{N+1}(\gamma\varepsilon^2+\delta)\mu \otimes \mathcal{L}^1\left(Q^+_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{t})\right). \tag{32}$$

Indeed, consider the cylinder

$$Q = B_{\eta\rho_0/4}(x^*) \times (\bar{t}, \bar{t}+t^*]$$

with $t^* = (\varepsilon\eta\rho_0/4)^2$. Using the energy estimate (14) on Q with $k = M/4$, $R = \eta\rho_0/2$, $r = R/2$ and $\alpha = 1$, we obtain together with (31) that for any $s \in (\bar{t}, \bar{t}+t^*]$

$$\begin{aligned}
&\int_{B_{\eta\rho_0/4}(x^*)} \left(u - \frac{M}{4}\right)^2_- (x,s)\,d\mu(x) \\
&\le \frac{16\gamma}{\eta^2\rho_0^2}\int_{\bar{t}}^{\bar{t}+t^*} dt \int_{B_{\eta\rho_0/2}(x^*)} \left(u - \frac{M}{4}\right)^2_- (x,t)\,d\mu(x) + \int_{B_{\eta\rho_0/2}(x^*)} \left(u - \frac{M}{4}\right)^2_- (x,\bar{t})\,d\mu(x) \\
&\le \frac{M^2}{16}(\gamma\varepsilon^2+\delta)\mu(B_{\eta\rho_0/2}(x^*)).
\end{aligned}$$

Define

$$B(t) = \left\{x \in B_{\eta\rho_0/4}(x^*) : u(x,t) \le \frac{M}{8}\right\},$$

and we have that

$$\int_{B_{\eta\rho_0/4}(x^*)} \left(u - \frac{M}{4}\right)^2_- (x,s)d\mu(x) \ge \int_{B(s)} \left(u - \frac{M}{4}\right)^2_- (x,s)d\mu(x) \ge \frac{M^2}{64}\mu(B(s)).$$

Putting the preceding two estimates together we arrive at

$$\mu(B(s)) \le 4(\gamma\varepsilon^2+\delta)\mu(B_{\eta\rho_0/2}(x^*))$$

for every $s \in (\bar{t}, \bar{t}+t^*]$. Integrating this inequality over s we obtain the estimate

$$\mu \otimes \mathcal{L}^1\left(\left\{u \le \frac{M}{8}\right\} \cap Q^+_{\varepsilon\eta\rho_0/4}(x^*,\bar{t})\right) \le 2^{N+2}(\gamma\varepsilon^2+\delta)\mu \otimes \mathcal{L}^1\left(Q^+_{\varepsilon\eta\rho_0/4}(x^*,\bar{t})\right).$$

We have to apply Proposition 5.1; we then have that there exists $\bar{x}$ so that $Q^+_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{t}) \subset Q^-_{\rho_0}(y_0,\tau_0)$ satisying equation (32). To see this, we take a disjoint family of balls $\{B_{\varepsilon\eta\rho_0/4}(x_j)\}_{j=1}^m$ such that $B_{\varepsilon\eta\rho_0/4}(x_j) \subset B_{\eta\rho_0/4}(x^*)$ for every $j = 1,\dots,m$, and

$$B_{\eta\rho_0/4}(x^*) \subset \bigcup_{j=1}^m B_{\varepsilon\eta\rho_0/2}(x_j).$$

Given this disjoint family, there exists j_0 such that (32) is satisfied with $\bar{x} = x_{j_0}$. Otherwise we would get a contradiction summing over $j = 1,\dots,m$.

Step 5. Due to our construction, we are able to state

$$\operatorname*{osc}_{Q^+_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{t})} u \le k = 2^\xi M.$$

We also have that if $\bar{s} = \bar{t} + (\varepsilon\eta\rho_0/4)^2$ then $Q^+_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{t}) = Q^-_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{s})$; we apply Proposition 5.2 with

$$\rho = \frac{\varepsilon\eta\rho_0}{4}, \quad \theta = 1, \quad m_- = 0, \quad \omega = k, \quad a = \frac{1}{2}, \quad \sigma = 2^{-\xi-3},$$

so we can deduce that there exists $\nu_- > 0$ such that if

$$\mu \otimes \mathcal{L}^1\left(\left\{u \le \frac{M}{8}\right\} \cap Q^-_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{s})\right) \le \nu_- \mu \otimes \mathcal{L}^1(Q^-_{\varepsilon\eta\rho_0/4}(\bar{x},\bar{s})) \tag{33}$$

then

$$u(x,t) \geq \frac{M}{16}, \qquad \mu \otimes \mathcal{L}^1 - \text{a.e. in } Q_r^-(\bar{x},\bar{s}),$$

where $r = \varepsilon\eta\rho_0/8$.

