Jiatong University, Xian 22–25 Oct 2018 Juha Kinnunen

Lectures on parabolic equations of the p-Laplacian type

The evolutionary p-Laplace (or p-parabolic) equation is

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1<p<\infty. \quad (u=u(x,t))$$

In these lectures, we consider the case $p \geq 2$.

When $p=2$, we have the heat equation $\frac{\partial u}{\partial t} - \Delta u = 0$.

More general structure conditions

Our arguments also apply to more general equations

$$\frac{\partial u}{\partial t} - \operatorname{div} A(\nabla u) = 0,$$

where A is a Carathéodory function and satisfies the structural conditions

$$A(\nabla u)\cdot\nabla u \geq \alpha |\nabla u|^p$$

and

$$|A(\nabla u)| \leq \beta |\nabla u|^{p-1},$$

where $0<\alpha<\beta<\infty$. Sometimes A is also assumed to be monotone.

Physical interpretation

Nonlinear diffusion.

$\boxed{p>2}$: Slow diffusion (degenerate)

$\boxed{p=2}$: The heat equation

$\boxed{p<2}$: Fast diffusion (singular)

Structure

(1) u_1, u_2 solutions $\not\Rightarrow$ $u_1 + u_2$ solution

(2) u solution $\Rightarrow$ $u + c$, $c \in \mathbb{R}$, solution

(3) u solution $\not\Rightarrow$ cu, $c \in \mathbb{R}$, solution

(4) u_1, u_2 (super) solutions $\Rightarrow$ $\min\{u_1, u_2\}$ supersolution.
In particular, $\min\{u, c\}$, $c \in \mathbb{R}$, supersolution

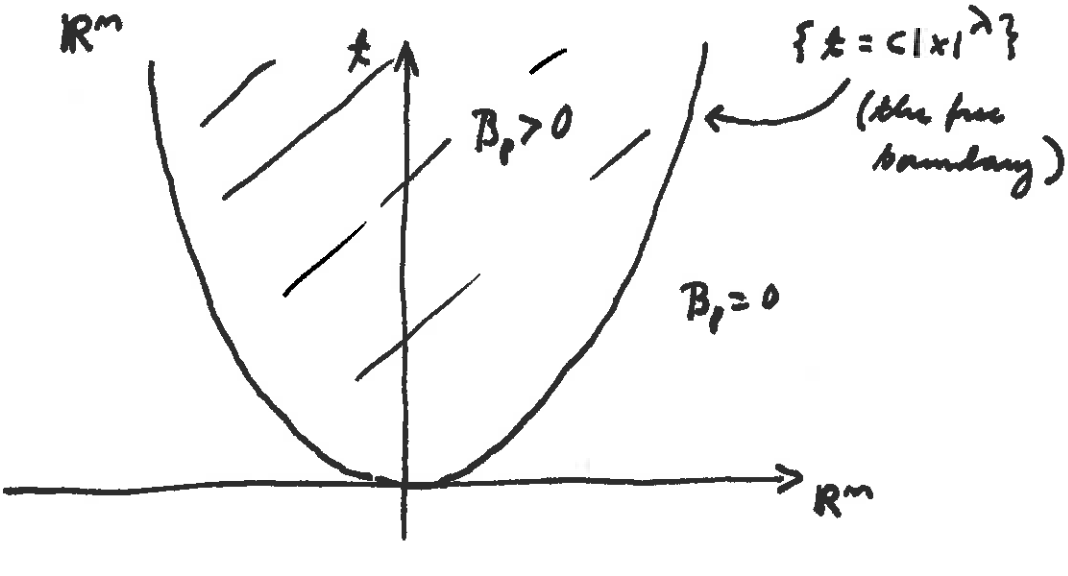
The Barenblatt solution

$B_p : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$,

$$B_p(x,t) = t^{-\frac{n}{\lambda}} \left(c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}},$$

where $\lambda = n(p-2) + p$, $p > 2$, and the constant c is usually chosen so that

$$\int B_p(x,t)\, dx = 1 \quad \text{for every } t > 0.$$



Note: Disturbances propagate with finite speed. The Barenblatt solution is not a classical solution.

Remark. For the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1<p<\infty,$$

the corresponding "fundamental" solution is

$$u(x) = \begin{cases} |x|^{\frac{p-n}{p-1}}, & p \neq n, \\ -\log|x|, & p = n \end{cases}$$

in $\mathbb{R}^n \setminus \{0\}$.

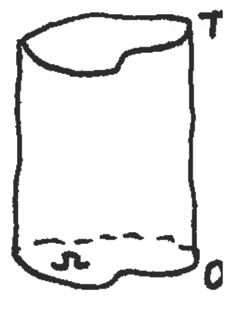
Weak solutions

Let $\Omega \subset \mathbb{R}^n$ be an open set and $T>0$.

$u : \Omega \times (0,T) \to [-\infty, \infty]$

$u \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega)) \Leftrightarrow u \in L^p(t_1,t_2; W^{1,p}(\Omega'))$

$\forall\, 0<t_1<t_2<T, \ \Omega' \subset\subset \Omega$



$$\Leftrightarrow \int_{t_1}^{t_2}\int_{\Omega'} (|u|^p + |\nabla u|^p)\,dx\,dt < \infty$$

$$\nabla u(x,t) = \left(\frac{\partial u}{\partial x_1}(x,t), \dots, \frac{\partial u}{\partial x_n}(x,t)\right) \quad \text{(the spatial gradient)}$$

Note: No smoothness in time.

