THE DE GIORGI MEASURE AND AN OBSTACLE PROBLEM RELATED TO MINIMAL SURFACES IN METRIC SPACES

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Abstract. We study the existence of a set with minimal perimeter that separates two disjoint sets in a metric measure space equipped with a doubling measure and supporting a Poincaré inequality. A measure constructed by De Giorgi is used to state a relaxed problem, whose solution coincides with the solution to the original problem for measure theoretically thick sets. Moreover, we study properties of the De Giorgi measure on metric measure spaces and show that it is comparable to the Hausdorff measure of codimension one. We also explore the relationship between the De Giorgi measure and the variational capacity of order one. The theory of functions of bounded variation on metric spaces is used extensively in the arguments.

1. Introduction

We study minimal surfaces in a metric measure space X that is equipped with a doubling measure and supports a Poincaré inequality. More precisely, we extend the results of De Giorgi, Colombini, and Piccinini in [11] (see also [10] and [21]) from the Euclidean setting to the metric setting. In this context, the minimization problem reads as follows. Let E and F be disjoint Borel sets in X. Find a Borel set G_0 such that $E \subset G_0$, $G_0 \cap F = \emptyset$ and

$$P(G_0) = \inf P(G),$$

where the infimum is taken over all Borel sets $G \subset X$ with the properties that $E \subset G$ and $G \cap F = \emptyset$. In other words, find a set with a minimal perimeter that separates the sets E and F. This is an obstacle problem in geometric measure theory. In order to be able to talk about the perimeter measure, we need the theory of functions of bounded variation on metric spaces developed by Miranda [19], Ambrosio [1], [2] and Ambrosio, Miranda and Pallara [3].

A rather standard argument based on compactness and lower semicontinuity properties of the perimeter measure shows that the minimizer

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exists. However, since the perimeter measure does not see sets of measure zero, the minimization problem is relevant only for the obstacles that are thick enough. The main reason is that there are too many admissible sets in the minimization problem. It is possible to restrict the class of admissible sets, for example, by considering those sets that are thick and whose complements are thick as well. Unfortunately, in general the minimization problem does not have a solution in this class. On the other hand, we can study a relaxed problem that takes the thin parts of the obstacles into account by introducing a penalty factor. The Hausdorff measure of codimension one would be a natural choice, but it turns out that a geometric measure constructed by De Giorgi is easier to deal with in questions related to lower semicontinuity. The relaxed obstacle problem stated in terms of the De Giorgi measure has a solution and for thick obstacles the solution coincides with the solution of the original problem.

We show that the results of De Giorgi, Colombini and Piccinini hold true in metric measure spaces and that they are independent of the Euclidean structure and the Lebesgue measure. In particular, we do not have integration by parts, divergence formula or tangents of sets available in a general metric measure space. We also study properties of the De Giorgi measure on metric measure spaces and show that it is comparable to the Hausdorff measure of codimension one. Moreover, we explore the relationship between the De Giorgi measure and the variational capacity of order one. Our arguments are based on the so-called boxing inequality, which has been studied in the metric context in [15] and [20]. We also apply Ambrosio's result in [2] (see also [1] and [3]), which states that the perimeter measure is concentrated on the measure theoretic boundary. We present robust arguments that are based on general principles.

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2. Preliminaries

In this paper, (X, d, μ) is a complete metric measure space with $\mu(X) = \infty$. The measure is assumed to be doubling. This means that there exists a constant $c_D \geq 1$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r)).$$

A complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact.

We define Sobolev spaces on X using upper gradients, see Shanmugalingam [22].

Definition 2.1. A nonnegative Borel function g on Ω is an upper gradient of an extended real valued function u on Ω if for all $x, y \in \Omega$ and for all paths γ joining x and y in Ω,

$$|u(x) - u(y)| \le \int_\gamma g\, ds$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

Let $1 \le p < \infty$. If u is a function that is integrable to power p in X, let

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \Big)^{1/p},$$

where the infimum is taken over all upper gradients of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Throughout this paper, we assume that X supports a weak $(1,1)$-Poincaré inequality, i.e. there exist constants $c_P > 0$ and $\tau \ge 1$ such that for all balls $B(x,r)$ of X, all locally integrable functions u on X and for all upper gradients g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le c_P r \fint_{B(x,\tau r)} g \, d\mu,$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Let $\Omega \subset X$ be an open set. For $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω as

$$\begin{aligned} &\|Du\|(\Omega) \\ &\quad = \inf \Big\{ \liminf_{i \to \infty} \int_\Omega g_{u_i} \, d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \Big\}, \end{aligned}$$

where g_{u_i} is an upper gradient of u_i in Ω. As usual, we say that a property holds locally if it holds in every compact set. We say that a function $u \in L^1(\Omega)$ is of bounded variation, $u \in BV(\Omega)$, if $\|Du\|(\Omega) < \infty$. Moreover, a measurable set $E \subset X$ is said to have finite perimeter in Ω if $\|D\chi_E\|(\Omega) < \infty$. The theory of functions of bounded variation on metric measure spaces has been developed in [19]. See also [1], [2] and [3].

It is essential for us that the total variation is the restriction of a Borel measure to open sets. For the following result we refer to Theorem 3.4 in [19].

Theorem 2.2. *Let $u \in BV_{\mathrm{loc}}(X)$. For every set $A \subset X$, we define*

$$\|Du\|(A) = \inf\big\{\|Du\|(\Omega) : \Omega \supset A,\, \Omega \subset X \text{ is open}\big\}.$$

Then $\|Du\|(\cdot)$ is a locally finite Borel outer measure. If $u \in BV(X)$, then $\|Du\|(X) < \infty$.

Let E be a set of finite perimeter in X. For every set $A \subset X$, we denote

$$P(E, A) = \|D\chi_E\|(A).$$

For short, we also write $P(E) = P(E, X)$.

Remark 2.3. Since X is proper, the above result implies by measure theory that if E is a set with finite perimeter in X, $A \subset X$ is a Borel set and $\varepsilon > 0$, then there is a compact set $K \subset A$ such that $P(E, A) < P(E, K) + \varepsilon$. See [18].

For sets $E, F \subset X$, we denote

$$E \triangle F = (E \setminus F) \cup (F \setminus E).$$

If $E_i \subset X$, $i = 1, 2, \dots$, are μ-measurable, then we say that $E_i \to E$ in $L^1(\Omega)$, if $\mu((E_i \triangle E) \cap \Omega) \to 0$, or equivalently

$$\int_\Omega |\chi_{E_i} - \chi_E| \, d\mu \to 0,$$

as $i \to \infty$. Analogously, $E_i \to E$ in $L^1_{\mathrm{loc}}(\Omega)$, if $E_i \to E$ in $L^1(K)$ for every compact subset K of Ω as $i \to \infty$.

Basic properties of the perimeter measure are collected in the following lemma. The properties (i)-(vi) below follow easily from the definitions. The property (vii) follows from the lower semicontinuity, Lemma 2.7. For the proofs, we refer to [19].

Lemma 2.4.
(i) *If $\mu((E \triangle F) \cap \Omega) = 0$, then $P(E, \Omega) = P(F, \Omega)$.*
(ii) *$P(E \cup F, \Omega) + P(E \cap F, \Omega) \le P(E, \Omega) + P(F, \Omega)$.*
(iii) *$P(E, \Omega) = P(X \setminus E, \Omega)$.*
(iv) *$P(E \setminus F, \Omega) \le P(E, \Omega) + P(F, \Omega)$.*
(v) *If $\Omega_1, \Omega_2 \subset X$ are open with $\Omega_1 \cap \Omega_2 = \emptyset$ and E is a Borel set such that $P(E, \Omega_1 \cup \Omega_2)$ is finite, then*

$$P(E, \Omega_1 \cup \Omega_2) = P(E, \Omega_1) + P(E, \Omega_2).$$

Moreover, $P(E, \Omega_1 \cup \Omega_2)$ is finite if and only if $P(E, \Omega_1)$ and $P(E, \Omega_2)$ are both finite.
(vi) *If $\Omega_1, \Omega_2 \subset X$ are open, $\Omega_1 \subset \Omega_2$ and $\operatorname{dist}(E, \Omega_2 \setminus \Omega_1) > 0$, then $P(E, \Omega_1) = P(E, \Omega_2)$.*
(vii) *For Borel sets E_i, $i = 1, 2, \dots$,*

$$P\Big(\bigcup_{i=1}^\infty E_i, \Omega\Big) \le \sum_{i=1}^\infty P(E_i, \Omega). \tag{2.5}$$

For the following compactness and lower semicontinuity properties of the perimeter measure in the metric setting, see Theorem 3.7 and Proposition 3.6 in [19]. The assumption that the space supports a weak $(1,1)$-Poincaré inequality is used in the proof of the following lemma. We recall that if Ω and Ω' are open sets, then $\Omega' \Subset \Omega$ denotes that $\overline{\Omega'}$ is a compact subset of Ω.

Lemma 2.6. *Let $\Omega \subset X$ be open. Let E_i, $i = 1, 2, \dots$ be a sequence of Borel sets such that for any $\Omega' \Subset \Omega$ there exists a constant $M(\Omega') < \infty$ for which $P(E_i, \Omega') \le M(\Omega')$ for every $i = 1, 2, \dots$. Then there exists a subsequence E_{i_j}, $j = 1, 2, \dots$ and a Borel set E such that $E_{i_j} \to E$ in $L^1_{\mathrm{loc}}(\Omega)$.*

Lemma 2.7. *Let u_i, $i = 1, 2, \dots$, be a sequence of functions in $BV_{\mathrm{loc}}(\Omega)$ converging to u in $L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\|Du\|(\Omega) \le \liminf_{i\to\infty} \|Du_i\|(\Omega).$$

If E_i, $i = 1, 2, \dots,$ is a sequence of Borel sets converging to a Borel set E in $L^1_{\mathrm{loc}}(\Omega)$, then

$$P(E, \Omega) \le \liminf_{i\to\infty} P(E_i, \Omega).$$

The following coarea formula will be useful for us. For the proof, we refer to Proposition 4.2 in [19].

Theorem 2.8. *If $u \in BV(X)$ and $A \subset X$ is a Borel set, then*

$$\|Du\|(A) = \int_{-\infty}^{\infty} P(\{x \in X : u(x) > t\}, A)\, dt.$$

We shall also need the following version of the Leibniz rule for functions of bounded variation.