Fix ε and δ in (32) small enough so that (33) is satisfied and $\bar{t} + (\varepsilon\eta\rho/4)^2 < 0$. With this choice of δ, we obtain the constants η and r that depend only on δ. Expansion of positivity, Proposition 5.3, implies

$$u(x,t) \geq \lambda \frac{M}{16},$$

for all $x \in B_{2r}(\bar{x})$ and $t \in [\hat{t} + \hat{\theta}r^2, \hat{t} + \tilde{\theta}r^2]$ for some $\hat{t} \in (\bar{t}, \bar{t} + (\varepsilon\eta\rho_0/4)^2]$, where $\tilde{\theta}$ depends only on γ, whereas λ depends on γ and $\hat{\theta} \in (0, \tilde{\theta})$ that we shall fix later. We can repeat the argument with r replaced by $2r$ and initial time varying in the interval $[\hat{t} + \hat{\theta}r^2, \hat{t} + \tilde{\theta}r^2]$ to obtain the following estimate

$$u(x,t) \geq \lambda^2 \frac{M}{16},$$

for all $x \in B_{4r}(\bar{x})$ and $t \in [\hat{t} + 5\hat{\theta}r^2, \hat{t} + 5\tilde{\theta}r^2]$. Thus iterating this procedure, we can show by induction that for any $m \in \mathbf{N}$

$$u(x,t) \geq \lambda^m \frac{M}{16}, \tag{34}$$

for all $x \in B_{2^m r}(\bar{x})$ and $t \in [s_m, t_m]$, where

$$s_m = \hat{t} + \hat{\theta}r^2 \frac{4^m - 1}{3} \quad \text{and} \quad t_m = \hat{t} + \tilde{\theta}r^2 \frac{4^m - 1}{3}.$$

We fix m in such a way that $2\rho < 2^m r \leq 4\rho$; since $\bar{x} \in B_\rho(x_0)$, we then have the inclusion $B_\rho(x_0) \subset B_{2^m r}(\bar{x})$. Recalling that $r = \varepsilon\eta(\rho - s_0)/16$, we obtain

$$(\rho - s_0)^{-\xi} = \left(\frac{2^4}{\varepsilon\eta} r\right)^{-\xi} = \frac{(\varepsilon\eta)^\xi}{2^{4\xi} r^\xi} \geq (\varepsilon\eta)^\xi 2^{(m-6)\xi} \rho^{-\xi}.$$

Hence equation (34) can be rewritten as follows

$$u(x,t) \geq \lambda^m \frac{M}{16} = \lambda^m \frac{(\rho - s_0)^{-\xi}}{16} \geq (2^\xi \lambda)^m (\varepsilon\eta)^\xi 2^{-6\xi - 4} \rho^{-\xi} = (2^\xi \lambda)^m (\varepsilon\eta)^\xi 2^{-6\xi-4} u(x_0, 0).$$

for any $x \in B_\rho(x_0)$ and $t \in [s_m, t_m]$.

We now fix $c_2 > 0$ and choose $\hat{\theta}$ in such a way that $\frac{16}{3}\hat{\theta} < c_2$. With this choice, since $2^m r \leq 4\rho$, we have

$$s_m \leq \hat{\theta} r^2 \frac{4^m}{3} \leq \hat{\theta} \frac{16}{3} \rho^2 < c_2 \rho^2. \tag{35}$$

Once $\hat{\theta}$ has been fixed, we have λ; we now fix $\xi = -\log_2 \lambda$. With these choices also the radius r is fixed and so m is chosen in such a way that

$$1 - \log_2 r \leq m \leq 2 - \log_2 r.$$

We draw the conclusion that

$$u(x,t) \geq c_0 u(x_0, 0)$$

with $c_0 := (\varepsilon\eta)^\xi 2^{-6\xi-4}$ for all $x \in B_\rho(x_0)$ and $t \in [s_m, t_m]$.