Definition. $u \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega))$ is a weak solution to the evolutionary p-Laplace equation $\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, if

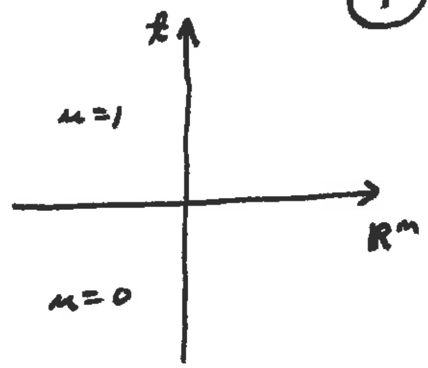
$$\int_0^T \int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi - u\frac{\partial \varphi}{\partial t}\right) dx\,dt = 0$$

for every $\varphi \in C_0^\infty(\Omega \times (0,T))$. u is a weak supersolution if "$\geq$" holds for $\varphi \geq 0$. We may also consider solutions in $\Omega \times (-\infty,\infty)$.

Examples

(1) $u : \mathbb{R}^{n+1} \to \mathbb{R}$, $u(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \le 0, \end{cases}$

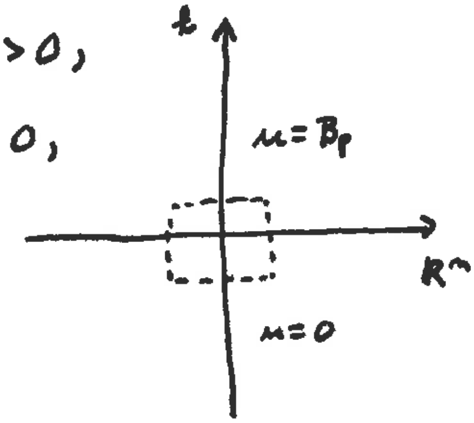
is a weak supersolution.



(2) $B_p : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is a weak solution.

(3) $u : \mathbb{R}^{n+1} \to \mathbb{R}$, $u(x,t) = \begin{cases} B_p(x,t), & t > 0, \\ 0, & t \le 0, \end{cases}$

is not a weak supersolution.



<u>Reason</u>: $\displaystyle\int_{[-1,1]^{n+1}} |\nabla u(x,t)|^p \, dx\,dt = \infty.$

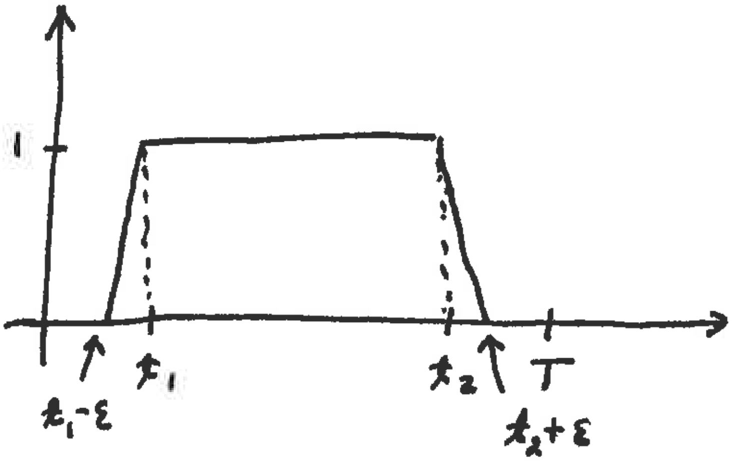
Comments on the definition

(1) $0 < t_1 < t_2 < T \Rightarrow$

$$\int_{t_1}^{t_2}\int_{\Omega} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) dx\,dt + \Big/_{t_1}^{t_2} \int_{\Omega} u(x,t)\varphi(x,t)\,dx = 0$$

(φ vanishes only on the lateral boundary)

<u>Reason</u>: Choose $\varphi \eta_\varepsilon$ as a test function, where $\eta_\varepsilon = \eta_\varepsilon(t)$ as in the picture



(2) The mollified equation is

$$\int_0^T \int_\Omega \left(\left(|\nabla u|^{p-2} \nabla u \right)_\varepsilon \cdot \nabla \varphi + \frac{\partial u_\varepsilon}{\partial t} \varphi \right) dx\,dt = 0$$

for every $\varphi \in C_0^\infty(\Omega \times (0,T))$ (and other function spaces) with $\varepsilon > 0$ small enough. Here $(\;)_\varepsilon$ refers to the standard convolution mollification in the time direction.

<u>Advantage</u>: We may use u as a test function.

In general, $\frac{\partial u}{\partial t}$ does not have a meaning.

<u>Variational approach</u>

$$\int_0^T \int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) dx\,dt = 0 \quad \forall \varphi \in C_0^\infty(\Omega \times (0,T))$$

$\Longleftrightarrow$

$$\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} \,dx\,dt + \frac{1}{p} \int_0^T \int_\Omega |\nabla u|^p \,dx\,dt \le \frac{1}{p} \int_0^T \int_\Omega |\nabla u + \nabla \varphi|^p \,dx\,dt$$

$$\forall \varphi \in C_0^\infty(\Omega \times (0,T))$$

<u>Proof</u>: $\boxed{\text{"}\Rightarrow\text{"}}$:

$$\int_0^T \int_\Omega |\nabla u|^p \,dx\,dt = \int_0^T \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla u \,dx\,dt$$

$$= \int_0^T \int_\Omega |\nabla u|^{p-2} \nabla u \cdot (\nabla u + \nabla \varphi) \,dx\,dt - \int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} \,dx\,dt$$

$$\Rightarrow \int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}\,dx\,dt + \int_0^T\int_\Omega |\nabla u|^p\,dx\,dt \le \int_0^T\int_\Omega |\nabla u|^{p-1}|\nabla u+\nabla\varphi|\,dx\,dt$$

$$\le \frac{1}{p'}\int_0^T\int_\Omega |\nabla u|^{(p-1)\cdot\frac{p}{p-1}}\,dx\,dt + \frac{1}{p}\int_0^T\int_\Omega |\nabla u+\nabla\varphi|^p\,dx\,dt$$