Lemma 2.9. *Let $\Omega \subset X$ be an open set, $E \subset X$ be a Borel set with finite perimeter and $u : \Omega \to [0, 1]$ be a function in $N^{1,1}(\Omega)$. Then*

$$\int_0^1 P(\{x \in E : u(x) > t\}, \Omega)\, dt \le \int_{E\cap\Omega} g_u\, d\mu + P(E, \Omega).$$

Proof. For $0 \le t \le 1$ denote $E_t = \{x \in E : u(x) > t\}$. The coarea formula implies that

$$\int_0^1 P(E_t, \Omega)\, dt = \|D(u\chi_E)\|(\Omega).$$

Hence it is enough to show that

$$\|D(u\chi_E)\|(\Omega) \le \int_{E\cap\Omega} g_u\, d\mu + P(E, \Omega).$$

Let v_i, $i = 1, 2, \dots$, be locally Lipschitz functions such that $v_i \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$ and that

$$P(E, \Omega) = \lim_{i\to\infty} \int_\Omega g_{v_i}\, d\mu.$$

Since truncations do not increase the BV energy and $0 \le \chi_E \le 1$, we can assume that $0 \le v_i \le 1$.

Let $\Omega' \Subset \Omega$. Then

$$\int_{\Omega'} |uv_i - u\chi_E|\, d\mu \le \int_{\Omega'} |v_i - \chi_E|\, d\mu \to 0$$

as $i \to \infty$, and hence $uv_i \to u\chi_E$ in $L^1(\Omega')$. Since $ug_{v_i} + v_i g_u$ is an upper gradient of uv_i, by Lemma 2.7 and the dominated convergence theorem,

$$\begin{aligned}
\|D(u\chi_E)\|(\Omega') &\le \limsup_{i\to\infty} \int_{\Omega'} (ug_{v_i} + v_i g_u)\, d\mu \\
&\le P(E, \Omega) + \limsup_{i\to\infty} \int_{\Omega'} v_i g_u\, d\mu \\
&\le P(E, \Omega) + \int_{E\cap\Omega'} g_u\, d\mu.
\end{aligned}$$

The claim follows by exhausting Ω with an increasing sequence of relatively compact sets $\Omega' \Subset \Omega$. □

Now we apply the theory of functions of bounded variation to show that the obstacle problem described in the introduction has a solution.

Theorem 2.10. *Let $\Omega \subset X$ be open, and let E and F be disjoint Borel sets in X. Then there exists a set G_0 with $\Omega \cap E \subset G_0$ and $G_0 \cap F \cap \Omega = \emptyset$ such that*

$$P(G_0, \Omega) \le P(G, \Omega)$$

for every set G with $\Omega \cap E \subset G$ and $G \cap F \cap \Omega = \emptyset$.

Proof. Denote

$$\lambda = \inf\{P(G, \Omega) : G \text{ Borel set}, \Omega \cap E \subset G, G \cap F \cap \Omega = \emptyset\}.$$

First we observe that there exists a minimizing sequence of Borel sets G_i, with $\Omega \cap E \subset G_i$ and $G_i \cap F \cap \Omega = \emptyset$ for every $i = 1, 2, \dots$, such that

$$\lim_{i\to\infty} P(G_i, \Omega) = \lambda.$$

In particular, this implies that there exists a constant $M < \infty$ such that $P(G_i, \Omega) \le M$ for every $i = 1, 2, \dots$. By Lemma 2.6, we obtain a subsequence G_{i_j}, $j = 1, 2, \dots$, and a Borel set G_0 such that

$$\chi_{G_{i_j}} \to \chi_{G_0} \text{ in } L^1_{\mathrm{loc}}(\Omega)$$

as $j \to \infty$. By passing to a subsequence, if necessary, we can also assume that

$$\chi_{G_{i_j}} \to \chi_{G_0} \text{ almost everywhere in } \Omega$$

as $j \to \infty$. By changing G_0 on a set of measure zero, we may assume that $\Omega \cap E \subset G_0$ and $G_0 \cap F \cap \Omega = \emptyset$. Hence $\lambda \le P(G_0, \Omega)$. On the other hand, from Lemma 2.7 we conclude that

$$P(G_0, \Omega) \le \liminf_{j\to\infty} P(G_{i_j}, \Omega) = \lambda.$$

This shows that $\lambda = P(G_0, \Omega)$ and hence G_0 is a minimizing set. $\square$

We recall the following two-dimensional Euclidean example with the Lebesgue measure from [10].

Example 2.11. Let $\Omega = \mathbb{R}^2$,

$$E = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1,\, x_2 = 0\}$$

and $F = \{x \in \mathbb{R}^2 : |x| \ge 4\}$. Then

$$\inf P(G, \Omega) = 0,$$

where the infimum is taken over all Borel sets G with $\Omega \cap E \subset G$ and $G \cap F \cap \Omega = \emptyset$. In particular, E itself will do as a minimizing set and $P(E, \Omega) = 0$ since the Lebesgue measure of E is zero.

Since E is a connected set with at least two points in the 2-dimensional space $\mathbb{R}^2$, E has positive 1-capacity. Therefore the answer that the infimum is zero is unsatisfactory as a quantity that is geometric measure theoretic but captures the potential theory corresponding to $p = 1$. The main reason that the infimum is zero in the previous example is the fact that there are too many admissible test sets. In the next section we introduce a smaller class that will serve our needs better.

3. The De Giorgi measure

In this section, we define De Giorgi measure as in [11] and study its basic properties in metric spaces.

Let $E \subset X$ be a Borel set. The upper density of E at a point $x \in X$ is defined by

$$\overline{D}(E, x) = \limsup_{r\to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}$$

and the lower density by

$$\underline{D}(E, x) = \liminf_{r\to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.$$

If $\overline{D}(E, x) = \underline{D}(E, x)$ then the limit exists and we denote it by $D(E, x)$. By the differentiation theory for doubling measures, we have

$$D(E, x) = 1 \quad \text{for } \mu\text{-almost every } x \in E$$

and

$$D(E,x)=0 \quad \text{for } \mu\text{-almost every } x\in X\setminus E,$$

(see for example the discussion in Section 2.7 of [12] or [18]).

We denote by $\mathcal{G}$ the collection of all $E\subset X$ such that E is μ-measurable,

$$\overline{D}(E,x)>0 \quad \text{for every } x\in E,$$

and

$$\overline{D}(X\setminus E,x)>0 \quad \text{for every } x\in X\setminus E,$$

(that is, $\underline{D}(E,x)<1$). In other words, $E\in\mathcal{G}$ if both E and $X\setminus E$ are thick in the sense that the upper density of the set is positive at all points belonging to the set. Clearly $E\in\mathcal{G}$ if and only if $X\setminus E\in\mathcal{G}$.

It is possible to associate to every Borel set $E\subset X$ several sets $G\in\mathcal{G}$ so that $\mu(G\triangle E)=0$. Therefore, it makes sense to try to find a set in this class that differs from the original set as little as possible. To this end, we define

$$\widetilde{E}=\{x\in E:\overline{D}(E,x)>0\}\cup\{x\in X:\underline{D}(E,x)=1\}. \tag{3.1}$$

Observe that $\widetilde{E}$ is a Borel set, $\widetilde{E}\in\mathcal{G}$ and, by the differentiation theory of measures, we have $\mu(E\triangle\widetilde{E})=0$. It is clear that $E\in\mathcal{G}$ if and only if $\widetilde{E}=E$. Moreover, for every $G\in\mathcal{G}$ with $\mu(G\triangle E)=0$ we have $\widetilde{E}\setminus E\subset G\setminus E$ and $E\setminus\widetilde{E}\subset E\setminus G$. We also record that if E is open, then $E\subset\widetilde{E}$.

The proof of the following lemma follows directly from the definitions.

Lemma 3.2. *With the notation as in* (3.1),

(i) *If Ω is open and E_1,E_2 are Borel sets with $E_1\cap\Omega=E_2\cap\Omega$, then $\widetilde{E}_1\cap\Omega=\widetilde{E}_2\cap\Omega$.*
(ii) *If $E_i\in\mathcal{G}$, $i=1,2,\dots$, then*

$$\bigcup_{i=1}^{\infty}E_i\subset\widetilde{\bigcup_{i=1}^{\infty}E_i}.$$

Let Ω be an open set, and let E and F be disjoint Borel sets in X. We would like to reformulate the obstacle problem in the following way. Find a set $G_0\in\mathcal{G}$ with $\Omega\cap E\subset G_0$ and $G_0\cap F\cap\Omega=\emptyset$ such that

$$P(G_0,\Omega)\le P(G,\Omega)$$

for every set $G\in\mathcal{G}$ with $\Omega\cap E\subset G$ and $G\cap F\cap\Omega=\emptyset$. The example at the end of the previous section shows that there may be no minimizing set in this class.

In order to be able to obtain the existence of such a minimizing set $G_0\in\mathcal{G}$, we need to relax the conditions $E\cap\Omega\subset G$ and $F\cap\Omega\cap G=\emptyset$. Example 2.14 tells us that in order to obtain meaningful answers, we should not relax the condition to the point of allowing measure zero subsets of E to leak outside of G nor measure zero subsets of F to leak into G. A finer notion than measure zero is needed here. We apply the following geometric measure proposed by De Giorgi.

Definition 3.3. Let $\Omega \subset X$ be an open set and $\varepsilon > 0$. For an arbitrary $E \subset X$, we define

$$\sigma_\varepsilon(E, \Omega) = \inf \Big\{ P(G, \Omega) + \frac{\mu(G \cap \Omega)}{\varepsilon} \,:\, G \in \mathcal{G}, E \cap \Omega \subset G \Big\}.$$

The De Giorgi measure of E with respect to Ω is

$$\sigma(E, \Omega) = \sup_{\varepsilon > 0} \sigma_\varepsilon(E, \Omega) = \lim_{\varepsilon \to 0} \sigma_\varepsilon(E, \Omega).$$

If $\Omega = X$, we denote $\sigma(E) = \sigma(E, X)$.

