On the definition and properties of super-parabolic functions

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The p-parabolic equation

The evolutionary p-Laplace (or p-parabolic) equation is

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty.$$

When $p = 2$ we have the standard heat equation.

In this talk, we assume that $p \geq 2$.

More general structure conditions

Our arguments also apply to more general equations of the type

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(\nabla u) = 0,$$

where $\mathcal{A}$ is a Carathéodory function and satisfies the standard structural conditions

$$\mathcal{A}(\nabla u) \cdot \nabla u \geq \alpha |\nabla u|^p$$

and

$$|\mathcal{A}(\nabla u)| \leq \beta |\nabla u|^{p-1},$$

where α and β are positive constants.

Sometimes $\mathcal{A}$ is also assumed to be monotone.

Structure

The equation is nonlinear: The sum of two solutions is NOT a solution, in general.

Constants CAN be added to solutions.

Solutions CANNOT be scaled.

The minimum of two (super)solutions is a supersolution. In particular, the truncations

$$\min(u, k), \quad k = 1, 2, \ldots,$$

are supersolutions.

Classes of supersolutions

(i) Weak supersolutions (test functions under the integral).

(ii) Superparabolic functions (defined through a comparison principle).

(iii) (Very) weak solutions to a measure data problem.

The plan of the talk

I. Superparabolic (supercaloric) functions.

II. Existence and connections to a measure data problem.

III. Nonlinear parabolic (thermal) capacity and the infinity set of a superparabolic function.

I. Superparabolic functions

Weak solutions

Let $\Omega \subset \mathbb{R}^n$ be an open set and $T > 0$. A function

$$u \in L^p_{\text{loc}}(0, T\, W^{1,p}_{\text{loc}}(\Omega)),$$

is a weak solution of

$$\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in $\Omega_T = \Omega \times (0, T)$ if

$$\int_0^T \int_\Omega \Big(|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi - u\frac{\partial \varphi}{\partial t}\Big)\, dx\, dt = 0$$

for all $\varphi \in C_0^\infty(\Omega_T)$.

If the integral ≥ 0 for all $\varphi \geq 0$, then u is a supersolution.

Regularity

(i) By parabolic regularity theory, solutions satisfy an intrinsic parabolic Harnack inequality and are locally Hölder continuous.

(DiBenedetto, Gianazza and Vespri, Acta Math., 2008)

(ii) Supersolutions are lower semicontinuous.

(Kuusi, Diff. Int. Eq., 2009)

OBSERVE: No regularity in time is assumed, in particular, for supersolutions.

The fundamental solution

The Barenblatt solution $\mathcal{B}_p : \mathbb{R}^{n+1} \to [0, \infty)$ is defined as

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-\frac{n}{\lambda}} \left(c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\lambda = n(p-2)+p$, $p > 2$, and the constant c is usually chosen so that

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x,t)\, dx = 1$$

for every $t > 0$.

When $p \to 2$, the Barenblatt solution converges to the heat kernel.

Motivation

The Barenblatt solution should be the worst possible (super)solution, although the principle of superposition is not available in the nonlinear case.

Properties

The Barenblatt solution is a weak solution in the upper half space

$$\{(x,t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ t > 0\}$$

and it satisfies the equation

$$\frac{\partial \mathcal{B}_p}{\partial t} - \operatorname{div}(|\nabla \mathcal{B}_p|^{p-2} \nabla \mathcal{B}_p) = \delta$$

in $\mathbb{R}^{n+1}$, where the right-hand side is Dirac's delta at the origin, that is

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \left(|\nabla \mathcal{B}_p|^{p-2} \nabla \mathcal{B}_p \cdot \nabla \varphi - \mathcal{B}_p \frac{\partial \varphi}{\partial t} \right) dx\, dt = \varphi(0)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$.

In contrast with the heat kernel, which is strictly positive, the Barenblatt solution has a bounded support at every $t > 0$. Hence, disturbancies propagate with finite speed.

An unexpected feature

The Barenblatt solution is not a supersolution in an open set that contains the origin. Indeed,

$$\int_{-1}^{1}\int_{Q}|\nabla\mathcal{B}_p(x,t)|^p\,dx\,dt=\infty,$$

where $Q=[-1,1]^n\subset\mathbb{R}^n$. Hence

$$\mathcal{B}_p\notin L^p_{\text{loc}}(-\infty,\infty;W^{1,p}_{\text{loc}}(\mathbb{R}^n))$$

and is so-called VERY weak solution to a measure data problem.

In contrast, the truncated functions

$$\min(\mathcal{B}_p(x,t),k),\qquad k=1,2,\ldots,$$

belong to the correct space and are supersolutions.

OBSERVE: An increasing limit of superolutions may fail to be a supersolution.

