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A characterization of BMO self-maps of a metric measure space

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Abstract This paper studies functions of bounded mean oscillation (BMO) on metric spaces equipped with a doubling measure. The main result gives characterizations for mappings that preserve BMO. This extends the corresponding Euclidean results by Gotoh to metric measure spaces. The argument is based on a generalization Uchiyama's construction of certain extremal BMO-functions and John-Nirenberg's lemma.

Keywords Bounded mean oscillation · Doubling condition · John–Nirenberg lemma · Analysis on metric measure spaces

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1 Introduction

Let X be a complete metric space equipped with a metric d and a Borel regular outer measure μ satisfying the doubling condition. A locally integrable function $f : X \to \mathbb{R}$ is of bounded mean oscillation, denoted as $f \in \mathrm{BMO}(X)$, if

$$\|f\|_* = \sup \fint_B |f - f_B|\, d\mu < \infty,$$

where the supremum is taken over all balls $B \subset X$. We discuss invariance properties of BMO-functions. More precisely, we extend a characterization of Gotoh [7,8] of mappings that preserve BMO to the metric setting. A μ-measurable map $F\colon X \to X$ is a BMO-map if $F^{-1}(E)$ is a μ-null set for each μ-null set $E \subset X$, and for every $f \in \mathrm{BMO}(X)$ the composed map $C_F(f) = f \circ F$ is in $\mathrm{BMO}(X)$. The first condition guarantees the uniqueness of the BMO-map. Moreover, the composition operator C_F is a bounded operator from $\mathrm{BMO}(X)$ to $\mathrm{BMO}(X)$.

The class of BMO-functions is used, for example, in harmonic analysis, partial differential equations and quasiconformal mappings. Indeed, the first invariance property for BMO-functions was obtained by Reimann [16], where he showed that a homeomorphism is a BMO-map if and only if it is quasiconformal, provided the homeomorphism is assumed to be differentiable almost everywhere. Later Astala showed in [1] that the differentiability assumption is superfluous for a suitably localized result. The advantage of the approach by Gotoh [7] is that it applies to general measurable functions and hence is more suitable to extensions to the metric setting. The Euclidean theory for BMO-functions is well understood, but not so much in a general metric measure space. For related metric space results we refer to [3,12,14,15] and also to [2, Section 3.3].

We generalize the construction of certain extremal BMO-functions by Uchiyama [19] (see also [5, Section 2]) to doubling spaces. The result is stated in Theorem 2.1 and it constitutes the first part of the present paper. In the second part, we consider characterizations of BMO-maps between doubling spaces. Our main result is stated in Theorem 3.1. The characterizations in Theorem 3.1 are along the lines of the ones due to Gotoh [7,8].

2 Construction of certain BMO-functions

Throughout the paper, X is a complete metric space equipped with a metric d and a Borel regular outer measure μ satisfying the doubling condition. An open ball

$$B(x,r) = \{y \in X : d(y,x) < r\}, \quad x \in X,\ r > 0,$$

is simply denoted by B, we write $\mathrm{rad}(B)$ for the radius of the ball B, and $\lambda B = \{y \in X : d(y,x) < \lambda r\}$, $\lambda > 0$, is the ball with the same center, but the radius dilated by the factor λ.

In this paper, the doubling condition means that there exists a constant $c_D > 1$ such that for all $x \in X$, $0 < r < \infty$ and $y \in X$ such that $B(x,2r) \cap B(y,r) \neq \emptyset$, we have

$$\mu(B(x,2r)) \le c_D \mu(B(y,r)).$$

Notice that this condition is usually required to hold only for $x = y$, but if this standard doubling condition is valid with some uniform constant c_μ, then $\mu(B(x,2r)) \le \mu(B(y,8r)) \le c_\mu^3 \mu(B(y,r))$, i.e. our version of the standard doubling condition is satisfied with $c_D = c_\mu^3$. The standard doubling condition implies that if $B(x, R) \subset X$, $y \in B(x, R)$, and $0 < r \le R < \infty$, then

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge c_\mu^{-2} \left(\frac{r}{R}\right)^{\log_2 c_\mu}.$$

We refer, for instance, to [2, Lemma 3.3].

We recall that a locally integrable function $f : X \to \mathbb{R}$ has bounded mean oscillation, denoted as $f \in \mathrm{BMO}(X)$, if

$$\|f\|_* = \sup ⨍_B |f - f_B|\, d\mu < \infty,$$

where the supremum is taken over all balls $B \subset X$. We will identify functions which only differ by a constant; we shall call $\|f\|_*$ the BMO-norm of f. Here both f_B and the barred integral $⨍_B f\, d\mu$ denote the integral average of f over a ball B.

The following theorem is a metric space counterpart of a construction of certain BMO-functions in Uchiyama [19] and Garnett–Jones [5].

Theorem 2.1 *Let $\lambda > 1$ and let $E_1, \ldots, E_N$, $N \ge 2$, be μ-measurable subsets of X such that*

$$\min_{1 \le j \le N} \frac{\mu(E_j \cap B)}{\mu(B)} \le c_D^{-4\lambda} \tag{2.1}$$

for any ball $B \subset X$. Then there exist functions $\{f_j\}_{j=1}^N$ such that

$$\sum_{j=1}^{N} f_j(x) = 1, \tag{2.2}$$

and for each $1 \le j \le N$

$$0 \le f_j(x) \le 1, \tag{2.3}$$

and

$$f_j(x) = 0 \quad \mu\text{-almost everywhere on } E_j, \tag{2.4}$$

and moreover,

$$\|f_j\|_* \le \frac{c_1}{\lambda}. \tag{2.5}$$

Here c_1 is a constant that only depends on c_D and N. Conversely, if there exists $\{f_j\}_{j=1}^N$ that satisfy (2.2)–(2.4) and

$$\|f_j\|_* \le \frac{c_2}{\lambda}$$

holds with a sufficiently small constant c_2, only depending on c_D and N, for every $1 \le j \le N$, then (2.1) holds.