Theorem 3.4. *Let $\Omega \subset X$ be an open set. Then the set functions $\sigma_\varepsilon(\cdot, \Omega)$ with $\varepsilon > 0$ and $\sigma(\cdot, \Omega)$ are outer measures.*

Proof. It is enough to show that $\sigma_\varepsilon(\cdot, \Omega)$ is countably subadditive for every $\varepsilon > 0$. Let E_i, $i = 1, 2, \ldots$, be subsets of X. For every $\eta > 0$ and for every $i = 1, 2, \ldots$, there exists $G_i \in \mathcal{G}$ such that $E_i \cap \Omega \subset G_i$ and

$$P(G_i, \Omega) + \frac{\mu(G_i \cap \Omega)}{\varepsilon} \le \sigma_\varepsilon(E_i, \Omega) + 2^{-i}\eta.$$

Denote

$$E = \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad G = \bigcup_{i=1}^{\infty} G_i.$$

As $G_i \in \mathcal{G}$, by Lemma 3.2 (ii) we have $G \subset \widetilde{G}$. Since $\mu(G \triangle \widetilde{G}) = 0$, Lemma 2.4 (i) implies that $P(\widetilde{G}, \Omega) = P(G, \Omega)$. By (2.5)

$$P(G, \Omega) \le \sum_{i=1}^{\infty} P(G_i, \Omega).$$

Since $E \cap \Omega \subset G \subset \widetilde{G}$ and $\widetilde{G} \in \mathcal{G}$ we have

$$\begin{aligned} \sigma_\varepsilon(E, \Omega) &\le P(\widetilde{G}, \Omega) + \frac{\mu(\widetilde{G} \cap \Omega)}{\varepsilon} = P(G, \Omega) + \frac{\mu(G \cap \Omega)}{\varepsilon} \\ &\le \sum_{i=1}^{\infty} P(G_i, \Omega) + \sum_{i=1}^{\infty} \frac{\mu(G_i \cap \Omega)}{\varepsilon} \\ &\le \sum_{i=1}^{\infty} \sigma_\varepsilon(E_i, \Omega) + \eta \sum_{i=1}^{\infty} 2^{-i} = \sum_{i=1}^{\infty} \sigma_\varepsilon(E_i, \Omega) + \eta. \end{aligned}$$

The claim follows by letting $\eta \to 0$. $\square$

Lemma 3.5. *If $\Omega_1, \Omega_2 \subset X$ are open and $\Omega_1 \cap \Omega_2 = \emptyset$, then*

$$\sigma_\varepsilon(E, \Omega_1) + \sigma_\varepsilon(E, \Omega_2) = \sigma_\varepsilon(E, \Omega_1 \cup \Omega_2)$$

for every $\varepsilon > 0$ and consequently

$$\sigma(E, \Omega_1) + \sigma(E, \Omega_2) = \sigma(E, \Omega_1 \cup \Omega_2)$$

for every $E \subset X$.

Proof. It is enough to prove the claim for σ_ε. Let $\eta > 0$ and let $G \in \mathcal{G}$ be such that $E \cap (\Omega_1 \cup \Omega_2) \subset G$ and

$$P(G, \Omega_1 \cup \Omega_2) + \frac{\mu(G \cap (\Omega_1 \cup \Omega_2))}{\varepsilon} \leq \sigma_\varepsilon(E, \Omega_1 \cup \Omega_2) + \eta.$$

Since $E \cap \Omega_1, E \cap \Omega_2 \subset G$ and $G \in \mathcal{G}$, we have by Lemma 2.4 (v) that

$$\begin{aligned}
&\sigma_\varepsilon(E, \Omega_1) + \sigma_\varepsilon(E, \Omega_2) \\
&\quad \leq P(G, \Omega_1) + \frac{\mu(G \cap \Omega_1)}{\varepsilon} + P(G, \Omega_2) + \frac{\mu(G \cap \Omega_2)}{\varepsilon} \\
&\quad = P(G, \Omega_1 \cup \Omega_2) + \frac{\mu(G \cap (\Omega_1 \cup \Omega_2))}{\varepsilon} \\
&\quad \leq \sigma_\varepsilon(E, \Omega_1 \cup \Omega_2) + \eta.
\end{aligned} \tag{3.6}$$

By letting $\eta \to 0$ we arrive at

$$\sigma_\varepsilon(E, \Omega_1) + \sigma_\varepsilon(E, \Omega_2) \leq \sigma_\varepsilon(E, \Omega_1 \cup \Omega_2).$$

Next, for $i = 1, 2$, let $G_i \in \mathcal{G}$ be sets for which $E \cap \Omega_i \subset G_i$ and

$$P(G_i, \Omega_i) + \frac{\mu(G_i \cap \Omega_i)}{\varepsilon} \leq \sigma_\varepsilon(E, \Omega_i) + \eta.$$

Define

$$G' = ((G_1 \cap \Omega_1) \cup (G_2 \cap \Omega_2)) \qquad \text{and} \qquad G = \widetilde{G'}.$$

Then $E \cap (\Omega_1 \cup \Omega_2) \subset G$ and $G \in \mathcal{G}$. With Lemma 2.4 (v), we have

$$\begin{aligned}
\sigma_\varepsilon(E, \Omega_1 \cup \Omega_2) &\leq P(G, \Omega_1 \cup \Omega_2) + \frac{\mu(G \cap (\Omega_1 \cup \Omega_2))}{\varepsilon} \\
&= P(G, \Omega_1) + P(G, \Omega_2) + \frac{\mu(G \cap \Omega_1)}{\varepsilon} + \frac{\mu(G \cap \Omega_2)}{\varepsilon}.
\end{aligned}$$

Since $\mu(G \triangle G') = 0$, from Lemma 2.4 (i), (ii) we conclude that

$$\begin{aligned}
P(G, \Omega_i) = P(G', \Omega_i) &\leq P(G_1 \cap \Omega_1, \Omega_i) + P(G_2 \cap \Omega_2, \Omega_i) \\
&= P(G_i \cap \Omega_i, \Omega_i)
\end{aligned}$$

for $i = 1, 2$. In the same way we see that $\mu(G \cap \Omega_i) = \mu(G_i \cap \Omega_i)$ for $i = 1, 2$. Thus

$$\begin{aligned}
\sigma_\varepsilon(E, \Omega_1 \cup \Omega_2) &\leq P(G_1, \Omega_1) + P(G_2, \Omega_2) + \frac{\mu(G_1 \cap \Omega_1)}{\varepsilon} + \frac{\mu(G_2 \cap \Omega_2)}{\varepsilon} \\
&\leq \sigma_\varepsilon(E, \Omega_1) + \sigma_\varepsilon(E, \Omega_2) + 2\eta.
\end{aligned}$$

Finally, by letting $\eta \to 0$ we arrive at

$$\sigma_\varepsilon(E, \Omega_1 \cup \Omega_2) \le \sigma_\varepsilon(E, \Omega_1) + \sigma_\varepsilon(E, \Omega_2)$$

This completes the proof. $\square$

Lemma 3.7. *Let $\Omega_1, \Omega_2 \subset X$ be open sets such that $\Omega_1 \subset \Omega_2$ and let $\varepsilon > 0$. Then $\sigma_\varepsilon(E, \Omega_1) \le \sigma_\varepsilon(E, \Omega_2)$ and consequently $\sigma(E, \Omega_1) \le \sigma(E, \Omega_2)$.*

Proof. By monotonicity of the measures $P(G, \cdot)$ and $\mu(\cdot)$,

$$P(G, \Omega_1) \le P(G, \Omega_2) \quad \text{and} \quad \mu(G \cap \Omega_1) \le \mu(G \cap \Omega_2)$$

for all test sets G in the definition of $\sigma_\varepsilon(E, \Omega_2)$. The claim follows from this. $\square$

Lemma 3.8. *If $\Omega_1, \Omega_2 \subset X$ are open, $\Omega_1 \subset \Omega_2$ and* $\operatorname{dist}(E, \Omega_2 \setminus \Omega_1) \ge \delta > 0$, *then*

$$\sigma_\varepsilon(E, \Omega_2) \le \Big(1 + \frac{2\varepsilon}{\delta}\Big) \sigma_\varepsilon(E, \Omega_1)$$

for every $\varepsilon > 0$. Consequently $\sigma(E, \Omega_1) = \sigma(E, \Omega_2)$.

Proof. We may assume that $\sigma_\varepsilon(E, \Omega_1) < \infty$. Let $\eta > 0$ and let $G \in \mathcal{G}$ be such that $E \cap \Omega_1 \subset G$ and

$$P(G, \Omega_1) + \frac{\mu(G \cap \Omega_1)}{\varepsilon} \le \sigma_\varepsilon(E, \Omega_1) + \eta.$$

We construct a test set for $\sigma_\varepsilon(E, \Omega_2)$ using the level sets of the Lipschitz function

$$u(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, E) \le \delta/4, \\ \dfrac{3}{2} - \dfrac{2}{\delta} \operatorname{dist}(x, E), & \text{if } \delta/4 < \operatorname{dist}(x, E) < 3\delta/4, \\ 0, & \text{if } \operatorname{dist}(x, E) \ge 3\delta/4. \end{cases}$$

Define $G_t = \{x \in G : u(x) > t\}$. Since u is $2/\delta$-Lipschitz and $0 \le u \le 1$, Lemma 2.9 implies that

$$\begin{aligned} \int_0^1 P(G_t, \Omega_1)\, dt &\le \int_{G \cap \Omega_1} g_u \, d\mu + P(G, \Omega_1) \\ &\le \frac{2}{\delta} \mu(G \cap \Omega_1) + P(G, \Omega_1), \end{aligned}$$

and hence, for some $0 < t_0 < 1$,

$$P(G_{t_0}, \Omega_1) \le \frac{2}{\delta} \mu(G \cap \Omega_1) + P(G, \Omega_1). \tag{3.9}$$

By the definition of u, we see that

$$G \cap (E_{\delta/4} \cap \Omega_1) = G_{t_0} \cap (E_{\delta/4} \cap \Omega_1),$$

where $E_{\delta/4} = \{x \in X : \operatorname{dist}(x, E) < \delta/4\}$.

Since $E_{\delta/4} \cap \Omega_1$ is open and

$$\Omega_1 \cap E \subset E_{\delta/4} \cap \Omega_1 \subset G_{t_0},$$

it follows that $E \cap \Omega_1 \subset \widetilde{G}_{t_0}$. Since

$$\operatorname{dist}(E, \Omega_2 \setminus \Omega_1) \geq \delta > 0$$

and $\Omega_1 \subset \Omega_2$, we have $E \cap \Omega_1 = E \cap \Omega_2$. Hence $E \cap \Omega_2 \subset \widetilde{G}_{t_0}$.