Question

What is the class of supersolutions that is closed under the increasing convergence?

Superparabolic functions

A function $v : \Omega_T \to (-\infty, \infty]$ is called p-superparabolic, if

(i) v is finite in a dense subset of Ω_T,

(ii) v is lower semicontinuous,

(iii) v satisfies the following comparison principle in every $D_{t_1,t_2} = D \times (t_1, t_2)$ with $D_{t_1,t_2} \Subset \Omega_T$: if h is a solution in D_{t_1,t_2} and continuous in $\overline{D_{t_1,t_2}}$ and if $h \leq v$ on the parabolic boundary of D_{t_1,t_2}, then $h \leq v$ in D_{t_1,t_2}.

(Kilpeläinen and Lindqvist, SIAM J. Math Anal., 1996)

(K. and Lindqvist, J. reine Angew. Math., 2008)

When $p = 2$ we have supercaloric functions (supertemperatures).

Remarks

(1) A superparabolic function does not, a priori, belong to a Sobolev space. The only connection to the equation is through the comparison principle.

(2) It is enough to compare in boxes instead of all cylindrical subdomains

(Korte, Kuusi and Parvianen, J. Evol. Eq., 2010).

Examples

(1) A lower semicontinuous representative of a supersolution is a superparabolic function.

(2) The Barenblatt solution is a superparabolic function, but not a supersolution.

(3) Any function of the form

$$v(x,t) = f(t),$$

where f is a monotone increasing lower semicontinuous function is superparabolic.

Increasing convergence

The class of superparabolic functions is closed under increasing limits, provided the limit function is finite on a dense set.

OBSERVE: Supersolutions DO NOT have this property, unless the limit function is assumed to be bounded or to belong to the correct parabolic Sobolev space.

(K. and Lindqvist, Ann. Mat. Pura Appl., 2006)

(Korte, Kuusi and Parviainen, J. Evol. Eq., 2010)

A delicate point

The time derivative can be assumed to be an object belonging to the dual of the parabolic Sobolev space, but this approach does not give a class of supersolutions which is closed under bounded increasing convergence.

EXAMPLE. The function $v : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$v(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a supersolution and it can easily be approximated by an increasing sequence of smooth supersolutions which only depend on the time variable. However, the time derivative of v does not belong to the dual of the parabolic Sobolev space.

Viscosity = Superparabolic

Superparabolic functions are precisely the viscosity supersolutions.

(Juutinen, Lindqvist and Manfredi, SIAM J. Mat. Anal., 2001).

In particular, all results for superparabolic functions hold also for viscosity supersolutions.

Questions

Is a superparabolic function a solution in any other sense than the viscosity sense?

What can be said about the Sobolev space regularity of a superparabolic function?

A summability result

Suppose that v is a superparabolic function. Then $v \in L^q_{\mathrm{loc}}(\Omega_T)$ for every q with

$$0 < q < p - 1 + \frac{p}{n}.$$

Moreover, the weak gradient exists and $\nabla v \in L^q_{\mathrm{loc}}(\Omega_T)$, whenever

$$0 < q < p - 1 + \frac{1}{n+1}$$

and the function is a very weak solution. This result is optimal, as the Barenblatt solution shows.

In addition, every bounded superparabolic function is a weak supersolution.

(K. and Lindqvist, Ann. Sc. Norm. Sup. Pisa, 2005)

When $p = 2$ the result follows from representation formulas.

Warning

The exponent for the gradient is strictly smaller than the natural exponent p. Hence, a superparabolic function is only a VERY weak supersolution.

It seems to be difficult to obtain estimates for the very weak solutions, since we cannot use the very weak solution as a test function.

(K. and Lewis, Duke Math. J., 2000)

(K. and Lewis, Ark. Mat. , 2002)

The infimal convolution

Let $0 \leq v \leq L$. For $\epsilon > 0$, define

$$v_\varepsilon(x,t) = \inf_{(y,\tau)\in\Omega_T} \left\{ v(y,\tau) + \frac{|x-y|^2 + |t-\tau|^2}{2\varepsilon} \right\},$$

Properties

(i) $v_\varepsilon \to v$ as $\varepsilon \to 0$.

(ii) v_ε is locally Lipschitz continuous in Ω_T.

(iii) v_ε is a supersolution in the set

$$\{(x,t) \in \Omega_T : \text{dist}((x,t), \partial\Omega_T) > \sqrt{2L\varepsilon}\}.$$

Proof of the summability result

Step 1: First assume that v is bounded.

Step 2: Approximate v with infimal convolutions v_ε.

Step 3: v_ε is a supersolution.

Step 4: Caccioppoli estimates for v_ε.

Step 5: Caccioppoli estimates are passed over from v_ε to v. This concludes the proof for bounded functions.