Before the proof of the theorem, we fix some notation and state few lemmas that will be needed later. Let q be a large integer, depending only on c_D and N, such that

$$1 + N c_D^6 q \le 2^q. \tag{2.6}$$

For every $k \in \mathbb{Z}$, let $r_k = 2^{-kq}$ and let $\mathcal{D}_k$ be a maximal set of points such that $d(x, y) \ge \frac{1}{2} r_k$ whenever $x, y \in \mathcal{D}_k$. Let $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$. Moreover, let

$$\mathcal{B}_k = \{B(x, r_k) : x \in \mathcal{D}_k\}.$$

From the maximality of the set $\mathcal{D}_k$ it follows that for every $k \in \mathbb{Z}$,

$$X = \bigcup_{B \in \mathcal{B}_k} B.$$

We say that a function $a \in C(X)$ is adapted to a ball $B = B(x, r)$, if

$$\operatorname{supp} a \subset B(x, 2r) \quad \text{and} \quad |a(x) - a(y)| \leq \frac{d(x, y)}{r}.$$

For a ball B, we set

$$g_j(B) = \log_{c_D} \frac{\mu(B)}{\mu(E_j \cap B)}, \quad 1 \leq j \leq N. \tag{2.7}$$

Let us state the following simple lemma for the function g_j.

Lemma 2.2 *Let k be a positive integer. If $B_1 \subset B_2$ and $c_D^k \mu(B_1) \geq \mu(B_2)$ for the balls B_1 and B_2 in X, then*

$$g_j(B_1) \geq g_j(B_2) - k.$$

Proof Clearly

$$\begin{aligned} g_j(B_1) &= \log_{c_D} \tfrac{\mu(B_1)}{\mu(B_1 \cap E_j)} \\ &\geq \log_{c_D} \tfrac{c_D^{-k} \mu(B_2)}{\mu(B_2 \cap E_j)} = g_j(B_2) - k. \end{aligned}$$

□

The next result is well known for the experts, but we recall it here.

Lemma 2.3 *Let $f \in \mathrm{BMO}(X)$. Then*

$$\tfrac{1}{2} \|f\|_* \leq \sup \left| \int_X f g \, d\mu \right| \leq \|f\|_*,$$

where the supremum is taken over all functions g for which there exists a ball B such that

$$\operatorname{supp} g \subset B, \quad \|g\|_\infty \leq \frac{1}{\mu(B)}, \quad \text{and} \quad \int_X g \, d\mu = 0.$$

Conversely, if f is a locally integrable function on X and the supremum above is finite, then $f \in \mathrm{BMO}(X)$ with the above norm estimate.

Proof First notice that for any g as above, we have

$$\left| \int_X f g \, d\mu \right| = \left| \int_X (f - f_B) g \, d\mu \right| \leq \fint_B |f - f_B| \, d\mu \leq \|f\|_*.$$

This gives the upper bound.

To see the lower bound, let $\varepsilon > 0$ and let B be a ball such that

$$\|f\|_* \leq \fint_B |f - f_B| \, d\mu + \varepsilon.$$

Let $h \in L^\infty(B)$ with $\|h\|_{L^\infty(B)} \leq 1$ be a function for which

$$\int_B |f - f_B| \, d\mu = \int_B (f - f_B) h \, d\mu. \tag{2.8}$$

Since $\int_B (f - f_B)\,d\mu = 0$, we have

$$\int_B |f - f_B|\,d\mu = \int_B (f - f_B)(h - h_B)\,d\mu. \tag{2.9}$$

Define

$$g = \frac{(h - h_B)\chi_B}{2\mu(B)}.$$

Then

$$\operatorname{supp} g \subset B, \quad \|g\|_{L^\infty(B)} \le \frac{1}{\mu(B)} \quad \text{and} \quad \int_X g\,d\mu = 0.$$

Moreover

$$\begin{aligned}\int_X fg\,d\mu &= \frac{1}{2\mu(B)} \int_B f(h - h_B)\,d\mu \\ &= \frac{1}{2\mu(B)} \int_B (f - f_B)(h - h_B)\,d\mu.\end{aligned} \tag{2.10}$$

By combining the Eqs. (2.9) and (2.10) we conclude that

$$\int_B |f - f_B|\,d\mu = 2\mu(B) \int_X fg\,d\mu$$

and

$$\int_X fg\,d\mu = \frac{1}{2} ⨍_B |f - f_B|\,d\mu \ge \frac{1}{2}(\|f\|_* - \varepsilon).$$

The claim follows by passing $\varepsilon \to 0$.

The Eq. (2.8) together with the above inequalities also indicates that the finiteness of $\sup \left|\int_X fg\,d\mu\right|$ implies $f \in \mathrm{BMO}(X)$. □

The proof of the metric space version of the following John-Nirenberg lemma can be found for example in Theorem 3.15 in [2]. See also [3] and [15].

Lemma 2.4 *Let $B \subset X$ be a ball and $f \in \mathrm{BMO}(5B)$. Then for every $\lambda > 0$*

$$\mu(\{x \in B : |f(x) - f_B| > \lambda\}) \le 2\mu(B) \exp\left(-\frac{A\lambda}{\|f\|_*}\right).$$

The positive constant A depends only on the doubling constant c_D.

We are ready for the proof of the main result of this section.

Proof of Theorem 2.1 The necessity part of the theorem is an immediate consequence of Lemma 2.4. Fix $\lambda > 1$ and let B be a ball. By (2.2), there exists j_0 such that

$$(f_{j_0})_B \ge \frac{1}{N}.$$

Thus, by Lemma 2.4 and (2.4), we have

$$\begin{aligned}\frac{\mu(B \cap E_{j_0})}{\mu(B)} &\le \frac{\mu(\{x \in B : |f_{j_0}(x) - (f_{j_0})_B| \ge 1/N\})}{\mu(B)} \\ &\le 2e^{-A/(N\|f\|_*)} \le 2\exp\left(-\frac{A\lambda}{Nc_2}\right) \le c_D^{-4\lambda},\end{aligned}$$

if c_2 is chosen to be small enough. This completes the proof of the necessity part of Theorem 2.1.