Since

$$\operatorname{dist}(G_{t_0}, \Omega_2 \setminus \Omega_1) \geq \delta/4, \quad \mu(G_{t_0} \triangle \widetilde{G_{t_0}}) = 0 \quad \text{and} \quad G_{t_0} \subset \Omega_1 \cup (X \setminus \Omega_2),$$

we have

$$P(G_{t_0}, \Omega_1) = P(G_{t_0}, \Omega_2) = P(\widetilde{G_{t_0}}, \Omega_2)$$

and

$$\mu(\widetilde{G_{t_0}} \cap \Omega_2) = \mu(\widetilde{G_{t_0}} \cap \Omega_1).$$

Here we also applied Lemma 2.4 (vi) and (i). These facts, together with (3.9), imply that

$$\begin{aligned}
\sigma_\varepsilon(E, \Omega_2) &\leq P(\widetilde{G_{t_0}}, \Omega_2) + \frac{\mu(\widetilde{G_{t_0}} \cap \Omega_2)}{\varepsilon} \\
&\leq \frac{2}{\delta}\,\mu(G \cap \Omega_1) + P(G, \Omega_1) + \frac{\mu(\widetilde{G_{t_0}} \cap \Omega_1)}{\varepsilon} \\
&\leq \Big(\frac{2\varepsilon}{\delta} + 1\Big)\frac{\mu(G \cap \Omega_1)}{\varepsilon} + P(G, \Omega_1) \\
&\leq \Big(\frac{2\varepsilon}{\delta} + 1\Big)\big(\sigma_\varepsilon(E, \Omega_1) + \eta\big).
\end{aligned}$$

Letting $\eta \to 0$ gives the first claim and then $\varepsilon \to 0$ implies that $\sigma(E, \Omega_2) \leq \sigma(E, \Omega_1)$. The reverse inequality follows from Lemma 3.7. This proves the second claim. $\square$

Now we are ready to show that the De Giorgi measure is a Borel regular outer measure.

Theorem 3.10. *Let $\Omega \subset X$ be open. Then $\sigma(\cdot, \Omega)$ is a Borel measure and*

$$\sigma(E, \Omega) = \inf\{\sigma(B, \Omega) : B \text{ is a Borel set, } E \subset B\} \tag{3.11}$$

for every $E \subset X$.

Remark 3.12. Since X is proper, the previous result implies that if $\Omega \subset X$ is open and $E \subset X$ satisfies $\sigma(E, \Omega) < \infty$, then for every $\varepsilon > 0$ there is a compact set $K \subset E$ such that $\sigma(E, \Omega) < \sigma(K, \Omega) + \varepsilon$. See [18].

Proof of Theorem 3.10. We begin by showing that σ is a Borel measure. By the Carathéodory criterion, it is enough to show that

$$\sigma(E_1 \cup E_2, \Omega) = \sigma(E_1, \Omega) + \sigma(E_2, \Omega)$$

whenever $d = \operatorname{dist}(E_1, E_2) > 0$, see for example Theorem 1.1.11 in [4]. Let E_1, E_2 be such sets. Since σ is an outer measure, the inequality

$$\sigma(E_1 \cup E_2, \Omega) \le \sigma(E_1, \Omega) + \sigma(E_2, \Omega)$$

follows from subadditivity. Since $d > 0$, there are disjoint open sets U_1 and U_2 such that $E_1 \subset U_1$, $E_2 \subset U_2$, and

$$\operatorname{dist}(E_1, X \setminus U_1) \ge d/4 > 0 \quad \text{and} \quad \operatorname{dist}(E_2, X \setminus U_2) \ge d/4 > 0.$$

By Lemma 3.8 and monotonicity of σ,

$$\begin{aligned} \sigma(E_1, \Omega) + \sigma(E_2, \Omega) &= \sigma(E_1, U_1 \cap \Omega) + \sigma(E_2, U_2 \cap \Omega) \\ &\le \sigma(E_1 \cup E_2, U_1 \cap \Omega) + \sigma(E_1 \cup E_2, U_2 \cap \Omega). \end{aligned}$$

Since U_1 and U_2 are open and disjoint, Lemmas 3.5 and 3.7 imply that

$$\begin{aligned} \sigma(E_1 \cup E_2, U_1 \cap \Omega) + \sigma(E_1 \cup E_2, U_2 \cap \Omega) &= \sigma(E_1 \cup E_2, (U_1 \cup U_2) \cap \Omega) \\ &\le \sigma(E_1 \cup E_2, \Omega). \end{aligned}$$

Hence σ is a Borel measure.

For (3.11), let $E \subset X$, $\varepsilon > 0$ and let $G_i \in \mathcal{G}$, $i = 1, 2, \ldots$ be such that $E \cap \Omega \subset G_i$ and that

$$\sigma_\varepsilon(G_i, \Omega) \le P(G_i, \Omega) + \frac{\mu(G_i \cap \Omega)}{\varepsilon} \le \sigma_\varepsilon(E, \Omega) + \frac{1}{i}.$$

Define

$$G(\varepsilon) = \bigcap_{i=1}^{\infty} G_i.$$

Then $E \cap \Omega \subset G(\varepsilon)$ and since each G_i is a Borel set, $G(\varepsilon)$ is a Borel set as well. Thus

$$\sigma_\varepsilon(E, \Omega) \le \sigma_\varepsilon(G(\varepsilon), \Omega) \le \inf_i \sigma_\varepsilon(G_i, \Omega) = \sigma_\varepsilon(E, \Omega). \tag{3.13}$$

Now, for each $k = 1, 2, \ldots$, let $G(1/k)$ be as above (for $\varepsilon = 1/k$), and define

$$G = \bigcap_{k=1}^{\infty} G(1/k).$$

Then $E \cap \Omega \subset G$ and by (3.13),

$$\sigma_{1/k}(G, \Omega) \le \sigma_{1/k}(G(1/k), \Omega) \le \sigma_{1/k}(E, \Omega).$$

Letting $k \to \infty$ and observing that G is a Borel set we arrive at

$$\sigma(E, \Omega) \ge \sigma(G, \Omega) \ge \inf\{\sigma(B, \Omega) : B \text{ Borel}, E \subset B\}.$$

The opposite inequality follows because σ is an outer measure. □

The following observation will be useful for us later.

Lemma 3.14. *Let $\Omega \subset X$ be open and $E \subset \Omega$. Then $\sigma(E, \Omega) = \sigma(E)$. Consequently, for every $F \subset X$, we have $\sigma(F, \Omega) = \sigma(F \cap \Omega)$.*

Proof. The inequality $\sigma(E, \Omega) \leq \sigma(E)$ follows from Lemma 3.7. We define Borel sets Ω_i and Γ_i, $i = 1, 2, \ldots$, by setting

$$\Omega_i = \{x \in X : \operatorname{dist}(x, X \setminus \Omega) > 1/i\},$$

$\Gamma_1 = \Omega_1$, and $\Gamma_i = \Omega_i \setminus \Omega_{i-1}$, $i = 2, 3, \ldots$.

Since the sets Γ_i are disjoint,

$$\Omega = \bigcup_{i=1}^{\infty} \Gamma_i,$$

and since Borel sets are σ-measurable by Theorem 3.10, we have

$$\sigma(E) = \sigma\Big(E \cap \Big(\bigcup_{i=1}^{\infty} \Gamma_i\Big)\Big) = \sum_{i=1}^{\infty} \sigma(E \cap \Gamma_i). \tag{3.15}$$

Moreover, as $E \subset \Omega$ and

$$\sum_{i=1}^{n} \sigma(E \cap \Gamma_i) = \sigma(E \cap \Omega_n)$$

for all $n = 1, 2, \ldots$, (3.15) implies that

$$\sigma(E) = \lim_{n \to \infty} \sigma(E \cap \Omega_n).$$

Since $\operatorname{dist}(E \cap \Omega_n, X \setminus \Omega) \geq 1/n > 0$, we may apply Lemma 3.8 to $E \cap \Omega_n$ and we obtain

$$\sigma(E \cap \Omega_n) = \sigma(E \cap \Omega_n, \Omega) \leq \sigma(E, \Omega).$$

The claim follows by letting $n \to \infty$. The second part of the theorem follows from the first one, since

$$\sigma(F, \Omega) = \sigma(F \cap \Omega, \Omega) = \sigma(F \cap \Omega).$$ □

4. The De Giorgi measure and the Hausdorff measure

In this section, we show that the Hausdorff measure of codimension one and the De Giorgi measure are equivalent.

Let $E \subset X$ and $R > 0$. We define

$$\mathcal{H}_R(E) = \inf\Big\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : r_i \leq R,\ E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \Big\}$$

and

$$\mathcal{H}(E) = \lim_{R \to 0} \mathcal{H}_R(E).$$

The number $\mathcal{H}(E)$, which is possibly infinite, is called the Hausdorff measure of codimension one of E.

Let $E \subset X$. We say that $x \in X$ belongs to the measure theoretic boundary of E, and denote $x \in \partial^* E$, if

$$\overline{D}(E, x) > 0 \qquad \text{and} \qquad \overline{D}(X \setminus E, x) > 0.$$

According to the next result, the total variation measure is concentrated on the measure theoretic boundary. For the proof, we refer to Theorem 5.3 in [2]. See also [3].

Theorem 4.1. *Let E be a set of finite perimeter in X and denote*

$$\Sigma_\gamma = \{x \in X : \min\{\overline{D}(E, x), \overline{D}(X \setminus E, x)\} \geq \gamma\} \subset \partial^* E.$$

Then there is $\gamma > 0$, depending only on the doubling constant and the constants in the weak $(1, 1)$–Poincaré inequality, such that $P(E) = P(E, \Sigma_\gamma)$. Moreover,

$$\mathcal{H}(\partial^* E \setminus \Sigma_\gamma) = 0 \quad \textit{and} \quad \mathcal{H}(\partial^* E) < \infty.$$

We also need the following version of the so-called boxing inequality. For the proof in the metric setting, see [15] and [20].

Theorem 4.2. *Let $E \subset X$ be a set of finite perimeter with $\mu(E) < \infty$, τ the dilation constant in the weak $(1, 1)$–Poincaré inequality and $E_\gamma = \{x \in X : \overline{D}(E, x) > \gamma\}$. Then there exists a collection of disjoint balls $B(x_i, \tau r_i)$, $i = 1, 2, \ldots$, such that*

$$E_\gamma \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i),$$

$$\frac{\gamma}{2c_D} < \frac{\mu(E \cap B(x_i, r_i))}{\mu(B(x_i, r_i))} \leq \frac{\gamma}{2}$$

for $i = 1, 2, \ldots$, and

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \leq c\, P(E).$$

The constant c depends only on the doubling constant c_D, the constants in the weak $(1, 1)$-Poincaré inequality and $\gamma > 0$.