Step 6: The unbounded case is reached via $\min(v, k)$, $k \to \infty$.

Step 7: Estimates do not blow up as $k \to \infty$ under the assumption that the boundary values are zero on the parabolic boundary.

Step 8: A construction which reduces the proof to the zero boundary values.

The main estimate

Assume that v is a superparabolic function with zero boundary values on the parabolic boundary. Let $v_k = \min(v, k)$. Then

$$\int_0^T \int_\Omega |\nabla v_k|^p \, dx \, dt + \operatorname*{ess\,sup}_{0<t<T} \int_\Omega v_k^2 \, dx \le Mk$$

for every $k = 1, 2, \ldots$

Proof. Choose the test functions

$$\varphi_j = (v_j - v_{j-1}) - (v_{j+1} - v_j), \quad j = 1, 2, \ldots, k$$

Summability

Let

$$E_k = \{(x,t) \in \Omega_T : k \le v(x,t) < 2k\}, \quad k = 1, 2, \ldots$$

and $\kappa = 1 + \frac{2}{n}$. Then

$$\begin{aligned}
k^{\kappa p}|E_k| &\le \iint_{E_k} v_{2k}^{\kappa p}\,dx\,dt \\
&\le \int_0^T \int_\Omega v_{2k}^{\kappa p}\,dx\,dt \\
&\le C \int_0^T \int_\Omega |\nabla v_{2k}|^{\kappa p}\,dx\,dt \Big(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega v_{2k}^2\,dx \Big)^{\frac{p}{n}} \\
&\le C M^{1+\frac{p}{n}} (2k)^{1+\frac{p}{n}}.
\end{aligned}$$

This implies

$$|E_k| \le C k^{1-p-\frac{p}{n}}, \quad k = 1, 2, \ldots$$

Summability concluded

$$\begin{aligned}\int_0^T \int_\Omega v^q \, dx \, dt &\le |\Omega_T| + \sum_{k=1}^{\infty} \iint_{E_{2^{k-1}}} v^q \, dx \, dt \\ &\le |\Omega_T| + \sum_{k=1}^{\infty} 2^{kq} |E_{2^{k-1}}| \\ &\le |\Omega_T| + C \sum_{k=1}^{\infty} 2^{k(q+1-p-\frac{p}{n})} < \infty.\end{aligned}$$

The estimate for the gradient is similar.

II. Equations with measure data

Question

The Barenblatt solution is a very weak solution of a measure data problem with Dirac's delta.

Is every superparabolic function a solution to a measure data problem?

Theorem. Let v be a superparabolic function. Then there exists a positive Radon measure μ such that v satisfies

$$\frac{\partial v}{\partial t} - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = \mu$$

in a very weak sense, that is,

$$\int_0^T \int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla \varphi - v\frac{\partial \varphi}{\partial t}\right) dx\, dt = \int_0^T \int_\Omega \varphi\, d\mu$$

for all $\varphi \in C_0^\infty(\Omega_T)$.

The measure μ is called the Riesz measure of v.

Proof. The summability result implies that

$$v,\ \nabla v \in L^{p-1}_{\mathrm{loc}}(\Omega_T).$$

Since the truncations are supersolutions,

$$\int_0^T \int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla \varphi - v\frac{\partial \varphi}{\partial t}\right) dx\, dt \geq 0$$

for every $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi \geq 0$. The claim follows from the Riesz representation theorem.

Existence

Let μ be a finite positive Radon measure in Ω_T. Then there is a superparabolic function v in Ω_T such that

$$\frac{\partial v}{\partial t} - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = \mu$$

in the weak sense.

(K., Lukkari and Parviainen, J. Funct. Anal., 2010)

(K., Lukkari and Parviainen, submitted)

REMARK. In addition, we can show that v has zero boundary values on the parabolic boundary of Ω_T.

Remarks

(1) If μ belongs to the dual of the parabolic Sobolev space, then by functional analysis there exists a unique solution of the above problem with given boundary values.

However, Dirac's delta does not belong to this class, so that the fundamental solutions are not covered by this.

(2) Uniqueness of the solution of the measure data problem with given boundary values is an open problem, since we cannot use the very weak solution as a test function.

Proof of the existence

Step 1: Approximate μ with smoother measures μ_i.

Step 2: Solve the corresponding problem with μ_i.

Step 3: Show that there exists a hyperparabolic function v such that

$$v_i \to v \quad \text{and} \quad \nabla \min(v_i, k) \to \nabla \min(v, k)$$

as $i \to \infty$.

Step 4: Using the summablity results, show that $v < \infty$ on a dense subset.

III. Nonlinear parabolic capacity

Motivation

The concept of capacity is of fundamental importance in potential theory.