Young, $\frac{1}{p}+\frac{1}{p'}=1$

$\boxed{\text{"}\Leftarrow\text{"}}$: $\varphi = \varepsilon\phi$, $\varepsilon>0$, $\phi\in C_0^\infty(\Omega\times(0,T))$

$$\Rightarrow \varepsilon\int_0^T\int_\Omega u\frac{\partial\phi}{\partial t}\,dx\,dt + \frac{1}{p}\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt \le \frac{1}{p}\int_0^T\int_\Omega |\nabla u+\varepsilon\nabla\phi|^p\,dx\,dt$$

$$\Rightarrow \int_0^T\int_\Omega u\frac{\partial\phi}{\partial t}\,dx\,dt + \frac{1}{p}\int_0^T\int_\Omega \underbrace{\frac{1}{\varepsilon}\left(|\nabla u|^p - |\nabla u+\varepsilon\nabla\phi|^p\right)}_{\to\, -p|\nabla u|^{p-1}\frac{\nabla u}{|\nabla u|}\cdot\nabla\phi,\ \varepsilon\to 0}\,dx\,dt \le 0$$

"$\geq$" follows by choosing $\varphi = -\varepsilon\phi$. □

<u>Regularity</u>

(1) Weak solutions are locally Hölder continuous both in the space and time variables if $p > \frac{2n}{n+2}$. (Chen & Di'Benedetto). They are even $C^{1,\alpha}_{loc}(\Omega\times(0,T))$.

(2) Supersolutions are lower semicontinuous. (Kuusi)

Harnack estimates

$\boxed{\text{Elliptic case}}$: $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, $1<p<\infty$

$u \geq 0$ weak solution $\Longrightarrow$

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u, \quad C = C(n,p)$$

(Moser and Trudinger)

Argument:

$$\sup_{B(x,r)} u \leq C\left(\fint_{B(x,2r)} u^{\delta}\, dy\right)^{\frac{1}{\delta}}$$

(for weak subsolutions)

Moser iteration based on energy estimates and the Sobolev inequality

$$\leq C\left(\fint_{B(x,r)} u^{-\delta}\, dy\right)^{-\frac{1}{\delta}}$$

The John–Nirenberg lemma

$$\leq C \inf_{B(x,r)} u$$

(for weak supersolutions)

Moser iteration

$\boxed{\text{Parabolic case}}$: $\frac{\partial u}{\partial t} - \Delta u = 0$ (the heat equation)

Warning: There is no Harnack's inequality at a fixed moment of time.

Reason: Consider the fundamental solution.

$u \geq 0$ is a weak solution $\Rightarrow$

$$C^{-1} \sup_{B(x,r)} u(\cdot, t - r^2) \leq u(x,t) \leq C \inf_{B(x,r)} u(\cdot, t + r^2)$$

(Hadamard and Pini)

<u>Geometry</u>: $B(x,r) \times [t - r^2, t + r^2]$ (waiting time)

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

The equation is invariant under the scaling $(x,t) \mapsto (\gamma x, \gamma^p t)$, $\gamma > 0$, which suggests that the Harnack estimates could hold true in the geometry $B(x,r) \times [t - r^p, t + r^p]$. However, the Barenblatt solution shows that this is not true.

$u \geq 0$ is a weak solution, $p > 2$ $\Rightarrow$

$$C^{-1} \sup_{B(x,r)} u(\cdot, t - \theta r^p) \leq u(x,t) \leq C \inf_{B(x,r)} u(\cdot, t + \theta r^p),$$

$C = C(n,p)$

for all intrinsic cylinders

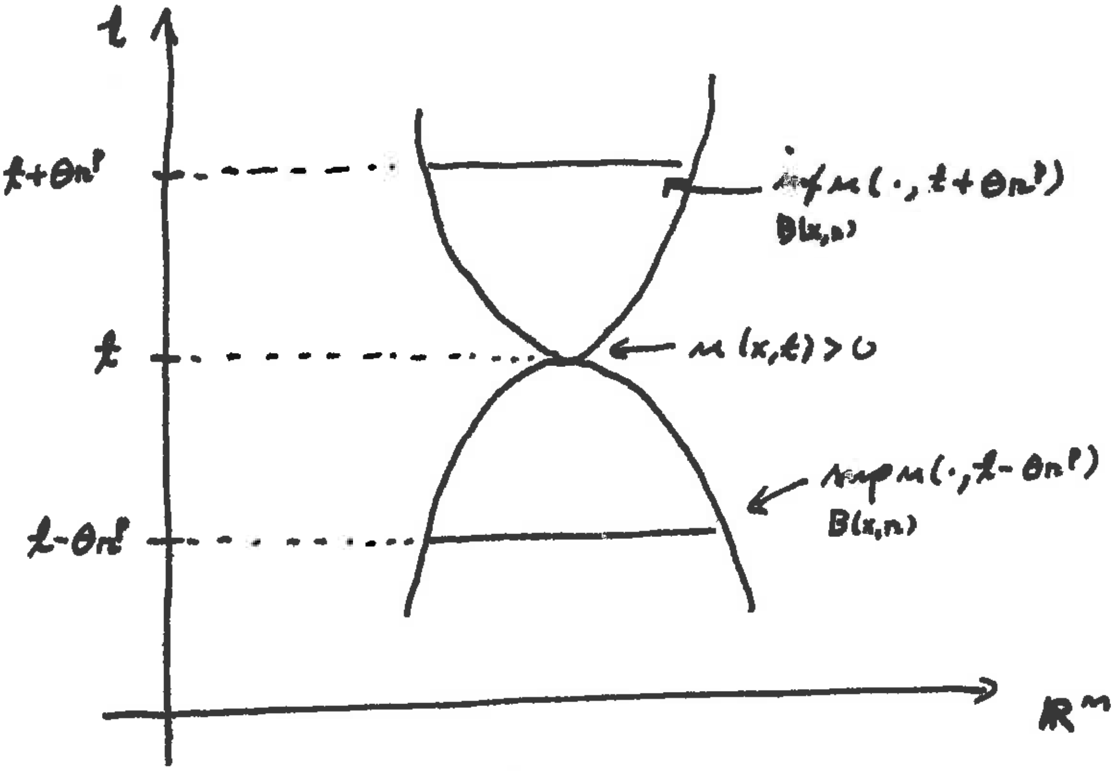
$$B(x, 4r) \times [t - \theta (4r)^p, t + \theta (4r)^p], \quad \theta = \left(\frac{C(n,p)}{u(x,t)} \right)^{p-2}$$