Then we consider the sufficiency. By (2.1), we have

$$\mu\left(\bigcap_{j=1}^{N} E_j\right) = 0.$$

Thus, if $\lambda > 1$ is smaller than a given number, then the functions

$$f_j = \frac{\chi_{E_j^c}}{\sum_{k=1}^{N} \chi_{E_k^c}}, \quad 1 \le j \le N,$$

satisfy the desired properties (we denote the characteristic function of a set A by χ_A). So we may assume that λ is large enough.

First, we assume that

$$E_1, \dots, E_N \subset B_0 \tag{2.11}$$

for some $B_0 \in \mathcal{B}_0$. We will inductively construct the sequences of BMO functions $\{f_{j,h}\}_{h=1}^{\infty}$, $1 \le j \le N$, such that

$$\sum_{j=1}^{N} f_{j,h}(x) = \lambda, \tag{2.12}$$

$$0 \le f_{j,h}(x) \le \lambda, \tag{2.13}$$

$$f_{j,h}(x) \le g_j(B) \quad \text{for every } x \in B, \text{ if } B \in \mathcal{B}_h, \tag{2.14}$$

and

$$\|f_{j,h}\|_* \le c_1. \tag{2.15}$$

If the functions $f_{j,h}$ above have been constructed, there exists a sequence $1 \le h_1 < h_2 < \dots$ such that $\{f_{j,h_k}\}_{k=1}^{\infty}$ converge weak* in L^{∞} as $k \to \infty$, since $\|f_{j,h}\|_{\infty} \le \lambda$ by (2.13). We set

$$f_j = \text{weak}^* - \lim_{k\to\infty} \frac{f_{j,h_k}}{\lambda}, \quad 1 \le j \le N.$$

Then (2.2) and (2.3) follow from (2.12) and (2.13). Let g be as in Lemma 2.3. Then

$$\left|\int f_j g \, d\mu\right| = \frac{1}{\lambda}\left|\lim_{k\to\infty} \int f_{j,h_k} g \, d\mu\right| \le \frac{1}{\lambda} \limsup_{k\to\infty} \|f_{j,h_k}\|_* \le \frac{c_1}{\lambda}.$$

Thus (2.5) with constant $2c_1$ follows from Lemma 2.3. Since, by Lebesgue's theorem,

$$\lim_{r\to 0} \sup_{\substack{B \ni x \\ \mathrm{rad}(B) \le r}} g_j(B) = 0$$

for μ-almost every $x \in E_j$, we have by (2.14)

$$\lim_{h\to 0} f_{j,h}(x) = 0$$

for μ-almost every $x \in E_j$. Thus (2.4) follows. Hence $\{f_j\}_{j=1}^{N}$ are the desired functions.

To remove the restriction (2.11), we take balls $B_p \in \mathcal{B}_{-p}$, $p = 1, 2, \dots$, such that $B_{p-1} \subset B_p$ for every p, and we can construct $f_{j,p}$ such that all other conditions are as for B_0, except that

$$f_{j,p} = 0 \quad \text{on } E_j \cap B_p.$$

Then there exists a sequence $1 \le p_1 < p_2 \dots$ such that $\{f_{j,p_k}\}_{k=1}^{\infty}$ converge weak* in L^∞. Then

$$f_j = \text{weak}^* - \lim_{k\to\infty} f_{j,p_k}, \quad 1 \le j \le N,$$

are the desired functions.

Thus, to complete the proof Theorem 2.1 we shall construct a sequence of functions that satisfy the conditions (2.12)–(2.15). The proof is organized as follows. In Lemma 2.5, we will construct the sequence $\{f_{j,h}\}_{h=0}^{\infty}$, $1 \le j \le N$, and show that these functions satisfy the conditions (2.12)–(2.14). And finally, in Lemma 2.7, we show that the condition (2.15) is valid for the functions.

Lemma 2.5 *Let $E_1, \dots, E_N$ satisfy* (2.1) *and* (2.11). *Then there exist $\{f_{j,h}\}$ and $A_{j,h} \subset \mathcal{B}_h$ having the properties* (2.12)–(2.14) *and satisfying the following conditions*

$$|f_{j,h}(x) - f_{j,h}(y)| \le 2^{(h+1)q} d(x, y), \tag{2.16}$$

$$A_{j,h} = \{B \in \mathcal{B}_h : \sup_B f_{j,h-1} > g_j(B)\}, \tag{2.17}$$

$$f_{j,h}(x) \ge f_{j,h-1}(x) - c_D^3 q, \tag{2.18}$$

and

$$f_{j,h}(x) \ge f_{j,h-1}(x) \quad \text{for } x \notin \bigcup_{B\in A_{j,h}} 2B. \tag{2.19}$$

Proof By (2.1), we have

$$\max_{1\le j\le N} g_j(B_0) \ge 4\lambda.$$

Set

$$s(B_0) = \min\{j : 1 \le j \le N, g_j(4B_0) \ge 4\lambda\},$$

$$f_{s(B_0),0} = \lambda, \quad \text{and} \quad f_{j,0} = 0 \quad \text{for } j \ne s(B_0).$$

Assume now that the functions $f_{1,k-1}, \dots, f_{N,k-1}$ have been defined and satisfy the conditions (2.12)–(2.14), (2.16), (2.18) and (2.19). Define $A_{j,k}$ by (2.17). For any ball B, let b_B denote a function that is adapted to B, $0 \le b_B \le 1$ and $b_B = 1$ on B. Let $A_{j,k} = \{B_m\}_{m=1}^{p}$. Set $a_{B_1} = \min\{qb_{B_1}, f_{j,k-1}\}$ and

$$a_{B_m} = \min\left\{qb_{B_m}, f_{j,k-1} - \sum_{n=1}^{m-1} a_{B_n}\right\} \quad \text{for } m = 2, \dots, p.$$

Since the supports of $\{b_{B_m}\}$ overlap at most c_D^3 times, the functions $c_D^{-3}q^{-1}a_{B_m}$ are adapted to B_m. Set

$$\widetilde{f}_{j,k} = f_{j,k-1} - \sum_{B\in A_{j,k}} a_B = f_{j,k-1} - v_{j,k}.$$

Since

$$\widetilde{f}_{j,k} = \max\left\{f_{j,k-1} - \sum_{B\in A_{j,k}} qb_B, 0\right\},$$

we see that $\{\widetilde{f}_{j,k}\}$ satisfy (2.13), (2.18) and (2.19).

