Supercaloric functions and Perron's method for the porous medium equation

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Outline of the talk 1(2)

We discuss nonnegative (super)solutions of the porous medium equation (PME)

$$u_t - \Delta(u^m) = 0$$

in the slow diffusion case $m > 1$ in cylindrical domains.

Motivation: Supersolutions arise in obstacle problems, problems with measure data, Perron-Wiener-Brelot method, boundary regularity, polar sets, removable sets and other aspects in nonlinear potential theory.

Classes of supersolutions:

- Weak supersolutions (test functions under the integral)
- Supercaloric functions (defined through a comparison principle)
- Solutions to a measure data problem
- Viscosity supersolutions (test functions evaluated at contact points)

Outline of the talk 2(2)

- **Goal**
  - To discuss a nonlinear theory of supercaloric functions for the PME
- **Questions**
  - Connections of supercaloric functions to supersolutions
  - Sobolev space properties of supercaloric functions
  - Infinity sets (polar sets) of supercaloric functions
- **Toolbox**
  - Energy estimates
  - Regularity results
  - Harnack inequalities
  - Obstacle problems
- **Applications**
  - Existence results (the Perron-Wiener-Brelot (PWB) method)
  - Polar sets and capacity

Space-time cylinders

- Let Ω be an open and bounded subset of $\mathbb{R}^N$ and let $0 \le t_1 < t_2 \le T$.
- We denote space-time cylinders as

$$\Omega_T = \Omega \times (0, T) \quad \text{and} \quad D_{t_1,t_2} = D \times (t_1, t_2),$$

  where $D \subset \Omega$ is an open set.
- The parabolic boundary of D_{t_1,t_2} is

$$\partial_p D_{t_1,t_2} = (\overline{D} \times \{t_1\}) \cup (\partial D \times [t_1, t_2]),$$

  i.e. only the initial and lateral boundaries are taken into account.
- We call a cylinder D_{t_1,t_2} regular if the boundary of the base set D is smooth.

Sobolev spaces

- $H^1(\Omega)$ for the Sobolev space of $u \in L^2(\Omega)$ such that the weak gradient $\nabla u \in L^2(\Omega)$.
- The Sobolev space with zero boundary values $H^1_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.
- The parabolic Sobolev space $L^2(0, T; H^1(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x, t)$ belongs to $H^1(\Omega)$ for almost all $t \in (0, T)$ and

  $$\iint_{\Omega_T} \left(|u|^2 + |\nabla u|^2\right) \, dx \, dt < \infty.$$

  The definition of the space $L^2(0, T; H^1_0(\Omega))$ is similar.
- $u \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\Omega))$, if u belongs to the parabolic Sobolev space for all $D_{t_1,t_2} \Subset \Omega_T$.

The porous medium equation (PME)

Assume that $m > 1$. A nonnegative function u is a weak solution of the PME

$$u_t - \Delta(u^m) = 0$$

in Ω_T, if $u^m \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\Omega))$ and

$$\iint_{\Omega_T} (-u\varphi_t + \nabla(u^m) \cdot \nabla\varphi) \, dx \, dt = 0$$

for every $\varphi \in C_0^\infty(\Omega_T)$. If the integral ≥ 0 for all $\varphi \geq 0$, then u is a weak supersolution.

It is possible to consider more general equations if this type, but we focus on the prototype equation. We may also consider solutions defined, for example, in $\Omega \times (-\infty, \infty)$ or $\mathbb{R}^{N+1}$.

Standard reference: Juan Luis Vázquez, *The porous medium equation*, Oxford University Press 2007.

Structural properties

- The equation is nonlinear: The sum of two solutions is not a solution, in general.
- Solutions cannot be scaled.
- Constants cannot be added to solutions. Thus the boundary values cannot be perturbed in a standard way by adding an epsilon.
- The minimum of two supersolutions is a supersolution. In particular, the truncations

$$\min(u,k), \quad k = 1,2,\dots,$$

  are supersolutions.
- Caccioppoli estimates are obtained for u^m instead of u.

Continuity properties

- A weak solution is continuous after a possible redefinition on a set of measure zero (Dahlberg-Kenig 1984 and DiBenedetto-Friedman 1985).
- A weak supersolution is lower semicontinuous after a possible redefinition on a set of measure zero, see Benny Avelin and Teemu Lukkari, *Lower semicontinuity of weak supersolutions to the porous medium equation*, Proc. Amer. Math. Soc. 143 (2015), no. 8, 3475–3486.