A combination of Theorems 4.1 and 4.2 above gives us the following result.

Corollary 4.3. *Let K be a compact set containing $E \in \mathcal{G}$. Then for every $R > 0$ there exists $\delta_R > 0$, depending on R and K, such that if $\mu(E) < \delta_R$ then*

$$\mathcal{H}_R(E) \leq cP(E).$$

Proof. Fix $R > 0$. We may assume that $P(E) < \infty$. By Theorem 4.1, there exists $\gamma > 0$ such that

$$\mathcal{H}_R(\partial^* E \setminus E_\gamma) \leq \mathcal{H}(\partial^* E \setminus E_\gamma) = 0, \tag{4.4}$$

where E_γ is as in Theorem 4.2. Note that $\Sigma_\gamma \subset E_\gamma$. Let

$$\delta_R = \frac{\gamma}{4c_D} \inf_{x \in K} \mu(B(x, R/5\tau)).$$

Since μ is doubling, $\delta_R > 0$. By Theorem 4.2, there exists a covering $B(x_i, 5\tau r_i)$, $i = 1, 2, \dots$, of E_γ, such that

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \le cP(E)$$

and

$$\mu(B(x_i, r_i)) < \frac{2c_D}{\gamma} \mu(E)$$

for every $i = 1, 2, \dots$. Hence, if $\mu(E) < \delta_R$, we obtain

$$\mu(B(x_i, r_i)) < \frac{1}{2} \inf_{x \in K} \mu(B(x, R/5\tau)) \le \frac{1}{2} \mu(B(x_i, R/5\tau))$$

for every $i = 1, 2, \dots$. Thus $5\tau r_i < R$ for all $i = 1, 2, \dots$ and consequently

$$\mathcal{H}_R(E_\gamma) \le \sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \le cP(E).$$

This with (4.4) completes the proof, since $E \in \mathcal{G}$ implies that $E \subset \partial^* E \cup E_\gamma$ and consequently

$$\mathcal{H}_R(E) \le \mathcal{H}_R(\partial^* E \setminus E_\gamma) + \mathcal{H}_R(E_\gamma) \le cP(E). \qquad \square$$

Now we are ready to prove the main result in this section.

Theorem 4.5. *There exist positive constants c_1 and c_2 such that for any set $E \subset X$, we have*

$$c_1 \mathcal{H}(E) \le \sigma(E) \le c_2 \mathcal{H}(E).$$

Proof. We begin with the second inequality. We may assume that $\mathcal{H}(E) < \infty$. For every $0 < \varepsilon < 1$ and $\eta > 0$, there exist balls $B(x_i, r_i)$, $i = 1, 2, \dots$, such that

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),$$

$r_i \le \varepsilon$ for every $i = 1, 2, \dots$ and

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \le \mathcal{H}(E) + \eta.$$

By the coarea formula, we have

$$\int_{r_i}^{2r_i} P(B(x_i, t))\, dt \le \mu(B(x_i, 2r_i)).$$

Hence for every $i = 1, 2, \dots$ there exists $\widetilde{r}_i$ with $r_i \le \widetilde{r}_i \le 2r_i$ and

$$P(B(x_i, \widetilde{r}_i)) \le c\,\frac{\mu(B(x_i, r_i))}{r_i}.$$

Let

$$B = \bigcup_{i=1}^{\infty} B(x_i, \widetilde{r}_i).$$

Since $\mu(B \triangle \widetilde{B}) = 0$, Lemma 2.4 (i) implies that $P(\widetilde{B}) = P(B)$. Here $\widetilde{B}$ is defined in (3.1). This together with Lemma 2.4 (vii) gives

$$\begin{aligned} P(\widetilde{B}) = P(B) &\le \sum_{i=1}^{\infty} P(B(x_i, \widetilde{r}_i)) \\ &\le c \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \le c(\mathcal{H}(E) + \eta). \end{aligned}$$

In addition, since $r_i \le \varepsilon$, we have

$$\begin{aligned} \mu(\widetilde{B}) = \mu(B) &\le \sum_{i=1}^{\infty} \mu(B(x_i, \widetilde{r}_i)) \le c \sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \\ &\le c\,\varepsilon \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \le c\,\varepsilon(\mathcal{H}(E) + \eta). \end{aligned}$$

Since B is open, $E \subset B \subset \widetilde{B}$. In addition, $\widetilde{B} \in \mathcal{G}$. Hence we may use $\widetilde{B}$ as a test set in the definition of $\sigma_\varepsilon(\cdot, X)$ and we have

$$\begin{aligned} \sigma_\varepsilon(E) &\le P(\widetilde{B}) + \frac{\mu(\widetilde{B})}{\varepsilon} \\ &\le c(\mathcal{H}(E) + \eta) + \frac{1}{\varepsilon}\,\varepsilon(\mathcal{H}(E) + \eta) \le c(\mathcal{H}(E) + \eta). \end{aligned}$$

Letting $\varepsilon \to 0$ and then $\eta \to 0$, we obtain

$$\sigma(E) \le c\mathcal{H}(E).$$

Since $\mathcal{H}$ and σ are Borel regular measures, it is enough to prove the first inequality for bounded sets E. We may also assume that $\sigma(E) < \infty$. By the definition of $\sigma(E)$, for every $\varepsilon > 0$ and $\eta > 0$ there exists a set $G \in \mathcal{G}$ such that $E \subset G$ and

$$P(G) + \frac{\mu(G)}{\varepsilon} \le \sigma(E) + \eta. \tag{4.6}$$

Let us fix $\eta > 0$ and $R > 0$ and set $\varepsilon = \delta_R(\sigma(E) + \eta)^{-1}$, where δ_R is as in Corollary 4.3. Thus by (4.6) and by the choice of ε, we have

$$\mu(G) \le \varepsilon(\sigma(E) + \eta) = \delta_R.$$

Consequently, we can apply Corollary 4.3 and (4.6) to conclude

$$\mathcal{H}_R(E) \le \mathcal{H}_R(G) \le cP(G) \le c(\sigma(E) + \eta).$$

Since R and η are arbitrary, it follows that

$$\mathcal{H}(E) \le c\sigma(E). \qquad \square$$

5. Existence of solution for a relaxed problem

Let $E, F \subset X$ be disjoint Borel sets. We are interested in finding a set with minimal perimeter separating E and F. As the main result in this section, we show that there exists a set $G_0 \in \mathcal{G}$ such that

$$I(G_0, \Omega, E, F) = \inf_{G \in \mathcal{G}} I(G, \Omega, E, F),$$

where

$$I(G, \Omega, E, F) = P(G, \Omega) + \sigma((E \setminus G) \cap \Omega) + \sigma((F \cap G) \cap \Omega).$$

Example 5.1. Let Ω, E and F be as in Example 2.11. Then the minimizing set $G_0 = \emptyset$ and

$$\inf_{G \in \mathcal{G}} I(G, \Omega, E, F) = 4.$$

The proof of the existence result is based on Proposition 5.9, which is a lower semicontinuity result. First we give two preliminary results.

Lemma 5.2. *Let $\Omega \subset X$ be open and $E \subset X$ be a Borel set. Suppose that $G \in \mathcal{G}$ satisfies*

$$D(G, x) = 0 \quad \textit{for every } x \in E \setminus G.$$

If $G_i \in \mathcal{G}$, $i = 1, 2, \dots$, are such that $G_i \to G$ in $L^1_{\mathrm{loc}}(\Omega)$ as $i \to \infty$, then

$$\sigma((E \setminus G) \cap \Omega) \le \liminf_{i \to \infty} \big(P(G_i, \Omega) + \sigma((E \setminus G_i) \cap \Omega)\big) + P(G, \Omega).$$

Remark 5.3. In the collection of sets $G' \in \mathcal{G}$ with $\mu(G' \triangle G) = 0$ the set G satisfying $D(G, x) = 0$ for every $x \in E \setminus G$ has the largest possible intersection with E. Indeed, let G' be such a set and let $x \in G' \cap E$. As $G' \in \mathcal{G}$ and $\mu(G' \triangle G) = 0$, we have $\overline{D}(G, x) = \overline{D}(G', x) > 0$. Since G has no density points in $E \setminus G$ and $x \in E$, we have that $x \in G \cap E$. Hence $G' \cap E \subset G \cap E$.

Proof. Let $\Omega' \Subset \Omega$ and $\varepsilon > 0$. By Theorem 3.4, σ_ε is an outer measure and therefore

$$\sigma_\varepsilon(E \setminus G, \Omega') \le \sigma_\varepsilon(E \setminus G_i, \Omega') + \sigma_\varepsilon((G_i \setminus G) \cap E, \Omega'). \tag{5.4}$$

We estimate the second term on the right-hand side.

First we claim that $(G_i \setminus G) \cap E \subset \widetilde{G_i \setminus G}$. Let $x_0 \in (G_i \setminus G) \cap E$. Then $x_0 \in E \setminus G$ and by assumption $\overline{D}(G, x_0) = 0$. As $G_i \in \mathcal{G}$ and $x_0 \in G_i$ we have $\overline{D}(G_i, x_0) > 0$. A combination of these facts implies that $\overline{D}(G_i \setminus G, x_0) > 0$, and hence $x_0 \in \widetilde{G_i \setminus G}$. This proves the first claim.