If $B \in A_{j,k}$ and $x \in B$, then by Lemma 2.2

$$\widetilde{f}_{j,k}(x) \le \max\{f_{j,k-1}(x) - q, 0\} \le \max\{g_j(\widetilde{B}) - q, 0\} \le g_j(B),$$

for every $\widetilde{B} \in \mathcal{B}_{k-1}$ such that $B \subset \widetilde{B}$.

If $B \in \mathcal{B}_k \setminus A_{j,k}$ and $x \in B$, then

$$\widetilde{f}_{j,k}(x) \le f_{j,k-1}(x) \le g_j(B)$$

by the definition of $A_{j,k}$. So $\{\widetilde{f}_{j,k}\}$ satisfies (2.14). These functions do not satisfy the property (2.12), and hence we shall modify the functions further. We set

$$f_{j,k} = \widetilde{f}_{j,k} + \sum_{\substack{B \in \bigcup_{m=1}^N A_{m,k} \\ s(B)=j}} a_B = \widetilde{f}_{j,k} + w_{j,k}.$$

The modified sequence $\{f_{j,k}\}$ satisfies (2.12). Also the conditions (2.13), (2.18), and (2.19) are met since $a_B \ge 0$.

Let us next look at the condition (2.14). If $B \in \mathcal{B}_k$ and $w_{j,k} = 0$ on B, then

$$f_{j,k} = \widetilde{f}_{j,k} \le g_j(B) \quad \text{on } B,$$

since $\widetilde{f}_{j,k}$ satisfies (2.14). If $B \in \mathcal{B}_k$ and $w_{j,k} \ne 0$ on B, then, by the definition of $w_{j,k}$, there exists a ball $\widetilde{B} \in \mathcal{B}_k$ such that

$$B \cap 2\widetilde{B} \ne \emptyset \quad \text{and} \quad g_j(4\widetilde{B}) \ge 4\lambda.$$

Then $B \subset 4\widetilde{B}$. By Lemma 2.2,

$$g_j(B) \ge g_j(4\widetilde{B}) - 2 \ge \lambda.$$

So by (2.13), we have

$$f_{j,k}(x) \le \lambda \le g_j(B)$$

and consequently (2.14) holds.

Let us show that the condition (2.16) holds. If $x, y \in \widetilde{B}$ and $\widetilde{B} \in \mathcal{B}_k$, then

$$\begin{aligned} &|(-v_{j,k}(x) + w_{j,k}(x)) - (-v_{j,k}(y) + w_{j,k}(y))| \\ &\le \textstyle\sum_{B \in \bigcup_{m=1}^N A_{m,k}} |a_B(x) - a_B(y)| \end{aligned} \tag{2.20}$$

Since the supports of $\{a_B\}_{B \in \bigcup_m A_{m,k}}$ overlap at most Nc_D^3 times, (2.20) is dominated by

$$Nc_D^3 \cdot c_D^3 q \cdot \frac{d(x,y)}{r_k} = Nc_D^6 q 2^{qk} d(x,y).$$

From this we conclude that

$$\begin{aligned} |f_{j,k}(x) - f_{j,k}(y)| &\le |f_{j,k-1}(x) - f_{j,k-1}(y)| + Nc_D^6 q 2^{qk} d(x,y) \\ &\le (1 + Nc_D^6 q) 2^{kq} d(x,y) \le 2^{(k+1)q} d(x,y), \end{aligned}$$

where we used (2.16) for $f_{j,k-1}$, and also the inequality (2.6). □

Lemma 2.6

$$f_{j,h}(x) \le g_j(B) - \frac{1}{3}\log_2 \frac{r}{r_h} + 8 \cdot 2^q + 6$$

for every $x \in B = B(y,r)$ for any B such that $r \le 4r_h$.

Proof There are at most c_D^3 balls in $B_1, \ldots, B_k$ with the centers in $\mathcal{D}_h$ such that $B_i \cap B \neq \emptyset$. Let

$$\delta = \min_{1 \le i \le k} g_j(B_i) = g_j(B_{i_0}).$$

By (2.14)

$$\inf_{x \in B} f_{j,h}(x) \le \delta,$$

and by (2.16) we have

$$f_{j,h}(x) \le \delta + 2^{(h+1)q} 2r \le \delta + 8 \cdot 2^q$$

whenever $x \in B$.

On the other hand,

$$\begin{aligned} g_j(B) &= \log_{c_D} \frac{\mu(B)}{\mu(B \cap E_j)} \\ &\ge \log_{c_D} \frac{\mu(B)}{\sum_i \mu(B_i \cap E_j)} \\ &\ge \log_{c_D} \frac{\mu(B)}{c_D^3 \max_i \{\mu(B_i \cap E_j)\}} \\ &= \log_{c_D} \frac{\mu(B)}{\mu(B_{i_0})} + \log_{c_D} \frac{\mu(B_{i_0})}{\mu(B_{i_0} \cap E_j)} + \log_{c_D} \frac{1}{c_D^3} \\ &\ge \log_{c_D} \frac{\mu(B)}{\mu(B_{i_0})} + \delta - 3 \\ &\ge \frac{1}{3} \log_2 \frac{r}{r_h} + \delta - 6. \end{aligned}$$

The desired result follows from the two previous estimates. □

We finish to proof of Theorem 2.1 by proving the following lemma.