Observe: No regularity in time is assumed, in particular, for weak supersolutions. For example,

$$u(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a weak supersolution.

Alternative definitions 1(2)

Sometimes it is assumed that $u^{\frac{m+1}{2}} \in L^2_{\mathrm{loc}}(0,T;H^1_{\mathrm{loc}}(\Omega))$ and

$$\iint_{\Omega_T} (-u\varphi_t + \nabla(u^m)\cdot\nabla\varphi)\,dx\,dt = 0$$

for every $\varphi \in C_0^\infty(\Omega_T)$, where

$$\nabla(u^m) = \frac{2m}{m+1} u^{\frac{m-1}{2}} \nabla(u^{\frac{m+1}{2}}).$$

Advantage: u can be used as a test function.

Remark: This definition gives the same class of bounded (super)solutions by Pekka Lehtelä and Stefan Sturm, *Regularity of weak solutions and supersolutions to the porous medium equation*, in preparation.

Alternative definitions 2(2)

$u^m \in L^1_{\text{loc}}(\Omega_T)$ is called a distributional solution of the PME, if

$$\iint_{\Omega_T} (-u\varphi_t - u^m \Delta\varphi)\, dx\, dt = 0$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

Advantage: Convergence results are immediate.

Remark: This definition gives the same class of functions by Pekka Lehtelä and Teemu Lukkari: *The equivalence of weak and very weak supersolutions to the porous medium equation*, Tohoku Math. J., to appear. The result is proved under the assumption that functions are continuous even though it would be more appropriate to consider locally bounded lower semicontinuous functions.

Takeaway

There are several ways to define weak (super)solutions of the PME, but they all give the same class of functions.

Example

The Barenblatt solution is

$$\mathcal{B}(x,t)=\begin{cases} t^{-\lambda}\left(C-\dfrac{\lambda(m-1)}{2mN}\dfrac{|x|^2}{t^{\frac{2\lambda}{N}}}\right)_+^{\frac{1}{m-1}}, & t>0,\\ 0, & t\le 0,\end{cases}$$

where $m>1$, $\lambda=\frac{N}{N(m-1)+2}$ and the constant C is usually chosen so that

$$\int_\Omega \mathcal{B}(x,t)\,dx=1$$

for all $t>0$.

Observe: There is a moving boundary and disturbances propagate with a finite speed.

Properties

- $\mathcal{B}$ is a weak solution in the upper half space

$$\{(x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N,\ t > 0\}.$$

- $\mathcal{B} \in L^q_{\mathrm{loc}}(\mathbb{R}^{N+1})$ whenever $q < m + \frac{2}{N}$, the weak gradient exists and $\nabla(\mathcal{B}^m) \in L^q_{\mathrm{loc}}(\mathbb{R}^{N+1})$ whenever $q < 1 + \frac{1}{1+mN}$.
- $\mathcal{B}$ is a weak solution to the measure data problem

$$\mathcal{B}_t - \Delta(\mathcal{B}^m) = C\delta,$$

where δ is Dirac's delta at the origin.
- However, $\mathcal{B}$ is not a weak supersolution, since

$$\int_{-1}^{1} \int_{|x|<1} |\nabla(\mathcal{B}(x,t)^m)|^2 \, dx \, dt = \infty,$$

and thus $\nabla(\mathcal{B}^m) \notin L^2_{\mathrm{loc}}(\mathbb{R}^{N+1})$.

Increasing limits of solutions

The class of solutions is closed under increasing limits in the following sense.

Lemma (K.-Lindqvist 2008)

Assume that u_k, $k = 1, 2, \ldots$, is a sequence of (continuous) weak solutions in Ω_T and that $0 \le u_1 \le u_2 \le \ldots$. If the limit function

$$u(x,t) = \lim_{k\to\infty} u_k(x,t)$$

is finite in a dense subset, then u is a (continuous) weak solution.

Proof.

The argument is based on an intrinsic Harnack inequality and Hölder continuity estimates of DiBenedetto. Ascoli's theorem and compactness arguments are applied to complete the proof. □

Increasing limits of supersolutions 1(3)

Warning: The class of supersolutions is not closed under increasing limits in general.

Example

- $$u_k(x,t) = k, \quad k = 1, 2, \dots,$$
  are solutions, but the limit function is identically infinity.
- $$\min(\mathcal{B}(x,t), k), \quad k = 1, 2, \dots,$$
  are weak supersolutions, but $\mathcal{B}$ is not a weak supersolution.