contained in the domain of definition of the solution.
(DiBenedetto, Gianazza, Vespri)

<u>Geometry</u>: $B(x,r) \times [t - \theta r^p, t + \theta r^p]$

depends on the solution

<u>Note</u>: The waiting time depends on the solution itself. The intrinsic geometry compensates the lack of homogeneity of the equation.



Motivation:

$$\frac{\partial}{\partial t}(cu) - \mathrm{div}\left(|\nabla(cu)|^{p-2}\nabla(cu)\right) = 0 \implies$$

$$c\frac{\partial u}{\partial t} - c^{p-1}\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \quad c>0$$

Local higher integrability of the gradient

$\boxed{\text{Elliptic case}}$: $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$, $1<p<\infty$

$u \in W^{1,p}_{loc}(\Omega)$ is a weak solution $\implies u \in W^{1,p+\varepsilon}_{loc}(\Omega)$

(Meyers & Elcrat, Giaquinta & Modica)

Note: The result holds true also for systems of the p-Laplacian type.

Tools

- Energy (Caccioppoli) estimates: $|\nabla u|$ can be controlled in terms of u: $\left(\fint_{B(x,r)} |\nabla u|^p dy\right)^{\frac{1}{p}} \le \frac{c}{r}\left(\fint_{B(x,2r)} |u - u_{B(x,2r)}|^p dy\right)^{\frac{1}{p}}$.
- Sobolev inequalities: u can be controlled in terms of $|\nabla u|$:
$$\left(\fint_{B(x,2r)} |u-u_{B(x,2r)}|^p dy\right)^{\frac{1}{p}} \le cr\left(\fint_{B(x,2r)} |\nabla u|^q dy\right)^{\frac{1}{q}}, \quad q<p.$$

Strategy:

(1) Energy estimates and Sobolev inequality $\Rightarrow$

$$\left(\fint_{B(x,r)} |\nabla u|^p dy\right)^{\frac{1}{p}} \le c\left(\fint_{B(x,2r)} |\nabla u|^q dy\right)^{\frac{1}{q}}, \quad q<p$$

(2) The self-improving property of reverse Hölder inequalities (Gehring, Fefferman & Muckenhoupt, Elcrat & Meyers) $\Rightarrow$

$$\left(\fint_{B(x,r)} |\nabla u|^{p+\varepsilon} dy\right)^{\frac{1}{p+\varepsilon}} \le c\left(\fint_{B(x,2r)} |\nabla u|^p dy\right)^{\frac{1}{p}}$$

for some $\varepsilon>0$.

$\boxed{\text{Parabolic case}}$: $\boxed{p=2}$:

A similar argument gives

$$\left(\fint_{Q_{r,r^2}} |\nabla u|^{2+\varepsilon} dx\,dt\right)^{\frac{1}{2+\varepsilon}} \le c\left(\fint_{Q_{2r,(2r)^2}} |\nabla u|^2 dx\,dt\right)^{\frac{1}{2}},$$

$$Q_{r,r^2} = B(x,r)\times(t-r^2, t+r^2)$$

(Giaquinta & Struwe)

$\boxed{p>2}$: $|\nabla u| \approx \lambda$, $\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Q_{r,r^2}

$$\Rightarrow \frac{\partial u}{\partial t} - \lambda^{p-2}\Delta u = 0 \text{ in } Q_{r,\cancel{\lambda^{2-p}}r^p}$$

$$\Rightarrow \frac{\partial u}{\partial t} - \Delta u = 0 \text{ in } Q_{r,\lambda^{2-p}r^2}$$

Note: The parabolic p-Laplace equation becomes the heat equation in the intrinsic geometry.

Advantages: Homogeneous estimates and scaling.

Theorem. $u \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega))$ is a weak solution to the parabolic p-Laplace equation $\Rightarrow \nabla u \in L^{p+\varepsilon}_{loc}(\Omega_T)$ for some $\varepsilon>0$.

Strategy: Intrinsic reverse Hölder inequality

$$\left(\fint_{Q_{r,\lambda^{2-p}r^2}} |\nabla u|^p \,dx\,dt\right)^{\frac{1}{p}} \le c \left(\fint_{Q_{2r,\lambda^{2-p}(2r)^2}} |\nabla u|^q \,dx\,dt\right)^{\frac{1}{q}}, \quad q<p,$$

$\lambda \approx |\nabla u|$

$\left(\lambda \approx \left(\fint |\nabla u|^p \,dx\,dt\right)^{\frac{1}{p}}\right)$

(K. & Lewis)

Tools

- Energy estimate:

$$\int_0^T\int_\Omega |\nabla u|^p|\varphi|^p \,dx\,dt + \operatorname*{ess\,sup}_{0<t<T}\int_\Omega |u|^2\varphi^p\,dx$$
$$\le c\int_0^T\int_\Omega |u|^2\left|\frac{\partial(\varphi^p)}{\partial t}\right| dx\,dt + c\int_0^T\int_\Omega |u|^p|\nabla\varphi|^p\,dx\,dt,$$

$\varphi \in C_0^\infty(\Omega_T)$

Reason: Choose $u\varphi^p$ as a test function.