Since
$$\mu\big((\widetilde{G_i \setminus G})\triangle(G_i \setminus G)\big) = 0,$$
Lemma 2.4 (i) and (iv) imply that
$$P(\widetilde{G_i \setminus G}, \Omega') = P(G_i \setminus G, \Omega') \le P(G_i, \Omega') + P(G, \Omega'). \tag{5.5}$$
Moreover, we have
$$\mu((\widetilde{G_i \setminus G}) \cap \Omega') = \mu((G_i \setminus G) \cap \Omega').$$
Since $(G_i \setminus G) \cap E \subset \widetilde{G_i \setminus G}$ and $\widetilde{G_i \setminus G} \in \mathcal{G}$, it follows from the definition of σ_ε and (5.5) that
$$\begin{aligned}\sigma_\varepsilon((G_i \setminus G) \cap E, \Omega') &\le P(\widetilde{G_i \setminus G}, \Omega') + \frac{\mu((\widetilde{G_i \setminus G}) \cap \Omega')}{\varepsilon} \\ &\le P(G_i, \Omega') + P(G, \Omega') + \frac{\mu((G_i \setminus G) \cap \Omega')}{\varepsilon}.\end{aligned} \tag{5.6}$$
By (5.4), (5.6) and Lemma 3.7, we conclude that
$$\sigma_\varepsilon(E \setminus G, \Omega') \le \sigma(E \setminus G_i, \Omega) + P(G_i, \Omega) + P(G, \Omega) + \frac{\mu((G_i \setminus G) \cap \Omega')}{\varepsilon}.$$
Since $G_i \to G$ in $L^1_{\mathrm{loc}}(\Omega)$, we have $\mu((G_i \setminus G) \cap \Omega') \to 0$ as $i \to \infty$. Letting first $i \to \infty$ and then $\varepsilon \to 0$, we obtain by Lemma 3.14 that
$$\begin{aligned}\sigma((E \setminus G) \cap \Omega') &= \sigma(E \setminus G, \Omega') \\ &\le \liminf_{i\to\infty}(\sigma((E \setminus G_i) \cap \Omega) + P(G_i, \Omega)) + P(G, \Omega).\end{aligned}$$
The claim follows by exhausting Ω with an increasing sequence of open sets $\Omega' \Subset \Omega$. $\square$

The proof of the following result is similar to the proof of the previous lemma and we leave it for the interested reader.

Lemma 5.7. *Let $\Omega \subset X$ be open and $F \subset X$ be a Borel set. Suppose that $G \in \mathcal{G}$ satisfies*
$$D(X \setminus G, x) = 0 \quad \textit{for every } x \in F \cap G.$$
If $G_i \in \mathcal{G}$, $i = 1, 2, \dots,$ are such that $G_i \to G$ in $L^1_{\mathrm{loc}}(\Omega)$ as $i \to \infty$, then
$$\sigma((F \cap G) \cap \Omega) \le \liminf_{i\to\infty}\big(P(G_i, \Omega) + \sigma((F \cap G_i) \cap \Omega)\big) + P(G, \Omega).$$

Remark 5.8. In the collection of sets $G' \in \mathcal{G}$ with $\mu(G'\triangle G) = 0$ the set G satisfying $D(X \setminus G, x) = 0$ for every $x \in F \cap G$ has the smallest possible intersection with F. Indeed, let G' be such a set and let $x \in F \setminus G'$. Since $G' \in \mathcal{G}$ and $\mu(G'\triangle G) = 0$, we obtain $\overline{D}(X \setminus G, x) = \overline{D}(X \setminus G', x) > 0$. As $x \in F$, we have $x \in F \setminus G$ and hence $F \setminus G' \subset F \setminus G$.

Now we are ready to prove the lower semicontinuity of $I(\cdot, \Omega, E, F)$. The next result is a metric space version of Theorem 3.2 on page 144 of [11].

Proposition 5.9. *Let $\Omega \subset X$ be an open set and assume that E and F are Borel sets for which $E \cap F = \emptyset$. Suppose that $G \in \mathcal{G}$ satisfies*

$$D(G, x) = 0 \quad \textit{for every } x \in E \setminus G$$

and

$$D(X \setminus G, x) = 0 \quad \textit{for every } x \in G \cap F.$$

If $G_i \in \mathcal{G}$, $i = 1, 2, \dots,$ are such that $G_i \to G$ in $L^1_{\mathrm{loc}}(\Omega)$ as $i \to \infty$, then

$$I(G, \Omega, E, F) \le \liminf_{i \to \infty} I(G_i, \Omega, E, F). \tag{5.10}$$

Proof. We may assume that the right–hand side of (5.10) is finite. Thus

$$\liminf_{i \to \infty} (P(G_i, \Omega) + \sigma((E \setminus G_i) \cap \Omega) + \sigma((F \cap G_i) \cap \Omega)) < \infty.$$

By Lemma 2.7, we have

$$P(G, \Omega) \le \liminf_{i \to \infty} P(G_i, \Omega) < \infty.$$

By Theorem 4.1, the measure $P(G, \cdot)$ is concentrated on $\partial^* G$. Fix $\varepsilon > 0$. There exists a compact set $K_1 \subset \partial^* G \cap \Omega$ such that

$$P(G, \Omega) \le P(G, K_1) + \varepsilon. \tag{5.11}$$

By Borel regularity of σ, see Theorem 3.10 and Remark 3.12, there exist compact sets $K_2 \subset (E \setminus G) \cap \Omega$ and $K_3 \subset (F \cap G) \cap \Omega$ such that

$$\sigma((E \setminus G) \cap \Omega) \le \sigma(K_2) + \varepsilon \tag{5.12}$$

and

$$\sigma((F \cap G) \cap \Omega) \le \sigma(K_3) + \varepsilon. \tag{5.13}$$

By the assumptions the sets $\partial^* G$, $E \setminus G$ and $F \cap G$ are disjoint. Thus there exist disjoint open sets $\Omega_i \subset \Omega$ such that $K_i \subset \Omega_i$, $i = 1, 2, 3$. Inequality (5.11) implies that

$$P(G, \Omega_2) \le \varepsilon \quad \text{and} \quad P(G, \Omega_3) \le \varepsilon.$$

We use estimates (5.11), (5.12), (5.13) and Lemma 3.14 and apply Lemmas 2.7, 5.2 and 5.7in Ω_1, Ω_2 and Ω_3, respectively, and obtain

$$\begin{aligned}
&P(G,\Omega)+\sigma((E\setminus G)\cap\Omega)+\sigma((F\cap G)\cap\Omega)\\
&\le P(G,\Omega_1)+\sigma((E\setminus G)\cap\Omega_2)+\sigma((F\cap G)\cap\Omega_3)+3\varepsilon\\
&\le \liminf_{i\to\infty} P(G_i,\Omega_1)\\
&\quad+\liminf_{i\to\infty}(\sigma((E\setminus G_i)\cap\Omega_2)+P(G_i,\Omega_2))+P(G,\Omega_2)\\
&\quad+\liminf_{i\to\infty}(\sigma((F\cap G_i)\cap\Omega_3)+P(G_i,\Omega_3))+P(G,\Omega_3)+3\varepsilon\\
&\le \liminf_{i\to\infty}(P(G_i,\Omega_1\cup\Omega_2\cup\Omega_3)\\
&\quad+\sigma((E\setminus G_i)\cap\Omega)+\sigma((F\cap G_i)\cap\Omega))+P(G,\Omega_2\cup\Omega_3)+3\varepsilon\\
&\le \liminf_{i\to\infty} I(G_i,\Omega,E,F)+5\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, this completes the proof. □

As a corollary of the above result, we obtain the existence of a minimizing set.

Theorem 5.14. *Let $\Omega\subset X$ be open. Let E and F be Borel sets in X such that $E\cap F=\emptyset$. Then there exists $G_0\in\mathcal{G}$ such that*

$$I(G_0,\Omega,E,F)=\inf\{I(G,\Omega,E,F):\,G\in\mathcal{G}\}.$$

Proof. Let $G\in\mathcal{G}$ be such that $I(G,\Omega,E,F)<\infty$. We define

$$\begin{aligned}
G'&=\{x\in X:\underline{D}(G,x)=1\},\\
G''&=\{x\in X:\overline{D}(G,x)>0\},
\end{aligned}$$

and

$$G^*=(G\cup(E\cap G''))\setminus(F\setminus G').$$

Then $G^*\in\mathcal{G}$ and G^* satisfies the assumptions of Proposition 5.9. Since $\mu(G^*\triangle G)=0$, by Theorem 2.4 (i) we have $P(G^*,\Omega)=P(G,\Omega)$. Moreover, $E\setminus G^*\subset E\setminus G$ and $F\cap G^*\subset F\cap G$. Therefore it follows that

$$I(G^*,\Omega,E,F)\le I(G,\Omega,E,F).$$

Now by Lemma 2.6, we can find $G_i\in\mathcal{G}$, $i=1,2,\dots$, such that $G_i\to G_0$ in $L^1_{\mathrm{loc}}(\Omega)$ for some $G_0\in\mathcal{G}$ and

$$\lim_{i\to\infty} I(G_i,\Omega,E,F)=\inf\{I(G,\Omega,E,F):\,G\in\mathcal{G}\}.$$

By the reasoning above, we can replace the set G_0 by G_0^* without increasing the limit, and the result follows by Proposition 5.9. □

Remark 5.15. If G_0 is a minimizing set and $I(G_0,\Omega,E,F)<\infty$, then

$$\sigma((E\setminus G_0)\cap\Omega)<\infty\quad\text{and}\quad\sigma((G_0\cap F)\cap\Omega)<\infty.$$

It follows easily from the definition of σ that sets with finite σ-measure are of μ-measure zero and, in particular, that

$$\mu((E \setminus G_0) \cap \Omega) = 0 \quad \text{and} \quad \mu((G_0 \cap F) \cap \Omega) = 0.$$

It is natural to ask whether the minimizing set G_0 of the above theorem is also minimal with respect to itself; that is, whether the minimizing process is a stable process. The next lemma gives an affirmative answer.

Lemma 5.16. *Let $\Omega \subset X$ be open. Let E and F be Borel sets in X such that $E \cap F = \emptyset$ and let $G_0 \in \mathcal{G}$ be a minimizing set given by Theorem 5.14. Then*

$$I(G_0, \Omega, G_0, F) = \inf\{I(G, \Omega, G_0, F) : G \in \mathcal{G}\}$$

and

$$I(G_0, \Omega, G_0 \cup E, F) = \inf\{I(G, \Omega, G_0 \cup E, F) : G \in \mathcal{G}\}.$$

Moreover, the infimums are same, that is

$$I(G_0, \Omega, G_0 \cup E, F) = I(G_0, \Omega, E, F).$$

Proof. First note that for every $G \in \mathcal{G}$, $E_1 \subset E_2$ and F, we have

$$I(G, \Omega, E_1, F) \leq I(G, \Omega, E_2, F).$$

By this observation and the definition of I, we have for every $G \in \mathcal{G}$

$$\begin{aligned} I(G, \Omega, G_0, F) &\geq I(G, \Omega, G_0 \cap E, F) \\ &\geq I(G, \Omega, E, F) - \sigma((E \setminus G_0) \cap \Omega) \\ &\geq I(G_0, \Omega, E, F) - \sigma((E \setminus G_0) \cap \Omega) = I(G_0, \Omega, G_0, F). \end{aligned}$$

This proves the first claim. Since $E \setminus G_0 = (E \cup G_0) \setminus G_0$, we also see that

$$\begin{aligned} I(G, \Omega, G_0 \cup E, F) &\geq I(G, \Omega, E, F) \\ &\geq I(G_0, \Omega, E, F) = I(G_0, \Omega, G_0 \cup E, F). \end{aligned}$$

Here we also used the last assertion, which holds since

$$\begin{aligned} I(G_0, \Omega, G_0 \cup E, F) &= P(G_0, \Omega) + \sigma(E \setminus G_0) + \sigma(F \cap G_0) \\ &= I(G_0, \Omega, E, F). \end{aligned}$$

This completes the proof. $\square$

The following result gives relations between three different obstacle problems. The first and third problems involve only the familiar perimeter measure of the competing sets G, whereas the second problem is the one studied above and it involves the De Giorgi measure in addition to the perimeter measure.