Increasing limits of supersolutions 2(3)

Delicate point: The time derivative can be assumed to be an object belonging to the dual of the parabolic Sobolev space, but this approach does not give a class of supersolutions which is closed under bounded increasing convergence.

Example

$$u : \mathbb{R}^{N+1} \to \mathbb{R}, \quad u(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a supersolution and it can easily be approximated by an increasing sequence of smooth supersolutions which only depend on the time variable. However, the time derivative of u does not belong to the dual of the parabolic Sobolev space because of a jump discontinuity.

The class of supersolutions is closed under increasing limits under the following assumptions.

Lemma (K.-Lindqvist 2008)

Assume that u_k, $k = 1, 2, \dots$, is a sequence of (lower semicontinuous) weak supersolutions in Ω_T and that $0 \le u_1 \le u_2 \le \dots$. If the limit function

$$u(x,t) = \lim_{k\to\infty} u_k(x,t)$$

is locally bounded, or $u^m \in L^2_{loc}(0,T;H^1_{loc}(\Omega))$, then u is a (lower semicontinuous) weak supersolution.

Supercaloric functions for the PME

We consider a class of m-supercaloric functions defined via a comparison principle. This class will be closed under increasing limit if the limit is finite in a dense subset.

Definition (K.-Lindqvist 2008)

A function $v : \Omega_T \to [0, \infty]$ is m-supercaloric, if

1. v is lower semicontinuous,
2. v is finite in a dense subset of Ω_T and
3. v satisfies the following comparison principle in every interior cylinder $D_{t_1,t_2} \Subset \Omega_T$: If $u \in C(\overline{D_{t_1,t_2}})$ is a weak solution of the PME in D_{t_1,t_2} and $v \geq u$ on $\partial_p D_{t_1,t_2}$, then $v \geq u$ in D_{t_1,t_2}.

m-subcaloric functions are defined analogously.

When $m = 1$ we have supercaloric functions (supertemperatures) for the heat equation.

Remarks 1(2)

- An m-supercaloric function is defined at every point, not just almost everywhere.
- By the Schwarz alternating method is enough to compare in boxes instead of all cylindrical subdomains, see Pekka Lehtelä and Teemu Lukkari: *The equivalence of weak and very weak supersolutions to the porous medium equation*, Tohoku Math. J., to appear.
- The minimum of m-supercaloric functions is m-supercaloric.
- An m-supercaloric function v in $\Omega \times \{t > t_0\}$ can be extended as zero in the past. In other words

$$\begin{cases} v(x,t), & t > t_0, \\ 0, & t \le t_0, \end{cases}$$

  is an m-supercaloric function in $\Omega \times \mathbb{R}$.
- An m-supercaloric function does not, a priori, belong to a Sobolev space. The only connection to the equation is through the comparison principle.

Remarks 2(2)

- A lower semicontinuous representative of a weak supersolution is m-supercaloric. This follows from the comparison principle.
- A locally bounded m-supercaloric function is a weak supersolution. In particular, the truncations $\min(v, k)$, $k = 1, 2, \ldots$, are supersolutions (K.-Lindqvist 2008). This follows by approximating a given m-supercaloric function pointwise by an increasing sequence of weak supersolutions, constructed through successive obstacle problems. By the boundedness assumption, the limit function is a weak supersolution.
- Since $\min(\mathcal{B}, k)$, $k = 1, 2, \ldots$, are weak supersolutions, the Barenblatt solution is m-supercaloric, but not a weak supersolution.
- Any function of the form $v(x, t) = f(t)$, where f is a monotone increasing lower semicontinuous function is m-supercaloric.

Takeaways

- There are no other bounded m-supercaloric functions than weak supersolutions, once the question of lower semicontinuity is taken into account. Thus if we are only interested in bounded functions these classes coincide.
- As we shall see, there are several ways to construct unbounded m-supercaloric functions, that are not weak supersolutions. Thus, in general, these are different classes of functions.

Unbounded m-supercaloric functions 1(2)

Example

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set, $m > 1$ and let $t_0 \in \mathbb{R}$. The friendly giant, obtained by separation of variables, is

$$v(x,t) = \frac{u(x)}{(t-t_0)^{\frac{1}{m-1}}}, \quad t > t_0,$$

where $u^m \in H_0^1(\Omega)$ is the unique positive weak solution to the nonlinear elliptic eigenvalue problem

$$\Delta(u^m) + \frac{1}{m-1} u = 0$$

in Ω. v is a solution in $\Omega \times (t_0, \infty)$ and the zero extension to $\Omega \times (-\infty, t_0]$ is m-supercaloric in $\Omega \times \mathbb{R}$.