- The Sobolev inequality: $u \in L^p(0,T; W_0^{1,p}(\Omega)) \Rightarrow$

$$\int_0^T \int_\Omega |u|^{p(1+\frac{2}{n})} \,dx\,dt \le c \int_0^T \int_\Omega |\nabla u|^p \,dx\,dt \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u|^2 dx \right)^{\frac{p}{n}}.$$

<u>Reason</u>: Apply the standard Sobolev inequality on time slices.

<u>Challenges</u>

(1) The integral average $⨍ u \,dx\,dt$ can be subtracted in the Caccioppoli estimate, but there is a slice-wise integral average in the Sobolev inequality, see 12.1.

(2) There are two competing exponents p and 2 in the energy estimate.

<u>Estimates below the natural exponent</u>

$\boxed{\text{Elliptic case}}$: $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, $1<p<\infty$.

A function $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation, if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \,dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we may choose $\varphi \in W_0^{1,p}(\Omega)$ as well. In particular, if $u \in W_0^{1,p}(\mathbb{R}^n)$, then

$$0 = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u \,dx = \int_\Omega |\nabla u|^p dx$$

$\Rightarrow \nabla u = 0. \Rightarrow u = 0$

$$\left| \fint_{B(x,r)} u(y,t_2)\,dy - \fint_{B(x,r)} u(y,t_1)\,dy \right|$$

$$= \left| \fint_{B(x,r)} u(y,t_2) - u(y,t_1)\,dy \right|$$

$$= \left| \fint_{B(x,r)} \int_{t_1}^{t_2} \frac{\partial u}{\partial t}(x,t)\,dt\,dy \right|$$

$$= \left| \fint_{B(x,r)} \int_{t_1}^{t_2} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) dt\,dy \right|$$

$$= \left| \int_{t_1}^{t_2} \frac{1}{c r^n} \int_{\partial B(x,r)} |\nabla u|^{p-2}\nabla u\cdot\nu\,dS \right|$$

the divergence (Gauss-Green) theorem

$$\leq \frac{C}{R^n} \int_{t_1}^{t_2} \int_{\partial B(x,r)} |\nabla u|^{p-1}\,dS\,dt$$

Choose $R < r \leq 2R$ s.t.

$$\int_{t_1}^{t_2} \int_{\partial B(x,r)} |\nabla u|^{p-1}\,dS\,dt \leq \frac{100}{R} \int_{t_1}^{t_2} \int_{B(x,2R)} |\nabla u|^{p-1}\,dx\,dt.$$

Reason: $\displaystyle\int_{t_1}^{t_2} \int_{\partial B(x,r)} |\nabla u|^{p-1}\,dS\,dt \geq \frac{100}{R} \int_{t_1}^{t_2} \int_{B(x,2R)} |\nabla u|^{p-1}\,dx\,dt$

$$\Rightarrow \int_{t_1}^{t_2} \int_{R}^{2R} \int_{\partial B(x,r)} |\nabla u|^{p-1}\,dS\,dr\,dt \geq \frac{100}{R} \int_{t_1}^{t_2} \int_{B(x,2R)} |\nabla u|^{p-1}\,dx\,dt \cdot R$$

$$\wedge|$$

$$\int_{t_1}^{t_2} \int_{B(x,2R)} |\nabla u|^{p-1}\,dx\,dt$$

The definition of a weak solution makes sense under a weaker assumption that $u \in W^{1,q}_{loc}(\Omega)$ for some $p-1 \le q \le p$. These are called very weak solutions.

<u>Warning</u>: For a very weak solution, we cannot use the solution itself as a test function.

<u>Lemma</u>. $u \in W^{1,q}(\mathbb{R}^n)$, $p-1 < q < p$, is a very weak solution $\Rightarrow$

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = 0$$

for every $\varphi \in W^{1,q}(\mathbb{R}^n)$ with $\nabla \varphi \in L^{\frac{q}{q-p+1}}(\mathbb{R}^n)$.

<u>Proof</u>: $\varphi_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, 2, \dots$, s.t. $\nabla \varphi_i \to \nabla \varphi$ in $L^{\frac{q}{q-p+1}}(\mathbb{R}^n)$

$$\left| \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \right| = \left| \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot (\nabla \varphi - \nabla \varphi_i) \, dx \right|$$

$$\le \int_{\mathbb{R}^n} |\nabla u|^{p-1} |\nabla \varphi - \nabla \varphi_i| \, dx$$

$$\underset{\text{Hölder}}{\le} \left(\int_{\mathbb{R}^n} |\nabla u|^q \, dx \right)^{\frac{p-1}{q}} \left(\int_{\mathbb{R}^n} |\nabla \varphi - \nabla \varphi_i|^{\frac{q}{q-p+1}} dx \right)^{\frac{q-p+1}{q}} \to 0. \qquad \square$$

<u>Theorem</u>. There is $q < p$ s.t. if $u \in W^{1,q}_{loc}(\Omega)$ is a very weak solution, then $u \in W^{1,p}_{loc}(\Omega)$ and thus u is a weak solution.

Proof: We consider the following global version of the claim:

$$u \in W^{1,q}(\mathbb{R}^n) \Rightarrow u = 0.$$

The following pointwise estimate holds true for all $u \in W^{1,q}(\mathbb{R}^n)$:

$$|u(x) - u(y)| \le c\,|x-y|\,\big(M|\nabla u|(x) + M|\nabla u|(y)\big), \text{ a.e. } x,y \in \mathbb{R}^n,$$

where M is the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy.$$

Let $E_\lambda = \{x \in \mathbb{R}^n : M|\nabla u|(x) > \lambda\}$, $\lambda > 0$.