Lemma 2.7 $\|f_{j,h}\|_* \le c_1$.

Proof Let $B = B(x, r)$ be any ball. If $r \le 2^{-hq}$ then, by (2.16), we have

$$\inf_{c \in \mathbb{R}} \fint_B |f_{j,h} - c| d\mu \le 2^q. \tag{2.21}$$

If $0 \le n < h$ and $2^{-(n+1)q} < r \le 2^{-nq}$, let

$$\beta_j = \fint_B f_{j,n} \, d\mu.$$

Notice that by Lemma 2.6,

$$\beta_j \le g_j(4B) + \frac{1}{3} q + 8 \cdot 2^q + 6. \tag{2.22}$$

We will show that

$$\fint_B |f_{j,h} - \beta_j| \, d\mu \le C. \tag{2.23}$$

Let

$$\begin{aligned}
&\{x \in B : |f_{j,h}(x) - \beta_j| \geq \alpha\} \\
&\quad = \{x \in B : f_{j,h}(x) < \beta_j - \alpha\} \cup \{x \in B : f_{j,h}(x) > \beta_j + \alpha\} \\
&\quad =: G(B, j, \alpha) \cup H(B, j, \alpha).
\end{aligned} \tag{2.24}$$

First, we estimate $\mu(G(B, j, \alpha))$. Let $\alpha > 2^{q+1}$. Note that $f_{j,n}(x) > \beta_j - 2^{q+1}$ on B by (2.16). So if $x \in G(B, j, \alpha)$ then, by (2.19), there exists $\widetilde{B} \in A_{j,k}$, $n < k \leq h$, such that $x \in 2\widetilde{B}$ and $f_{j,k}(x) < \beta_j - \alpha$. So by (2.18), we have

$$f_{j,k-1}(x) < \beta_j - \alpha + c_D^3 q,$$

and by (2.16)

$$f_{j,k-1}(y) < \beta_j - \alpha + c_D^3 q + 3$$

for every $y \in \widetilde{B}$. Thus, by the definition of $A_{j,k}$, we obtain

$$g_j(\widetilde{B}) < \beta_j - \alpha + c_D^3 q + 3.$$

By the above, we can use the standard 5-covering theorem ([2, Lemma 1.7]) and take disjoint balls $\{B_m\} \subset \bigcup_{n<k\leq h} A_{j,k}$ such that

$$B_m \subset 4B, \quad G(B, j, \alpha) \subset \bigcup_m 5B_m,$$

and

$$g_j(B_m) < \beta_j - \alpha + c_D^3 q + 3. \tag{2.25}$$

Thus

$$\begin{aligned}
\mu(G(B, j, \alpha)) &\leq c_D^3 \sum_m \mu(B_m) = c_D^3 \sum_m \mu(E_j \cap B_m) c_D^{g_j(B_m)} \\
&\leq C c_D^{\beta_j - \alpha} \sum_m \mu(E_j \cap B_m) \\
&\leq C c_D^{g_j(4B) - \alpha} \sum_m \mu(E_j \cap B_m) \\
&\leq C c_D^{g_j(4B) - \alpha} \mu(E_j \cap 4B) \leq C \mu(B) c_D^{-\alpha}.
\end{aligned} \tag{2.26}$$

Here we used first (2.7), then (2.25), (2.22) and finally (2.7) again.

Let us then estimate the measure $\mu(H(B, j, \alpha))$. Let $\alpha > (N-1)2^{q+1}$. Note that $\sum_{m=1}^N \beta_m = \lambda$ by (2.12). So if $x \in H(B, j, \alpha)$, then

$$\begin{aligned}
\sum_{1\leq m\leq N,\, m\neq j} f_{m,h}(x) &= \lambda - f_{j,h}(x) = \sum_{m=1}^N \beta_m - f_{j,h}(x) \\
&= \left(\sum_{1\leq m\leq N,\, m\neq j} \beta_m \right) - \left(f_{j,h}(x) - \beta_j \right) \\
&< \left(\sum_{1\leq m\leq N,\, m\neq j} \beta_m \right) - \alpha.
\end{aligned}$$

Thus

$$\sum_{\substack{1\leq m\leq N \\ m\neq j}} (\beta_m - f_{m,h}(x)) > \alpha.$$

So

$$x \in \bigcup_{\substack{1\le m\le N\\ m\ne j}} G(B, m, \alpha/(N-1)),$$

and consequently

$$H(B, j, \alpha) \subset \bigcup_{\substack{1\le m\le N\\ m\ne j}} G(B, m, \alpha/(N-1)).$$

By (2.26), we have

$$\mu(H(B, j, \alpha)) \le C(N-1)\mu(B)c_D^{-\alpha/(N-1)}. \tag{2.27}$$

Thus, if $2^{-hq} \le r \le 1$, then (2.23) follows from (2.26) and (2.27). If $r > 1$, then put $\beta_{s(B_0)} = \lambda$ and $\beta_j = 0$ for $j \ne s(B_0)$. Then (2.23) follows from the same argument. Thus Lemma 2.7 follows from (2.21) and (2.23).

The proof of Theorem 2.1 is now complete. □

3 Characterizations of BMO-maps

We say that a μ-measurable map $F\colon X \to X$ is a BMO-*map* if

(I) $F^{-1}(E)$ is a μ-null set for each μ-null set $E \subset X$,
(II) for every $f \in \mathrm{BMO}(X)$ the composed map $C_F(f) = f \circ F$ is in $\mathrm{BMO}(X)$.

We shall prove a metric space generalization of a theorem due to Gotoh [7, Theorem 3.1] which characterizes BMO-maps between doubling metric measure spaces. In the proof we apply Uchiyama's construction proved in Sect. 2. The condition (3.1) has a similar flavor as the conditions in [6] and [11] related to invariance properties of quasiconformal mappings.