Notice that by (35) we have got two alternatives. Either $c_2\rho^2 \in [s_m, t_m]$ or $c_2\rho^2 > t_m$. In the former case, the proof is completed by taking $c_1 = c_0^{-1}$. Whereas in the latter case, we can select $\tilde{t} \in [s_m, t_m]$ such that

$$u(x, \tilde{t}) \geq c_0 u(x_0, 0)$$

for all $x \in B_\rho(x_0)$. We can assume that $\hat\theta$ is small enough such that $\tilde t + \hat\theta\rho^2 < c_2\rho^2$. By expansion of positivity, Proposition 5.3, we then obtain that

$$u(x,t) \geq \lambda c_0 u(x_0,0)$$

for all $x \in B_{2\rho}(x_0)$ and $t \in [\tilde t + \hat\theta\rho^2, \tilde t + \tilde\theta\rho^2]$. If $c_2\rho^2 < \tilde t + \tilde\theta\rho^2$, then the proof is completed by selecting $c_1 = (\lambda c_0)^{-1}$. If this was not the case, we could restrict the previous inequality on $B_\rho(x_0)$, and so iterating the procedure, adding the condition that $\hat\theta \leq \tilde\theta$, using the fact that the estimate is already true on $[\tilde t + \hat\theta\rho^2, \tilde t + \tilde\theta\rho^2]$,

$$u(x,t) \geq \lambda^2 c_0 u(x_0,0)$$

for each $x \in B_\rho(x_0)$ and $t \in [\tilde t + \hat\theta\rho^2, \tilde t + 2\tilde\theta\rho^2]$. By induction, if k is an integer such that $\tilde t + k\tilde\theta\rho^2 \geq c_2\rho^2$, then

$$u(x,t) \geq \lambda^k c_0 u(x_0,0)$$

for every $x \in B_\rho(x_0)$ and $t \in [\tilde t + \hat\theta\rho^2, \tilde t + k\tilde\theta\rho^2]$. It is crucial to select such an index k which depends only on the class and not on the function. We then take k in such a way that $\tilde t + k\tilde\theta \geq c_2\rho^2$. As $-\rho^2 \leq \tilde t \leq c_2\rho^2$ the index k has to be chosen in such a way that both $1 + c_2 \leq k\tilde\theta$ and $\tilde t + k\tilde\theta$ remains in the domain of reference. Notice that $1 + c_2 \leq 2$. Hence there exists k such that $2 \leq k\tilde\theta \leq 3$, and we are done with the proof. □

References

[1] M.T. Barlow and R.F. Bass and T. Kumagai, *Stability of parabolic Harnack inequalities on metric measure spaces*, J. Math. Soc. Japan, 58, (2006), n.2, pp. 485–519.

[2] M.T. Barlow and A. Grigor'yan and T. Kumagai, *On the equivalence of parabolic Harnack inequalities and heat kernel estimates*, J. Math. Soc. Japan, to appear.

[3] A. Björn and J. Björn, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics 17, European Mathematical Society (EMS), Zürich, 2011.

[4] J. Björn *Boundary continuity for quasiminimizers on metric spaces*, Illinois J. Math., 46 (2002), pp. 383–403.

[5] L. Capogna and G. Citti and G. Rea *A subelliptic analogue of Aronson-Serrin's Harnack inequality*, manuscript (2011).

[6] J. Cheeger. *Differentiability of Lipschitz functions on measure spaces*, Geom. Funct. Anal. 9 (1999), no. 3, pp. 428–517.

[7] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), pp. 25–43.

[8] T. Delmotte *Graphs between the elliptic and parabolic Harnack inequalities*, Potential Anal., 16 (2002), pp. 151–168.

[9] E. DiBenedetto, *Harnack estimates in certain function classes*, Atti Sem. Mat. Fis. Univ. Modena, 37 (1989), no. 1, pp. 173–182.