$u|_{\mathbb{R}^n \setminus E_\lambda}$ is $c\lambda$-Lipschitz

$$\underset{\text{Rademacher}}{\overset{}{\Rightarrow}} \exists\ c\lambda\text{-Lipschitz } u_\lambda : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } u_\lambda = u \text{ on } \mathbb{R}^n \setminus E_\lambda$$

$$\underset{\text{Lemma}}{\Rightarrow} \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla u_\lambda \, dx = 0$$

$\nabla u_\lambda \in L^\infty(\mathbb{R}^n) \subset L^{p-\varepsilon}(\mathbb{R}^n)$

$$\Rightarrow \int_{\mathbb{R}^n \setminus E_\lambda} |\nabla u|^p\,dx = -\int_{E_\lambda} |\nabla u|^{p-2} \nabla u \cdot \nabla u_\lambda \, dx$$

$$\le c\lambda \int_{E_\lambda} |\nabla u|^{p-1}\,dx, \quad \lambda > 0$$

$$\Rightarrow \int_0^\infty \lambda^{-\varepsilon-1}\int_{M|Du|\le\lambda}|Du|^p\,dx\,d\lambda \le c\int_0^\infty \lambda^{-\varepsilon}\int_{M|Du|>\lambda}|Du|^{p-1}\,dx\,d\lambda$$

$$\begin{array}{ccc} \| & & \| \\ \frac{1}{\varepsilon}\int_{\mathbb{R}^n}|Du|^p(M|Du|)^{-\varepsilon}\,dx & & \frac{\varepsilon}{1-\varepsilon}\int_{\mathbb{R}^n}|Du|^{p-1}(M|Du|)^{1-\varepsilon}\,dx \end{array}$$

$$\Rightarrow \int_{\mathbb{R}^n}|Du|^p(M|Du|)^{-\varepsilon}\,dx \le c\,\frac{\varepsilon}{\varepsilon-1}\int_{\mathbb{R}^n}|Du|^{p-1}(M|Du|)^{1-\varepsilon}\,dx$$

$$\int_{\mathbb{R}^n}(M|Du|)^{p-\varepsilon}\,dx \le c\int_{\mathbb{R}^n}|Du|^{p-\varepsilon}\,dx$$

The maximal function theorem

$$= c\int_{|Du|\le\delta M|Du|}|Du|^{p-\varepsilon}\,dx + c\int_{|Du|>\delta M|Du|}|Du|^{p-\varepsilon}\,dx$$

$$\le c\,\delta^{p-\varepsilon}\int_{\mathbb{R}^n}(M|Du|)^{p-\varepsilon}\,dx + c\,\delta^{-\varepsilon}\int_{\mathbb{R}^n}|Du|^p(M|Du|)^{-\varepsilon}\,dx.$$

By choosing $\delta>0$ small enough

$$\int_{\mathbb{R}^n}(M|Du|)^{p-\varepsilon}\,dx \le c\int_{\mathbb{R}^n}|Du|^p(M|Du|)^{-\varepsilon}\,dx$$

$$\le c\,\frac{\varepsilon}{-\varepsilon+1}\int_{\mathbb{R}^n}|Du|^{p-1}(M|Du|)^{1-\varepsilon}\,dx$$

$$\le c\,\frac{\varepsilon}{1-\varepsilon}\int_{\mathbb{R}^n}(M|Du|)^{p-1}(M|Du|)^{1-\varepsilon}\,dx$$

$$= c\,\frac{\varepsilon}{1-\varepsilon}\underbrace{\int_{\mathbb{R}^n}(M|Du|)^{p-\varepsilon}\,dx}_{<\infty,\ \text{if } |Du|\in L^{p-\varepsilon}(\mathbb{R}^n)}$$

$$\Rightarrow \nabla u = 0 \overset{u \in W^{1,q}(\mathbb{R}^n)}{\Rightarrow} u = 0 \qquad \square$$

choose $\varepsilon > 0$ small enough

$\boxed{\text{Parabolic case}}$: A similar result holds true for the parabolic p-Laplace equation: If $u \in L^{p-\varepsilon}_{loc}(0,T; W^{1,p-\varepsilon}_{loc}(\Omega))$ is a very weak solution, then $u \in L^{p}_{loc}(0,T; W^{1,p}_{loc}(\Omega))$ and thus u is a weak solution. By the local higher integrability for the gradient, we have $u \in L^{p+\varepsilon}_{loc}(0,T; W^{1,p+\varepsilon}_{loc}(\Omega))$.

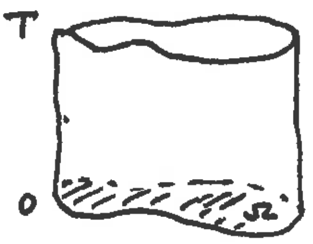
<u>Note</u>: From an exponent below the natural one, we reach an exponent above the natural one.

<u>Comparison principle</u>

Let $\Omega \subset \mathbb{R}^n$ be bounded. Assume that u is a weak supersolution and v is a weak subsolution in $\Omega \times (0,T)$. If u and $-v$ are lower semicontinuous in $\overline{\Omega \times (0,T)}$ and $v \le u$ on the parabolic boundary

$$\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup \partial\Omega \times [0,T]),$$

then $v \le u$ a.e. in $\Omega \times (0,T)$.



<u>Note</u>: Every supersolution has a lower semicontinuous representative, thus we may consider pointwise defined functions.

<u>Remark</u>. The comparison principle implies uniqueness of the solution to an initial-boundary value problem in $\Omega \times (0,T)$.