Proposition 5.17. *Let $\Omega \subset X$ be open, and let E and F be disjoint Borel sets in X and denote*

$$\lambda = \min\{P(G,\Omega) : G \text{ Borel set, } \Omega \cap E \subset G,\, G \cap F \cap \Omega = \emptyset\},$$

$$\gamma = \min\{I(G,\Omega,E,F) : G \in \mathcal{G}\} \text{ and}$$

$$\nu = \inf\{P(G,\Omega) : G \in \mathcal{G},\, E \cap \Omega \subset G, G \cap F \cap \Omega = \emptyset\}.$$

Then $\lambda \leq \gamma \leq \nu$. If, in addition,

$$\overline{D}(E \cap \Omega, x) > 0 \quad \text{for every } x \in E \cap \Omega$$

and

$$\overline{D}(F \cap \Omega, x) > 0 \quad \text{for every } x \in F \cap \Omega,$$

then $\lambda = \gamma = \nu$.

Proof. First we show that $\lambda \leq \gamma$. Without loss of generality we may assume that $\gamma < \infty$. By Theorem 5.14 there is $G_0 \in \mathcal{G}$ such that

$$P(G_0,\Omega) + \sigma(E \setminus G_0,\Omega) + \sigma(G_0 \cap F,\Omega) = \gamma.$$

It follows from Remark 5.15 that

$$\mu((E \setminus G_0) \cap \Omega) = 0 \quad \text{and} \quad \mu((G_0 \cap F) \cap \Omega) = 0.$$

The set

$$G_1 = (G_0 \cup (E \setminus G_0)) \setminus (G_0 \cap F)$$

is a Borel set with $E \subset G_1$ and $G_1 \cap F = \emptyset$. Since $\mu(G_0 \triangle G_1) = 0$, by Lemma 2.4 (i) we have

$$\lambda \leq P(G_1,\Omega) = P(G_0,\Omega) \leq \gamma.$$

Let us then show that $\gamma \leq \nu$. For every $G \in \mathcal{G}$ with $E \cap \Omega \subset G$ and $G \cap (F \cap \Omega) = \emptyset$, we have

$$I(G,\Omega,E,F) = P(G,\Omega)$$

since $\sigma((E \setminus G) \cap \Omega) = \sigma((F \cap G) \cap \Omega) = 0$. The claim follows.

Finally, assume that

$$\overline{D}(E \cap \Omega, x) > 0 \quad \text{for every } x \in E \cap \Omega$$

and

$$\overline{D}(F \cap \Omega, x) > 0 \quad \text{for every } x \in F \cap \Omega.$$

By Theorem 2.10, there is a Borel set G_0 such that $E \cap \Omega \subset G_0$, $F \cap \Omega \cap G_0 = \emptyset$ and $P(G_0,\Omega) = \lambda$. Then

$$E \cap \Omega \subset \widetilde{E \cap \Omega} \subset \widetilde{G}_0.$$

Similarly, we have

$$F \cap \Omega \subset \widetilde{F \cap \Omega} \subset \widetilde{X \setminus G_0} = X \setminus \widetilde{G}_0.$$

Thus $\widetilde{G}_0 \cap F \cap \Omega = \emptyset$ and it follows that

$$\nu \leq P(\widetilde{G}_0,\Omega) = P(G_0,\Omega) = \lambda. \qquad \square$$

6. Comparison of obstacle problems and capacities

In this section, we explore the relationship between relaxed obstacle problems, the variational capacity of order one and the BV-capacity. Recall that the Hausdorff measure of codimension one and the De Giorgi measure are equivalent. In [15] it was shown that if E is a compact subset of X, then the variational capacity of E is a geometric object in the following sense:

$$\operatorname{cap}_1(E) \approx \inf\{P(U,X)\,:\, E \subset U \text{ open}, \mu(U) < \infty\} \approx \mathcal{H}_\infty(E).$$

Here, the variational 1-capacity is defined as

$$\operatorname{cap}_1(E) = \inf \int_X g\, d\mu,$$

where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u = 1$ on E, and all upper gradients g of u. While this result does indicate that the variational 1-capacity is a geometric measure theoretic concept, it has two drawbacks; in general, the infimum above is not a minimum (see Example 2.11), and if the metric measure space X is 1-parabolic (see [15] for this concept) the quantity $\operatorname{cap}_1(E) = 0$ for all E and so does not impart useful geometric information. Here we consider a relative capacity that addresses the two concerns mentioned above.

Let B be a ball in X. For $E \subset B$, we define the BV capacity as

$$\operatorname{cap}_{BV}(E,B) = \inf \|Du\|(X),$$

where the infimum is taken over all $u \in BV(X)$ such that $u = 1$ in a neighbourhood of E and $u = 0$ in $X \setminus \overline{B}$. We set the 1-capacity $\operatorname{cap}_1(E,B)$ in an analogous manner by taking the infimum over all $u \in N^{1,1}(X)$ such that $u = 1$ in E and $u = 0$ in $X \setminus \overline{B}$. It turns out that the BV-capacity $\operatorname{cap}_{BV}(E,B)$ is *the same as* the infimum in three modifications of the obstacle problem studied in the previous section. The three obstacle problems, studied in Proposition 6.3, give rise to three apparently different quantities λ_0, ν_0, and γ_0, but the main theorem of this section, Theorem 6.1, relates all these quantities. To obtain this relationship we need to modify the relaxed obstacle problem developed in the previous section, as follows.

Let $B \subset X$ be a ball, and let $E \subset B$ and $F = X\setminus\overline{B}$ with $\operatorname{dist}(\overline{E},\overline{F}) > 0$ and denote

$$\begin{aligned}
\lambda_0 &= \inf\{P(G) : G \in \mathcal{G}',\, E \subset \operatorname{int} G,\, G \cap F = \emptyset\},\\
\gamma_0 &= \inf\{I(G,X,E,F) :\, G \in \mathcal{G},\, E \subset \operatorname{int} G\} \text{ and}\\
\nu_0 &= \inf\{P(G) :\, G \in \mathcal{G},\, E \subset \operatorname{int} G, G \cap F = \emptyset\}.
\end{aligned}$$

In Proposition 6.3 more general sets E, F will be considered.

Theorem 6.1. *Let $B = B(x,r)$ be a ball in X, $\overline{E} \subset B$, $F = X \setminus \overline{B}$ and $\Omega = X$. Then $\operatorname{cap}_{BV}(E,B) = \nu_0 = \lambda_0 = \gamma_0$, where the quantities λ_0, ν_0, and γ_0 are as above.*

In general $\operatorname{cap}_{BV}(E,B)$ and the variational 1-capacity $\operatorname{cap}_1(E,B)$ need not be equivalent. However, if in addition E is compact, then by the results in [15], $\operatorname{cap}_1(E,B) \approx \nu_0$.

Example 6.2. Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric but the measure μ given by $d\mu(x) = (2 - \chi_Q(x))\,dx$, where Q is the closed unit square centered at the origin and dx is the Lebesgue measure on $\mathbb{R}^2$. Then with $E = Q$ and $B = B(0,2)$, we see that

$$\operatorname{cap}_{BV}(E,B) = \nu_0 = 8,$$

whereas, we can find a sequence of $G_i \in \mathcal{G}$ with $E \subset \operatorname{int} G_i$ and $G_i \subset \overline{B}$ such that $\chi_{G_i} \to \chi_E$ in $L^1(X)$,

$$I(G_i, X, E, X \setminus \overline{B}) \to \nu_0,$$

and $E \in \mathcal{G}$, but

$$I(E, X, E, X \setminus \overline{B}) = P(E) = 4.$$

Thus in general we cannot expect the limiting set G_0 obtained from a minimizing sequence for ν_0 to satisfy $I(G_0, X, E, X \setminus \overline{B}) = \nu_0$.

The class $\mathcal{G}'$ denotes the collection of all Borel sets E that satisfy

$$\overline{D}(E,x) > 0 \quad \text{for every } x \in E.$$

Note that if F is an open set, then $F \in \mathcal{G}'$. In application to Theorem 6.1, we will take E to be a subset of a ball B and F to be the open set $X \setminus \overline{B}$.

Proposition 6.3. *Let $\Omega \subset X$ be open, and let E and F be Borel sets in X with $\operatorname{dist}(\overline{E}, \overline{F}) > 0$ and denote*

$$\lambda_0 = \inf\{P(G,\Omega) : G \in \mathcal{G}',\, \Omega \cap E \subset \operatorname{int} G,\, G \cap F \cap \Omega = \emptyset\},$$
$$\gamma_0 = \inf\{I(G,\Omega,E,F) :\, G \in \mathcal{G},\, \Omega \cap E \subset \operatorname{int} G\} \text{ and}$$
$$\nu_0 = \inf\{P(G,\Omega) :\, G \in \mathcal{G},\, E \cap \Omega \subset \operatorname{int} G, G \cap F \cap \Omega = \emptyset\}.$$

Then $\lambda_0 \le \gamma_0 \le \nu_0$. If, in addition $F \in \mathcal{G}'$, then $\lambda_0 = \gamma_0 = \nu_0$.