Remarks

- The infinity set of the friendly giant is the whole time slice $\Omega \times \{t_0\}$.
- This cannot occur for the classical heat equation when $m = 1$.
- Since the friendly giant v is a solution in $\Omega \times (t_0, \infty)$ it plays an important role as a minorant for m-supercaloric functions which blows up at time t_0.
- This can be used to show that an m-supercaloric function, with infinite initial values on the whole time slice $\Omega \times \{t_0\}$, blows up at a rate greater or equal to the friendly giant.

Unbounded m-supercaloric functions 2(2)

Example

Let

$$v(x,t) = u(x)e^{\frac{1}{(m-1)t}}, \quad t > 0,$$

where u is a solution to the same elliptic problem as in the previous example. Then

$$\begin{aligned} & v_t(x,t) - \Delta(v(x,t)^m) \\ & = e^{\frac{1}{(m-1)t}} \left(e^{\frac{1}{t}} - \frac{1}{t^2} \right) \frac{u(x)}{m-1} \geq 0. \end{aligned}$$

Thus v is a supersolution in $\Omega \times (t_0, \infty)$ and the zero extension to $\Omega \times (-\infty, t_0]$ is m-supercaloric in $\Omega \times \mathbb{R}$.

Observe: An m-supercaloric function may blow up exponentially near the infinity set.

Question: What are the Sobolev space properties of unbounded m-supercaloric functions?

Class $\mathfrak{B}$

First we consider m-supercaloric functions that have a similar behaviour as the Barenblatt solution.

Definition

We say that a nonnegative m-supercaloric function v belongs to class $\mathfrak{B}$, if $v \in L^q_{\mathrm{loc}}(\Omega_T)$ for some $q > m - 1$.

Example

The Barenblatt solution belongs to class $\mathfrak{B}$.

A characterization of class $\mathfrak{B}$

The following result is based on K.-Lindqvist 2008, 2016.

Theorem (K.-Lehtelä-Lindqvist-Parviainen, in preparation)

Assume that v is a nonnegative m-supercaloric function in Ω_T. Then the following claims are equivalent:

1. $v \in \mathfrak{B}$,
2. $v \in L^{m-1}_{loc}(\Omega_T)$,
3. $\nabla(v^m)$ *exists and* $\nabla(v^m) \in L^q_{loc}(\Omega_T)$ *whenever* $q < 1 + \frac{1}{1+mN}$,
4. $$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D v(x,t)\,dx < \infty$$

   whenever $D \times (\delta, T-\delta) \Subset \Omega_T$.

Remark

If $v \in \mathfrak{B}$, then

$$v \in L^q_{\mathrm{loc}}(\Omega_T) \quad \text{for every} \quad q < m + \frac{2}{N}.$$

This is a consequence of a reverse Hölder inequality for supersolutions of the PME, see Pekka Lehtelä, *A weak harnack estimate for supersolutions to the porous medium equation*, Differential Integral Equations, to appear.

The upper bound for the exponent is sharp as the Barenblatt solution shows.

Moral: The result shows that functions in class $\mathfrak{B}$ have similar Sobolev space properties as the Barenblatt solution.

A measure data problem

If $v \in \mathfrak{B}$, there exists a Radon measure μ on $\mathbb{R}^{N+1}$, such that v is a weak solution to the measure data problem

$$v_t - \Delta(v^m) = \mu.$$

Reason: By the discussion above,

$$v \in L^1_{\text{loc}}(\Omega_T) \quad \text{and} \quad \nabla(v^m) \in L^1_{\text{loc}}(\Omega_T).$$

Thus we may apply the Riesz representation theorem to the nonnegative linear operator

$$L_v(\varphi) = \iint_{\Omega_T} (-v\varphi_t + \nabla(v^m) \cdot \nabla\varphi)\, dx\, dt,$$

where $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$.

Class $\mathfrak{M}$

Next we consider the complementary class of $\mathfrak{B}$. We denote this class by $\mathfrak{M}$, which refers to the somewhat monstrous behaviour of these functions.

Definition

We say that a nonnegative m-supercaloric function v belongs to class $\mathfrak{M}$, if $v \notin L^q_{\mathrm{loc}}(\Omega_T)$ for every $q > m - 1$.