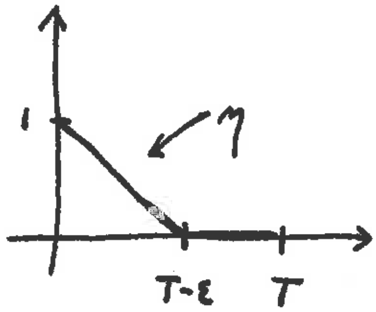
Proof: $\varphi \in C_0^\infty(\Omega \times (0,T))$, $\varphi \geq 0$

$$\int_0^T \int_\Omega \left(\left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \varphi + (v-u) \frac{\partial \varphi}{\partial t} \right) dx\, dt \geq 0$$

Replace u with $u+\varepsilon$, $\varepsilon > 0$.

Choose $\varphi = (v-u-\varepsilon)_+ \eta$, $0 < \varepsilon < T$,

$$\eta(t) = \begin{cases} \dfrac{T-\varepsilon-t}{T-\varepsilon}, & t < T-\varepsilon, \\ 0, & t \geq T-\varepsilon \end{cases}$$



$$\int_0^T \int_{v \geq u+\varepsilon} \underbrace{\eta \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot (\nabla u - \nabla v)}_{\geq 0} dx\, dt$$

$$\leq \int_\Omega \int_0^T (v-u-\varepsilon)_+^2 \frac{\partial \eta}{\partial t}\, dt\, dx + \frac{1}{2} \int_\Omega \int_0^T \eta \frac{\partial}{\partial t} \left((v-u-\varepsilon)_+^2 \right) dt\, dx$$

$$= \frac{1}{2} \int_\Omega \int_0^T (v-u-\varepsilon)_+^2 \frac{\partial \eta}{\partial t}\, dt\, dx$$

integrate by parts

$$= -\frac{1}{2(T-\varepsilon)} \int_\Omega \int_0^{T-\varepsilon} (v-u-\varepsilon)_+^2\, dt\, dx \leq 0$$

$\Rightarrow (v-u-\varepsilon)_+ = 0$ in $\Omega \times (0, T-\varepsilon)$

$\Rightarrow v \leq u+\varepsilon$ in $\Omega \times (0, T-\varepsilon)$

Supercaloric functions

A function $v : \Omega_T \to (-\infty, \infty]$ is p-supercaloric, if

(1) v is lower semicontinuous,

(2) $v < \infty$ in a dense subset of Ω_T and

(3) v satisfies the comparison principle in every interior cylinder $D_{t_1,t_2} \Subset \Omega_T$: If $u \in C(\overline{D_{t_1,t_2}})$ is a weak solution to the parabolic p-Laplace equation in D_{t_1,t_2} and $v \geq u$ on $\partial_p D_{t_1,t_2}$, then $v \geq u$ in D_{t_1,t_2}.

$\left(v \geq u \text{ on } \sqcup \Rightarrow v \geq u \text{ in } \blacksquare \right)$

Properties

(1) A lower semicontinuous representative of a weak supersolution is supercaloric.

<u>Reason</u>: The comparison principle.

(2) A locally bounded p-supercaloric function is a weak supersolution. In particular, the truncations

$$\min\{v, k\}, \quad k = 1, 2, \ldots,$$

are supersolutions. (K. & Lindqvist)

<u>Idea of the proof</u>: Approximate a given supercaloric function pointwise by an increasing sequence of weak supersolutions constructed through successive obstacle problems. By the boundedness assumption and energy estimates, the limit function is a weak supersolution.

Note: There are no other bounded p-supercaloric functions than weak supersolutions, once the question of lower semicontinuity is taken into account. Thus if we are only interested in bounded functions, these classes coincide.

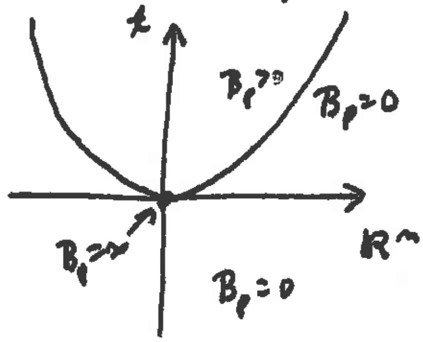
As we shall see, there are several ways to construct unbounded p-supercaloric functions that are not weak supersolutions. Thus in general, these are different classes of functions.

Examples

(1) The Barenblatt solution $B_p : \mathbb{R}^{n+1} \to [0,\infty)$,

$$B_p(x,t) = \begin{cases} t^{-\frac{n}{\lambda}}\left(C - \frac{p-2}{p}\lambda^{\frac{1}{1-p}}\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, & t>0, \\ 0, & t \le 0, \end{cases}$$

where $\lambda = n(p-2)+p$, $p>2$, is p-supercaloric in $\mathbb{R}^{n+1}$.



- B_p is a weak solution in $\mathbb{R}^{n+1}_+$
- $B_p \in L^{p-1+\frac{p}{n}-\varepsilon}_{loc}(\mathbb{R}^{n+1})$ for every $\varepsilon>0$
- $\nabla B_p \in L^{p-1+\frac{1}{n+1}-\varepsilon}_{loc}(\mathbb{R}^{n+1})$ for every $\varepsilon>0$
- $\displaystyle\int_{-1}^{1}\int_{B(0,1)} |\nabla B_p(x,t)|^p \,dx\,dt = \infty \Rightarrow B_p \notin L^p_{loc}(-\infty,\infty; W^{1,p}_{loc}(\mathbb{R}^n))$

  $\Rightarrow B_p$ is not a weak supersolution in $\mathbb{R}^{n+1}$.

- $\min\{B_p(x,t), k\}$, $k=1,2,\dots$, are weak supersolutions.