Proof. First we show that $\lambda_0 \le \gamma_0$. If $\gamma_0 = \infty$, the claim is obvious. Hence we may assume that $\gamma_0 < \infty$. Let $\varepsilon > 0$ and let $G_1 \in \mathcal{G}$ be a set with $\Omega \cap E \subset \operatorname{int} G_1$ and

$$I(G_1, \Omega, E, F) \le \gamma_0 + \varepsilon.$$

This implies that $\sigma(F \cap G_1 \cap \Omega) < \infty$ and consequently $\mu(F \cap G_1 \cap \Omega) = 0$. Therefore,

$$D(F \cap G_1 \cap \Omega, x) = 0 \quad \text{for every } x \in \Omega.$$

Let

$$G_2 = G_1 \setminus (F \cap \Omega).$$

Since $G_1 \in \mathcal{G} \subset \mathcal{G}'$, by the comment above we see that $G_2 \in \mathcal{G}'$. Moreover, $\Omega \cap E \subset \operatorname{int} G_2$ and $F \cap G_2 \cap \Omega = \emptyset$. Thus

$$\lambda_0 \le P(G_2, \Omega) = P(G_1, \Omega) \le I(G_1, \Omega, E, F) \le \gamma_0 + \varepsilon.$$

Here we also used the fact that $\mu(G_1 \triangle G_2) = 0$ and Lemma 2.4 (i). Letting $\varepsilon \to 0$, we obtain $\lambda_0 \le \gamma_0$.

Next we claim that $\gamma_0 \le \nu_0$. If $G \in \mathcal{G}$ is a set for which $E \cap \Omega \subset \operatorname{int} G$ and $F \cap G \cap \Omega = \emptyset$, then

$$I(G, \Omega, E, F) = P(G, \Omega).$$

The claim follows.

Finally, we show that if $F \in \mathcal{G}'$ then $\nu_0 \le \lambda_0$. Let $\varepsilon > 0$ and $G \in \mathcal{G}'$ be a set for which

$$E \cap \Omega \subset \operatorname{int} G, \quad F \cap G \cap \Omega = \emptyset \quad \text{and} \quad P(G, \Omega) \le \lambda_0 + \varepsilon.$$

We denote

$$G' = \{x \in \Omega : \underline{D}(G, x) = 1\}.$$

By the differentiation theory for measures we have $\mu(G' \setminus G) = 0$. Let

$$G_1 = G \cup G'.$$

As $\mu(G' \triangle G_1) = 0$, by Lemma 2.4 (i) we have $P(G, \Omega) = P(G_1, \Omega)$. Moreover, since $G \subset G_1$ and $G_1 \in \mathcal{G}$, we have $E \cap \Omega \subset \operatorname{int} G_1$. Observe that $F \cap G \cap \Omega = \emptyset$. As

$$\overline{D}(F, x) > 0 \quad \text{for every } x \in F \cap \Omega,$$

we see that $F \cap G' = \emptyset$. Thus $G_1 \in \mathcal{G}$, $F \cap G_1 \cap \Omega = \emptyset$ and $E \cap \Omega \subset \operatorname{int} G_1$. Consequently

$$\nu_0 \le P(G_1, \Omega) = P(G, \Omega) \le \lambda_0 + \varepsilon.$$

Letting $\varepsilon \to 0$ we see that $\nu_0 \le \lambda_0$. □

Proof of Theorem 6.1. First we show that $\nu_0 \le \operatorname{cap}_{BV}(E, B)$. Let $\varepsilon > 0$. Take $u \in BV(X)$ such that $0 \le u \le 1$ in X, $u \ge 1$ in a neighbourhood of E, $u = 0$ in F and

$$\|Du\|(2B) \le \operatorname{cap}_{BV}(E, B) + \varepsilon.$$

By the coarea formula

$$\int_0^1 P(\{x \in X : u(x) > t\}, 2B)\, dt = \|Du\|(2B).$$

Hence there exists t with $0 < t < 1$ such that

$$P(\{x \in X : u(x) > t\}, 2B) \le \|Du\|(2B).$$

Let $A = \{x \in X : u(x) > t\}$, and let $\widetilde{A}$ be the corresponding set given by equation (3.1). Then $A \subset B$ and hence $A \cap F = \emptyset$ and

$$E \subset \operatorname{int}\{x \in X : u(x) = 1\} \subset A.$$

Since $\underline{D}(A, x) = 0$ for every $x \in F$, we have $\widetilde{A} \cap F \cap 2B = \emptyset$. Thus

$$E = E \cap \Omega \subset \operatorname{int} \widetilde{A}, \quad \widetilde{A} \in \mathcal{G} \quad \text{and} \quad \widetilde{A} \cap F \cap 2B = \emptyset$$

with $P(\widetilde{A}, 2B) = P(A, 2B)$. This implies that

$$\nu_0 \leq P(\widetilde{A}, 2B) \leq \|Du\|(2B) \leq \operatorname{cap}_{BV}(E, B) + \varepsilon.$$

Letting $\varepsilon \to 0$ we arrive at $\nu_0 \leq \operatorname{cap}_{BV}(E, B)$.

In order to see the other inequality, let $G \in \mathcal{G}$ such that $E \cap \Omega \subset \operatorname{int} G$, $G \cap F \cap \Omega = \emptyset$ and

$$P(G, 2B) \leq \nu_0 + \varepsilon.$$

If $\nu_0 = \infty$, there is nothing to prove, so we suppose that ν_0 is finite. Then we have that $\chi_G \in BV(X)$ with $G \subset \overline{B}$. Since $\chi_G = 1$ in a neighbourhood of E, it follows immediately that

$$\operatorname{cap}_{BV}(E, B) \leq \|D\chi_G\|(X) = P(G, 2B) \leq \nu_0 + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\operatorname{cap}_{BV}(E, B) \leq \nu_0$. $\square$

As stated above, if E is a compact set, then $\operatorname{cap}_1(E, B) \approx \nu_0$. We also obtain a better comparison result as follows. Observe that since E is compact and is contained in the interior of the test set $G \in \mathcal{G}$ used in the computation of ν_0, there exists $\delta > 0$ such that the 3δ-neighborhood of E, denoted $E_{3\delta}$, is contained in G.

Let v_i, $i = 1, 2, \dots$, be a sequence in $N^{1,1}(X)$ such that $0 \leq v_i \leq 1$, $v_i \to \chi_G$ in $L^1(X)$ and

$$\lim_{i \to \infty} \int_X g_{v_i}\, d\mu = P(G, 2B) = P(G, X).$$

Since $v_i \to \chi_G$ only in $L^1(X)$, we do not know that $v_i = 1$ on E. Therefore we need to modify v_i to obtain a function in $N^{1,1}(X)$ that takes on the value 1 on E. To this end, let η be a Lipschitz function such that $0 \leq \eta \leq 1$, $\eta = 1$ on the set E_δ, and $\eta = 0$ on $X \setminus E_{2\delta}$, and set

$$u_i = \eta + (1 - \eta) v_i.$$

Then $u_i = 1$ on E_δ and $u_i = (1 - v_i)\eta + v_i$ on $X \setminus E_\delta$, and it follows that the minimal upper gradient $g_{u_i} = g_i$ satisfies

$$g_i \leq (1 - v_i) g_\eta + (1 + \eta) g_{v_i}$$

on $E_{2\delta} \setminus E_\delta$, $g_i = 0$ on E_δ, and $g_i = g_{v_i}$ on $X \setminus E_{2\delta}$. Furthermore, it can be seen that $u_i \to \chi_G$ in $L^1(X)$. Since $\partial G \subset X \setminus E_{3\delta}$, by Remark 3.2 in [19],

$$P(G, X \setminus E_{2\delta}) = P(G, X).$$

It follows from the lower semicontinuity, Lemma 2.7, that

$$P(G, X \setminus E_{2\delta}) \le \liminf_{i\to\infty} \int_{X\setminus E_{2\delta}} g_{v_i}\, d\mu \le \lim_{i\to\infty} \int_X g_{v_i}\, d\mu = P(G, X),$$

and therefore

$$\liminf_{i\to\infty} \int_{E_{2\delta}} g_{v_i}\, d\mu = 0.$$

By passing to a subsequence of $(v_i)_{i=1}^\infty$ if necessary, we may assume that

$$\lim_{i\to\infty} \int_{E_{2\delta}} g_{v_i}\, d\mu = 0.$$

A similar truncation allows us to assume also that for a fixed $\varepsilon > 0$, $v_i = 0$ on $X \setminus (1+\varepsilon)B$. Observe that

$$(1 - v_i) g_\eta \to 0 \quad \text{in} \quad L^1(X),$$

and so

$$\begin{aligned}
\mathrm{cap}_1(E, (1+\varepsilon)B) &\le \liminf_{i\to\infty} \int_X g_i\, d\mu \\
&\le \limsup_{i\to\infty} \int_{E_{2\delta}} g_i\, d\mu + \limsup_{i\to\infty} \int_{X\setminus E_{2\delta}} g_{v_i}\, d\mu \\
&\le P(G, X) + \limsup_{i\to\infty} \int_{E_{2\delta}\setminus E_\delta} (2g_{v_i} + g_\eta(1 - v_i))\, d\mu \\
&= P(G, X),
\end{aligned}$$

where we used the fact that g_η is bounded. From this it follows that

$$\mathrm{cap}_1(E, (1+\varepsilon)B) \le \mathrm{cap}_{BV}(E, B) = \nu_0 \le \mathrm{cap}_1(E, B).$$

As an easy corollary to Theorem 6.1 we obtain the following result connecting BV-capacity to Hausdorff measure.

Corollary 6.4. *Let $E \subset B$ with $\mathrm{cap}_{BV}(E, B) > 0$. Then there exists $G \in \mathcal{G}$ such that $E \subset \mathrm{int}\, G$ and*

$$\mathcal{H}(\partial^* G) \approx \mathrm{cap}_{BV}(E, B).$$

Proof. By Theorem 6.1, since $\mathrm{cap}_{BV}(E, B) > 0$, there exists $G \in \mathcal{G}$ such that $E \subset \mathrm{int}\, G$, $G \subset \overline{B}$, and $P(G) \le 2\, \mathrm{cap}_{BV}(E, B)$. Furthermore, with $u = \chi_G$ we have $u = 1$ on a neighborhood of E and $u = 0$ on $X \setminus \overline{B}$. Hence

$$\mathrm{cap}_{BV}(E, B) \le \|D\chi_G\|(X) = P(G),$$

and therefore

$$\mathrm{cap}_{BV}(E, B) \le P(G) \le 2\ \mathrm{cap}_{BV}(E, B).$$

The claim now follows from Theorem 4.1. $\square$

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