[10] E. DiBenedetto and U. Gianazza and V. Vespri, *Local clustering of the non–zero set of functions in $W^{1,1}(E)$*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 17 (2006), no. 3, pp. 223–225

[11] E. DiBenedetto and N.S. Trudinger, *Harnack inequalities for quasiminima of variational integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire, (4) 1 (1984), pp. 295–308.

[12] S. Fornaro and F. Paronetto and V. Vespri, *Disuguaglianza di Harnack per equazioni paraboliche*, Quaderni Dip. Mat. Univ. Lecce, 2/2008,
`http://siba-ese.unisalento.it/index.php/quadmat/view/614`

[13] U. Gianazza and V. Vespri, *Parabolic De Giorgi classes of order p and the Harnack inequality*, Calc. Var. Partial Differential Equations 26 (2006), no. 3, pp. 379–399.

[14] M. Giaquinta and E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math., 148 (1982), pp. 31–46.

[15] M. Giaquinta and E. Giusti, *Quasi–minima*, Ann. Inst. H. Poincaré Anal. Non Linéaire, (2) 1 (1984), pp. 79–107.

[16] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[17] A.A. Grigor'yan, *The heat equation on noncompact Riemannian manifolds*, Mat. Sb., (1) 182 (1991), pp. 55–87; translation in Math. USSR-Sb. 72 (1992), no. 1, 47–77.

[18] P. Hajłasz and P. Koskela, *Sobolev met Poincarè*, Mem. Amer. Math. Soc. (688) 145 (2000), Providence, RI, 2000.

[19] P. Hajłasz, *Sobolev spaces on metric-measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces*, (Paris, 2002), pp. 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[20] J. Heinonen, *Lectures on analysis on metric spaces*, Springer–Verlag, New York, 2001.

[21] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math., 181 (1998), no. 1, pp. 1–61.

[22] S. Keith and X. Zhong, *The Poincaré inequality is an open ended condition*, Ann. of Math. (2) 167 (2008), pp. 575–599.

[23] J. Kinnunen and O. Martio, *Potential theory of quasiminimizers*, Ann. Acad. Sci. Fenn. Math., 28 (2003), no. 2, pp. 459–490.

[24] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi–minimizers on metric spaces*, Manuscripta Math.,105 (2001), no. 3, pp. 401–423.

[25] P. Koskela and P. MacManus *Quasiconformal mappings and Sobolev spaces*, Studia Math., 131 (1998), no. 1, pp. 1–17.

[26] G.M. Lieberman, *Second order parabolic differential equations*, World Scientific Pubishling Co., Inc., River Edge, NJ, 1996.

[27] S. Marchi, *Boundary regularity for parabolic quasiminima*, Ann. Mat. Pura Appl., (4) 166 (1994), pp. 17–26.

[28] O. Martio and C. Sbordone, *Quasiminimizers in one dimension: integrability of the derivative, inverse function and obstacle problems*, Ann. Mat. Pura Appl. (4), 186 (2007), pp. 579–590.

[29] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices, 2 (1992), pp. 27–38.

[30] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana, 16 (2000) no. 2, pp. 243–279.

[31] N. Shanmugalingam, *Harmonic functions on metric spaces*, Illinois J. Math., 45 (2001), no. 3, pp. 1021–1050.

[32] K.-T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9), 75 (1996), pp. 273–297.

[33] K.-T. Sturm, *Diffusion processes and heat kernels on metric spaces*, Ann. Prob. (1), 26 (1998), pp. 1–55.

[34] K.-T. Sturm, *How to construct diffusion processes on metric spaces*, Pot. An., 8 (1998), pp.149–161.

[35] G. Wang, *Harnack inequalities for functions in the De Giorgi parabolic classes*, Lecture Notes Math., 1306 (1988), pp. 182–201.

[36] W. Wieser, *Parabolic Q–minima and minimal solutions to variational flow*, Manuscripta Math., 59 (1987), pp. 63–107.

[37] S. Zhou, *On the local behavior of parabolic Q–minima*, J. Partial Differential Equations, 6 (1993), pp. 255–272.

[38] S. Zhou, *Parabolic Q–minima and their applications*, J. Partial Differential Equations, 7 (1994), pp. 289–322.

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