(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded and regular domain. The friendly giant, obtained by separation of variables, is

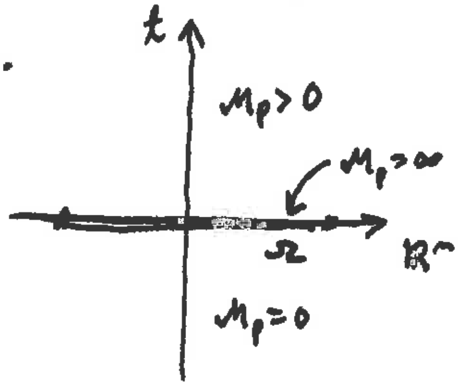
$$\mathcal{M}_p(x,t) = \begin{cases} t^{-\frac{1}{p-2}} u(x), & t>0,\ x\in\Omega, \\ 0, & t\le 0, \end{cases}$$

where $u \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is a weak solution to the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \frac{1}{p-2}u = 0$$

with $u=0$ on $\partial\Omega$ and $u>0$ in Ω. Such a solution exists by the direct methods in the calculus of variations and positivity follows the Harnack inequality. The friendly giant is p-supercaloric in $\Omega\times\mathbb{R}$.

- $\mathcal{M}_p$ is a weak solution in $\cancel{\mathbb{R}^{n+1}_{+}}$ $\Omega\times\mathbb{R}_+$.
- $\mathcal{M}_p \notin L^{p-2}_{loc}(\Omega\times\mathbb{R})$ ~~[illegible]~~.
- $\displaystyle\lim_{\substack{(y,t)\to(x,0)\\ t>0}} \mathcal{M}_p(y,t) = \infty$ for every $x\in\Omega$.
- $\mathcal{M}_p$ is not a weak supersolution.



(3) A p-supercaloric function may blow up arbitrarily fast. For example,

$$v(x,t) = \begin{cases} u(x)\, e^{\frac{1}{(p-2)t}}, & t>0,\ x\in\Omega, \\ 0, & t\le 0, \end{cases}$$

and u is as in (2), blows up exponentially.

Classification of supercaloric functions

There are two mutually exclusive classes of p-supercaloric functions in the case $p>2$.

Class $\mathcal{B}$: The following conditions are equivalent for a p-supercaloric function :

(1) $v \in L^{p-2}_{loc}(\Omega_T)$,

(2) $\nabla v \in L^{p-1+\frac{1}{n-1}-\varepsilon}_{loc}(\Omega_T)$ for every $\varepsilon>0$, $\left(p-1+\frac{1}{n-1} < p\right)$

(3) $v \in L^{p-1+\frac{p}{n}-\varepsilon}(\Omega_T)$ for every $\varepsilon > 0$. $\left(p-2 < p-1+\frac{p}{n}\right)$

Note : The Barenblatt solution belongs to class $\mathcal{B}$. This shows that the exponents are optimal.

Remark. There is a remarkable self-improving property: If $v \in L^{p-2}_{loc}(\Omega_T)$, then $v \in L^{q}_{loc}(\Omega_T)$ for every $q < p-1+\frac{p}{n}$. This can be shown by a Moser type iteration argument.

Class $\mathcal{M}$: The following conditions are equivalent for a p-supercaloric function :

(1) $v \notin L^{p-2}_{loc}(\Omega_T)$

(2) there is a time t_0, $0 < t_0 < T$, such that

$$\liminf_{\substack{(y,t)\to(y,t_0)\\ t>t_0}} v(y,t)\,(t-t_0)^{\frac{1}{p-2}} > 0$$

for every $x \in \Omega$.

Note: The Friendly giant belongs to class $\mathcal{M}$. This shows that functions in class $\mathcal{M}$ blow up at least with the rate given by the Friendly giant.

Remark. Functions in class $\mathcal{M}$ blow up at every point $x \in \Omega$ for a certain moment of time t_0.

Measure data problem

The Barenblatt solution satisfies the equation

$$\frac{\partial \mathcal{B}_p}{\partial t} - \operatorname{div}(|\nabla \mathcal{B}_p|^{p-2} \nabla \mathcal{B}_p) = \delta \quad \text{in } \mathbb{R}^{n+1},$$

that is

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \left(|\nabla \mathcal{B}_p|^{p-2} \nabla \mathcal{B}_p \cdot \nabla \varphi - \mathcal{B}_p \frac{\partial \varphi}{\partial t} \right) dx\, dt = \varphi(0)$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$. This is a special case of a more general result for functions in class $\mathcal{B}$.

Theorem. If v ~~belongs to class $\mathcal{B}$~~ is a p-superparabolic function which belongs to class $\mathcal{B}$, then there exists a nonnegative Radon measure μ s.t.

$$\frac{\partial v}{\partial t} - \operatorname{div}(|\nabla v|^{p-2} \nabla v) = \mu,$$

that is,

$$\int_0^T \int_\Omega \left(|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi - v \frac{\partial \varphi}{\partial t} \right) dx\, dt = \int_0^T \int_\Omega \varphi \, d\mu$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

<u>Proof</u>: The classification result above $\Rightarrow$ $v, \nabla v \in L^{p-1}_{loc}(\Omega_T)$

$\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$

$v_k = \min\{v, k\}$, $k = 1, 2, \ldots$, are weak supersolutions

$$L_\mu : C_0^\infty(\Omega_T) \to \mathbb{R}, \quad L_\mu(\varphi) = \underbrace{\int_0^T \int_\Omega \left(|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi - v \frac{\partial \varphi}{\partial t} \right) dx\, dt}_{< \infty}$$

$$= \lim_{k \to \infty} \int_0^T \int_\Omega \left(|\nabla v_k|^{p-2} \nabla v_k \cdot \nabla \varphi - v_k \frac{\partial \varphi}{\partial t} \right) dx\, dt \geq 0$$

Lebesgue dominated convergence

The Riesz representation theorem gives the claim. $\square$

$\boxed{\text{THE END}}$