Theorem 3.1 *Suppose that $F\colon X \to X$ is μ-measurable. Then the following conditions are equivalent:*

(i) *There exist positive finite constants K and α such that for an arbitrary pair of μ-measurable subsets E_1, E_2 of X we have*

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k)\cap B)}{\mu(B)} \le K\left(\sup_B \min_{k=1,2} \frac{\mu(E_k\cap B)}{\mu(B)}\right)^{\alpha}, \tag{3.1}$$

where the suprema are taken over all balls B in X;

(ii) *There exist constants $0 < \gamma < 1/4$ and $\lambda > 0$ such that for an arbitrary pair of μ-measurable subsets E_1, E_2 of X satisfying*

$$\sup_B \min_{k=1,2} \frac{\mu(E_k\cap B)}{\mu(B)} < \lambda,$$

we have

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k)\cap B)}{\mu(B)} < \gamma,$$

where the suprema are taken over all balls B in X;

(iii) *F is a* BMO-*map with the operator norm of C_F bounded by CK/α, where C depends only on the doubling constant.*

The condition (i) readily implies the condition (ii), and hence to show the equivalence of conditions (i)–(iii), it is enough to prove implications (i)⇒(iii), (ii)⇒(iii) and (iii)⇒(i), in Propositions 3.7, 3.8, and 3.9, respectively. The Uchiyama construction of BMO functions, presented in Section 2, is used in the proof of Proposition 3.9. For the proof of the bound for the operator norm, see Proposition 3.7.

Remark 3.2 Let us comment on the condition (i).

(1) Setting $E_1 = E_2 = X$ in (3.1) it can be seen that $K \geq 1$.
(2) If (3.1) is valid for some positive α_0 it clearly holds for all $0 < \alpha < \alpha_0$. And moreover, since the condition (3.1) is interesting mainly with small values of the exponent α, we shall assume, without loss of generality, that $\alpha \leq 1$.

We shall next prove several lemmas on BMO functions.

Lemma 3.3 *Let $f \in \mathrm{BMO}(X)$. Then*

$$\begin{aligned} &\min\{\mu(\{x \in B\colon f(x) \geq t\}),\ \mu(\{x \in B\colon f(x) \leq s\})\} \\ &\quad \leq 2\mu(B)\exp\left(-C\frac{t-s}{\|f\|_*}\right) \end{aligned}$$

for every $-\infty < s \leq t < \infty$, where C is a positive constant depending on the doubling constant c_D.

Proof By symmetry, we may assume that $f_B \leq (s+t)/2$. Then Lemma 2.4 implies that

$$\begin{aligned} \mu(\{x \in B\colon f(x) \geq t\}) &\leq \mu\left(\left\{x \in B\colon |f(x) - f_B| \geq \frac{t-s}{2}\right\}\right) \\ &\leq 2\mu(B)\exp\left(-\frac{A(t-s)}{2\|f\|_*}\right). \end{aligned}$$

If $f_B \geq (s+t)/2$, we get a similar estimate for $\mu(\{x \in B\colon f(x) \leq s\})$. □

A converse of the statement in Lemma 3.3 is presented in the following.

Lemma 3.4 *Let $f\colon X \to \mathbb{R}$ be a μ-measurable function with $|f| < \infty$ μ-almost everywhere in X. Assume there exist positive constants C_1, C_2 such that for every ball B in X we have*

$$\begin{aligned} &\min\{\mu(\{x \in B\colon f(x) \geq t\}),\ \mu(\{x \in B\colon f(x) \leq s\})\} \\ &\quad \leq C_1\mu(B)\exp(-C_2(t-s)) \end{aligned}$$

for every $-\infty < s \leq t < \infty$. Then $f \in \mathrm{BMO}(X)$ and

$$\|f\|_* \leq 4(C_1+1)C_2^{-1}\exp(2C_2).$$

In the proof of Lemma 3.4 we apply the following lemma which can be found in [7, Lemma 4.5].

Lemma 3.5 *Let $\lambda\colon \mathbb{R} \to [0,1]$ be a non-constant, non-decreasing function. Assume that there exists positive constants C_1, C_2 such that*

$$\min\{\lambda(s), 1-\lambda(t)\} \leq C_1\exp(-C_2(t-s))$$

for every $-\infty < s \le t < \infty$. *Then there exists* $t_0 \in \mathbb{R}$ *such that*

$$\max\{\lambda(t_0 - t), 1 - \lambda(t_0 + t)\} \le (C_1 + 1)\exp(2C_2)\exp(-C_2 t)$$

for each $t \ge 0$.

Proof of Lemma 3.4 We apply Lemma 3.5 by setting

$$\lambda(t) = \frac{\mu(\{x \in B\colon f(x) \le t\})}{\mu(B)}.$$

Then by the hypothesis $\lambda(t)$ meets the assumption in Lemma 3.5 with the same constants C_1 and C_2. Hence there exists $t_0 \in \mathbb{R}$ such that the second inequality of Lemma 3.5 is valid for every $t \ge 0$. This implies that

$$\nu(t) = \mu(\{x \in B\colon |f(x) - t_0| \ge t\}) \le 2(C_1 + 1)\mu(B)\exp(2C_2)\exp(-C_2 t)$$

for every $t \ge 0$. We obtain

$$\begin{aligned}\int_B |f - f_B|\, d\mu &\le 2\int_B |f - t_0|\, d\mu = 2\int_0^\infty \nu(t)\, dt \\ &\le 4(C_1 + 1)C_2^{-1}\exp(2C_2)\mu(B)\end{aligned}$$

from which the claim follows. □

In Euclidean spaces the following lemma is due to Strömberg [17]. A result similar to this has also been considered for nondoubling measures by Lerner in [13].

Lemma 3.6 *Let* $f\colon X \to \mathbb{R}$ *be* μ*-measurable. Assume that there exist constants* $0 < \gamma < (4c_D^3)^{-1}$*, and* $\lambda > 0$ *such that for each ball* B *in* X *we have*

$$\inf_{c\in\mathbb{R}} \mu(\{x \in B\colon |f(x) - c| \ge \lambda\}) \le \gamma\mu(B). \tag{3.2}$$

Then $f \in \mathrm{BMO}(X)$ *satisfying* $\|f\|_* \le C\lambda$*, where a positive constant* C *depends only on the doubling constant* c_D.