Example

The friendly giant, and other similar functions, belongs to class $\mathfrak{M}$.

Remark

As we have seen, the separation of variables method can be modified to produce m-supercaloric functions which blow up exponentially fast near the infinity set. These functions are lacking several properties, such as local integrability.

We shall see that classes $\mathfrak{B}$ and $\mathfrak{M}$ are mutually exclusive. Thus every nonnegative m-supercaloric function belongs either to $\mathfrak{B}$ or $\mathfrak{M}$. The decisive difference between these classes is a local integrability property, but there are several ways to characterize these classes.

For the p-parabolic equation with $p > 2$, see Tuomo Kuusi, Peter Lindqvist, Mikko Parviainen, *Shadows of infinities*, Ann. Mat. Pura Appl. (4) 195 (2016), no. 4, 1185–1206.

A characterization of class $\mathfrak{M}$

Theorem (K.-Lehtelä-Lindqvist-Parviainen, in preparation)

Assume that v is a nonnegative m-supercaloric function in Ω_T. Then the following claims are equivalent:

1. $v \in \mathfrak{M}$,
2. $v \notin L^{m-1}_{loc}(\Omega_T)$,
3. *there exists $\delta > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D v(x,t)\,dx = \infty,$$

   whenever $D \Subset \Omega$ with $|D| > 0$.

4. *there exists $(x_0, t_0) \in \Omega_T$, such that*

$$\liminf_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} v(x,t)(t-t_0)^{\frac{1}{m-1}} > 0.$$

Moral: The result shows that functions in class $\mathfrak{M}$ blow up at least with the rate given by the friendly giant.

Dichotomy: Either

$$v \in L^q_{\text{loc}}(\Omega_T) \quad \text{for every} \quad q < m + \frac{2}{N}$$

or

$$v \notin L^{m-1}_{\text{loc}}(\Omega_T).$$

Thus the local integrability of a solution is either up to $m + \frac{2}{N}$ or worse than $m - 1$. There is a gap between these exponents.

The infinity set

We consider the infinity set (polar set)

$$I(t_0) = \Big\{(x_0, t_0) : \lim_{t \to t_0+} u(x_0, t) = \infty\Big\}$$

at time $t_0 \in (0, T)$. More general approach directions can be considered as well.

Example

For the Barenblatt solution $I(0) = \{0\}$ and for the friendly giant $I(0) = \Omega$.

Past and future

It is essential that the limit in the definition of $l(t_0)$ is determined only by the future times $t > t_0$, while the past and present times $t \leq t_0$ are totally excluded.

This is in striking contrast to the pointwise value of the function, which can always be determined only by the past. An extension of Brelot's classical theorem for m-supercaloric functions (K.-Lindqvist 2008) states that

$$v(x_0, t_0) = \operatorname*{ess\,lim\,inf}_{\substack{(x,t)\to(x_0,t_0)\\ t<t_0}} v(x, t)$$

Here the notion of the essential limes inferior means that any set of $(N + 1)$-dimensional Lebesgue measure zero can be neglected in the calculation of the lower limit.

The infinity set and $\mathfrak{M}$

Theorem

Assume that v is a nonnegative m-supercaloric function in Ω_T. Then the following claims are equivalent:

- $v \in \mathfrak{M}$,
- *there exists $t_0 \in (0, T)$ such that*

$$\lim_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} v(x,t) = \infty \quad \textit{for every} \quad x_0 \in \Omega.$$

Theorem (K.-Lehtelä-Lindqvist-Parviainen, in preparation)

Assume that v is a nonnegative m-supercaloric function in Ω_T. Then for every $t \in (0, T)$ there are two alternatives: either

$$|I(t)| = 0 \quad \textit{or} \quad I(t) = \Omega.$$

Proof.

A chaining argument and weak Harnack's inequality. □

Moral: Even though we consider the slow diffusion case, infinities propagate with infinite speed.

Characterizations of $\mathfrak{B}$ and $\mathfrak{M}$

- $v \in \mathfrak{M}$ if and only if $I(t) = \Omega$ for some $t \in (0, T)$.
- $v \in \mathfrak{B}$ if and only if $|I(t)| = 0$ for every $t \in (0, T)$.
- If v is a nonnegative m-supercaloric function defined on whole $\mathbb{R}^{N+1}$, then $v \in \mathfrak{B}$. Thus class $\mathfrak{M}$ does not occur in the whole space.