(1) The Wiener type criterion for the boundary regularity is expressed in terms of capacity.

(2) Removable sets are characterized through capacity.

(3) The infinity set of a superharmonic function can be characterized as a set of zero capacity.

Definition

Let $\Omega \subset \mathbb{R}^n$ be a regular bounded open set and

$$\Omega_\infty = \Omega \times (0, \infty).$$

The parabolic capacity of $E \subset \Omega_\infty$ is

$$\begin{aligned} &\mathrm{cap}\,(E) \\ &= \sup\{\mu(\Omega_\infty) : 0 \le v_\mu \le 1,\, \mathrm{supp}\,\mu \subset E\}, \end{aligned}$$

where μ is a Radon measure, and v_μ is a solution of

$$\frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu$$

with zero boundary values on the parabolic boundary of Ω_∞.

When $p = 2$, we have the thermal capacity.

(K., Korte, Kuusi and Parviainen, to appear in Math. Ann.)

Remarks

(1) The potential v_μ exists by the previous existence result.

(2) Since v_μ is bounded, by the previous summability result v_μ is a supersolution and belongs to the correct parabolic Sobolev space. Hence, we may restrict to the case when μ belongs to the dual of the parabolic Sobolev space and then the solution v_μ with given boundary values is unique.

(3) The capacity is defined relative to the equation and to the reference domain Ω_∞.

Properties

(i) If $E_1 \subset E_2$, then $\text{cap}\,(E_1) \leq \text{cap}\,(E_2)$.

(ii) $\text{cap}\,(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \text{cap}\,(E_i)$.

(iii) If $E_1 \subset E_2 \subset \ldots$, then

$$\lim_{i\to\infty} \text{cap}\,(E_i) = \text{cap}\,(\bigcup_{i=1}^{\infty} E_i).$$

(iv) If $K_1 \supset K_2 \supset \ldots$ are compact sets, then

$$\lim_{i\to\infty} \text{cap}\,(K_i) = \text{cap}\,(\bigcap_{i=1}^{\infty} K_i).$$

Approximation by compact sets

For every Borel set $E \subset \Omega_\infty$ we have

$$\mathrm{cap}\,(E) = \sup\{\mathrm{cap}\,(K) : K \subset E,\ K \text{ compact}\}.$$

GAIN: It is enough to have estimates for capacities of compact sets.

Capacitary potential

Let K be a compact subset of Ω_∞. Then

$$\text{cap}(K) = \mu_{R_K}(K) < \infty.$$

Here the potential R_K is the solution of the obstacle problem with the obstacle χ_K, that is, the pointwise infimum of superparabolic functions v in Ω_∞ such that $v \geq \chi_K$.

The potential R_K is a weak supersolution with zero boundary data in Ω_∞. Moreover, it is a weak solution in $\Omega_\infty \setminus K$.

The infinity set

Let v be superparabolic in Ω_∞. Then

$$\text{cap}\Big(\{(x,t) \in \Omega_\infty : v(x,t) = \infty\}\Big) = 0.$$

(K., Korte, Kuusi and Parviainen, to appear in Math. Ann.)

The main estimate

Let v be superparabolic in Ω_∞ and $\lambda > 1$. Then there is a constant C, independent of K, such that

$$\begin{aligned}&\operatorname{cap}\left(\{v > \lambda\} \cap K\right)\\&\qquad \le C\mu_{R_K^v}(\Omega_\infty)\Big(\lambda^{1-p} + \lambda^{-1/(p-1)}\Big)\end{aligned}$$

for all compact $K \subset \Omega_\infty$. Here R_K^v denotes the solution of the obstacle problem with the obstacle $v\chi_K$.

The main idea of the proof

In the elliptic case,

$$\min\left(\frac{v}{\lambda}, 1\right)$$

provides an admissible function for the capacity of the set $\{v > \lambda\}$, since the class of superharmonic functions is closed under scaling.

Because the class of superparabolic functions is not closed under the scaling, we derive estimates for the scaled obstacle problems instead.

Open problems

(1) Wiener type criterion for the boundary requality.

(2) Potential estimates for superparabolic functions.

(3) Uniqueness for the measure data problem.

(4) $1 < p < 2$.

(5) The dependency of the capacity of the equation.

(6) The relations of the capacity to the Hausdorff measure.

(7) Removability problems for superparabolic fuunctions.

(8) Other equations.

Summary

In many cases supersolutions and superparabolic functions are identified, but this is not, strictly speaking, correct.

Fundamental Barenblatt solution is a prime example of a superparabolic function. It reflects the worst possible behaviour of a solution also in the nonlinear case.

Every superparabolic function is a solution to a measure data problem.

It is possible to develop theory for nonlinear parabolic capacity. This concept is useful in boundary regularity.