Proof Let f be μ-measurable on X, and fix γ and λ such that the hypothesis (3.2) is satisfied for each ball in X. Fix a ball $B \subset X$ and let c_0 be the number where the infimum in (3.2) is reached. For each $m = 1, 2, \ldots$ we write

$$\begin{aligned}S_m^+ &= \{x \in B\colon f(x) - c_0 > m\lambda\},\\ S_m^- &= \{x \in B\colon f(x) - c_0 < -m\lambda\},\\ S_m &= S_m^+ \cup S_m^- = \{x \in B\colon |f(x) - c_0| > m\lambda\},\\ E_m &= \{x \in B\colon m\lambda < |f(x) - c_0| \le (m+1)\lambda\},\end{aligned}$$

and

$$E_0 = \{x \in B\colon |f(x) - c_0| \le \lambda\}.$$

Let us estimate the measure of the set S_m^+. First notice that $S_m^+ \subset S_{m-1}^+$. For μ-almost every $x \in S_{m-1}^+$, there exists a ball $B_x = B(x, r_x)$ such that

$$\frac{1}{2c_D}\mu(B_x) < \mu(B_x \cap S_{m-1}^+) \le \frac{1}{2}\mu(B_x) \tag{3.3}$$

and

$$\mu(B(x,r) \cap S^+_{m-1}) > \frac{1}{2}\mu(B(x,r))$$

for all $r < \frac{1}{2}r_x$; see, for example, Theorem 3.1 and Remark 3.2 in [9].

By a well known 5-covering theorem ([2, Lemma 1.7]), we can cover the set S^+_{m-1} by finite or countable sequence of balls $\{B_i\}_i$ satisfying (3.3) such that the balls $\{\frac{1}{5}B_i\}_i$ are disjoint. It follows from (3.3) that the infimum in (3.2) is reached with some constant c such that

$$c_0 + (m-2)\lambda \le c \le c_0 + m\lambda$$

in each of the balls B_i, and hence $c - c_0 \le m\lambda$.

We conclude, by applying the in inequality (3.2) in balls B_i, that

$$\begin{aligned}\mu(S^+_{m+1}) &\le \sum_i \mu(B_i \cap S^+_{m+1}) \le \gamma \sum_i \mu(B_i) \le c_D^3 \gamma \sum_i \mu(\tfrac{1}{5}B_i)\\ &\le 2c_D^3\gamma \sum_i \mu(\tfrac{1}{5}B_i \cap S^+_{m-1}) \le 2c_D^3\gamma\mu(S^+_{m-1})\end{aligned}$$

Since $\mu(S_1^+) \le \mu(S_1) < \gamma\mu(B)$, it follows from the previous estimate that

$$\mu(S^+_{2m+2}) \le \mu(S^+_{2m+1}) \le (2c_D^3\gamma)^{m+1}\mu(B)$$

for each $m = 1, 2, \ldots$. Since a similar estimate holds for S^-_m, we altogether have

$$\mu(S_m) \le 2(2c_D^3\gamma)^{m/2}\mu(B).$$

We thus conclude

$$\begin{aligned}\fint_B |f - f_B|\, d\mu &\le \frac{2}{\mu(B)}\left(\sum_{m=0}^{\infty}\int_{E_m} |f - c_0|\, d\mu\right)\\ &\le \lambda + 2\sum_{m=1}^{\infty}(m+1)\lambda\frac{\mu(S_m)}{\mu(B)}\\ &\le \lambda\left(1 + 2\sum_{m=1}^{\infty}(m+1)(2c_D^3\gamma)^{m/2}\right)\\ &\le \lambda\left(1 + 2\sum_{m=1}^{\infty}(m+1)2^{-m/2}\right).\end{aligned}$$

Since the preceding estimate holds for any ball $B \subset X$, the claim follows.

Let us now turn to the proof of Theorem 3.1.

Proposition 3.7 [(i) $\Rightarrow$ (iii)] *Let $F\colon X \to X$ be μ-measurable and assume that there exist positive finite constants K and α such that the condition* (i) *of Theorem* 3.1 *holds. Then F is a* BMO*-map satisfying $\|C_F\| \le CK/\alpha$, where C depends on the doubling constant c_D.*

Proof The condition (i) implies that if E is a μ-null subset of X then also $\mu(F^{-1}(E)) = 0$.

Let $f \in \mathrm{BMO}(X)$ and set for each $-\infty < s \le t < \infty$

$$E_1 = \{x \in X\colon f(x) \le s\} \quad \text{and} \quad E_2 = \{x \in X\colon f(x) \ge t\}. \tag{3.4}$$

It follows from Lemma 3.3 that

$$\min\{\mu(E_1 \cap B),\ \mu(E_2 \cap B)\} \leq 2\mu(B) \exp\left(-C\frac{t-s}{\|f\|_*}\right)$$

for all balls B in X. The condition (i) implies

$$\min\{\mu(F^{-1}(E_1) \cap B),\ \mu(F^{-1}(E_2) \cap B)\} \leq 2^\alpha K\mu(B) \exp\left(-C\frac{\alpha(t-s)}{\|f\|_*}\right)$$

for all balls B in X. Since

$$F^{-1}(E_1) \cap B = \{x \in B \colon (f \circ F)(x) \leq s\}$$

and

$$F^{-1}(E_2) \cap B = \{x \in B \colon (f \circ F)(x) \geq t\},$$

it follows from Lemma 3.4 that $f \circ F \in \mathrm{BMO}(X)$ and (recall that $\alpha \leq 1$, see Remark 3.2)

$$\begin{aligned}\|C_F(f)\|_* &\leq \frac{4(2^\alpha K + 1)\|f\|_*}{C\alpha} \exp(2C\alpha/\|f\|_*)\\ &= \frac{CK\|f\|_*}{\alpha} \exp(C\alpha/\|f\|_*),\end{aligned}$$

where C is a positive constant depending on the doubling constant c_D. Applying the preceding estimate to τf, $\tau > 0$, and letting $\tau \to \infty$, we obtain that $\|C_F\| \leq CK/\alpha$. □

Proposition 3.8 ((ii) $\Rightarrow$ (iii)) *Let $F\colon X \to X$ be μ-measurable and assume that there exist constants $0 < \gamma < (4c_D^3)^{-1}$ and $\lambda > 0$ such that the condition* (ii) *of Theorem* 3.1 *holds. Then F is a* BMO*-map satisfying $\|C_F\| \leq C\lambda$, where C depends on the doubling constant c_D and γ.*

Proof The condition (ii) implies that if E is a μ-null subset of X then also $\mu(F^{-1}(E)) = 0$.

Let $f \in \mathrm{BMO}(X)$ and assume, without loss of generality, that $\|f\|_* = 1$. We define the sets E_1 and E_2 for each $-\infty < s < t < \infty$ as in (3.4). We apply Lemma 3.3 and obtain

$$\sup_B \min_{k=1,2} \frac{\mu(E_k \cap B)}{\mu(B)} \leq 2\exp(-C(t-s)) < \lambda,$$

whenever $t - s \geq C_1$, where C_1 only depends on λ and the constant C from Lemma 3.3. Hence the condition (ii) implies that

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k) \cap B)}{\mu(B)} < \gamma.$$

For every ball B in X we set

$$\begin{aligned}s_B = \sup\{s \in \mathbb{R} :\ &\mu(\{x \in B \colon f(F(x)) \leq s\})\\ &\leq \mu(\{x \in B \colon f(F(x)) \geq s + C_1\})\}.\end{aligned}$$

Since $|f(F(x))| < \infty$ for μ-almost every $x \in X$, we have that $s_B \neq \pm\infty$. Hence

$$\mu(\{x \in B \colon f(F(x)) \leq s_B - 1\}) < \gamma\mu(B)$$

and

$$\mu(\{x \in B \colon f(F(x)) \geq s_B + C_1 + 1\}) < \gamma\mu(B).$$

If we set $c_B = s_B + C_1/2$ and $\tau = 1 + C_1/2$, we obtain

$$\mu(\{x \in B\colon |f(F(x)) - c_B| \geq \tau\}) \leq 2\gamma\mu(B).$$

The claim follows from Lemma 3.6. □

We shall apply the Uchiyama construction in the proof of the following result.

Proposition 3.9 ((iii) $\Rightarrow$ (i)) *Let $F\colon X \to X$ be a* BMO*-map. Then there exist positive constants K and β, depending only on the doubling constant c_D, such that the condition* (i) *of Theorem* 3.1 *holds with $\alpha = \beta/\|C_F\|$.*

Proof Let E_1 and E_2 be μ-measurable subsets in X and let $\lambda > 0$ be such that

$$c_D^{-4\lambda} = \sup_B \min_{k=1,2} \frac{\mu(E_k \cap B)}{\mu(B)}.$$

By Theorem 2.1 there exist functions f_1 and f_2, both in $\mathrm{BMO}(X)$, such that $f_1 + f_2 = 1$, $0 \leq f_k \leq 1$, $f_k = 0$ on E_k, and $\|f_k\|_* \leq C_1/\lambda$ for $k = 1, 2$, where a positive constant C_1 depends on the doubling constant c_D. Define for $k = 1, 2$ the composed function $g_k = f_k \circ F$. Then $g_1 + g_2 = 1$, $0 \leq g_k \leq 1$, $g_k = 0$ on $F^{-1}(E_k)$, and $\|g_k\|_* \leq C_1\|C_F\|/\lambda$ for $k = 1, 2$.

Let us fix a ball B in X. Clearly, we may assume that $(g_1)_B \geq 1/2$. Then by Lemma 2.4 we obtain

$$\begin{aligned}\frac{\mu(F^{-1}(E_1) \cap B)}{\mu(B)} &\leq \frac{\mu(\{x \in B\colon |g_1(x) - (g_1)_B| \geq 1/2\})}{\mu(B)} \\ &\leq 2\exp(-C\lambda/\|C_F\|),\end{aligned}$$

where C is a positive constant depending on the doubling constant c_D. By plugging in the value of λ, we obtain

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k) \cap B)}{\mu(B)} \leq 2\left(\sup_B \min_{k=1,2} \frac{\mu(E_k \cap B)}{\mu(B)}\right)^{C/\|C_F\|}$$

which completes the proof. □

3.1 A_p-weights and BMO-maps

We close this paper by discussing the connection between Muckenhoupt A_p-weights and BMO-maps.

It is well known that if ω is an A_p-weight for some $1 \leq p < \infty$, then $\log \omega \in \mathrm{BMO}(X)$, and on the other hand, whenever $f \in \mathrm{BMO}(X)$, then $e^{\delta f}$ is an A_p-weight for some $\delta > 0$ and $1 \leq p < \infty$. We refer to [4] for this result in the Euclidean setting. It straightforward to verify that the result has its counterpart also in metric measure spaces with a doubling measure.

We can add the following condition to the list in Theorem 3.1:

(iv) For each A_p-weight ω, with some $1 \leq p < \infty$, the composed map $\omega^\delta \circ F$ is an $A_{p'}$-weight for some positive δ and $1 \leq p' < \infty$.

In Euclidean spaces, the condition (iv) can be stated in terms of A_∞-weights, see [8, Corollary 3.3], and these weights have several but equivalent characterizations. In general metric spaces A_∞-weights have first been defined and studied in [18]. In this generality, however, these different conditions are not necessarily equivalent. In particular, the class of A_∞-weights can be strictly larger than the union of A_p-weights [18]. Several characterizations for A_∞-weights and their relations in doubling metric measure spaces have also been studied in [10].

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