  This follows by contradiction from comparison and scaling arguments related to the friendly giant. In this case the function is equal to infinity on a set of positive measure on $\mathbb{R}^{N+1}$, which violates the finiteness assumption on a dense subset.

Takeaways

- A nonnegative m-supercaloric function has a Barenblatt type behaviour (class $\mathfrak{B}$) or it blows up at least with the rate given by the friendly giant (class $\mathfrak{M}$).
- Functions in class $\mathfrak{B}$ satisfy a natural Sobolev space properties. There is a measure data problem and the Riesz measure associated with class $\mathfrak{B}$.
- Functions in class $\mathfrak{M}$ are lacking several properties, such as local integrability. Thus these functions are not easily tractable.
- Class $\mathfrak{M}$ does not occur in the whole space.
- The infinity set on a time slice is either a set of measure zero or the whole time slice.
- $v \in \mathfrak{M}$ if and only if $I(t) = \Omega$ for some $t \in (0, T)$.
- $v \in \mathfrak{B}$ if and only if $|I(t)| = 0$ for every $t \in (0, T)$.

A boundary value problem

Assume that Ω_T is a bounded space-time cylinder and $g \in C(\partial_p \Omega_T)$ is a nonnegative boundary function. Consider the boundary value problem

$$\begin{cases} u_t - \Delta(u^m) = 0 \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \partial\Omega \times [0, T), \\ u(x,0) = g(x,0). \end{cases}$$

Problem: How to prove existence of a solution to this problem with general continuous boundary values and general cylindrical domains?

The Perron-Wiener-Brelot method

The Perron-Wiener-Brelot (PWB) method in potential theory gives an upper solution $\overline{H}_g$ and a lower solution $\underline{H}_g$ with

$$\underline{H}_g \leq \overline{H}_g.$$

Question: For which bondary functions g we have

$$\underline{H}_g = \overline{H}_g?$$

If this happens, the boundary function g is called resolutive and we denote the common function by H_g.

Warning: Even if the upper and lower PWB solutions coincide, they may take the wrong boundary values. A punctured ball gives an example for the Laplace equation.

Upper class and upper solution

Definition

Let $g : \partial_p \Omega_T \to \mathbb{R}$ be given. The upper class $\mathfrak{U}_g$ consists of m-supercaloric functions v which satisfy

$$\liminf_{x \to y} v(x) \geq g(y)$$

for all $y \in \partial_p \Omega_T$. The upper PWB solution is

$$\overline{H}_g(x) = \inf_{v \in \mathfrak{U}_g} v(x).$$

Definition

Let $g : \partial_p\Omega_T \to \mathbb{R}$ be given. The lower class $\mathfrak{L}_g$ consists of m-subcaloric functions u which satisfy

$$\limsup_{x\to y} u(x) \le g(y)$$

for all $y \in \partial_p\Omega_T$. The lower PWB solution is

$$\underline{H}_g(x) = \sup_{u\in\mathfrak{L}_g} u(x).$$

Remark

If there exists a function $u \in C(\overline{\Omega_T})$ solving the boundary value problem in the classical sense, then

$$u = \overline{H}_g = \underline{H}_g.$$

To see this, simply note that the function u belongs to both the upper class and the lower class.

As we will see, both $\overline{H}_g$ and $\underline{H}_g$ are weak solutions to the PME.

Upper and lower PWB solutions

Theorem (K.-Lindqvist-Lukkari 2016)

$\overline{H}_g$ and $\underline{H}_g$ are continuous weak solutions to the PME in Ω_T.

Proof.

The argument is based on the fact that monotone limits of Poisson modifications is a weak solution to the PME. □

A comparison principle tailored for the Perron solutions

Theorem (K.-Lindqvist-Lukkari 2016)

Let u be a m-subcaloric and v m-supercaloric such that

$$\limsup_{x \to y} u(x) \leq \liminf_{x \to y} v(y)$$

for all $y \in \partial_p \Omega_T$. Then $u \leq v$ in Ω_T.

Proof.

The proof of this result is not immediate, since constants may not be added to solutions. An approximation result, based on a test function introduced by Oleinik, is used to bypass this. □

Observe: The essential feature here is that the base Ω of the space-time cylinder Ω_T may be an arbitrary bounded open set.

Wiener's resolutivity theorem for the PME

The following theorem states that continuous functions are resolutive in cylindrical domains. This extends classical Wiener's resolutivity theorem for the PME.

Theorem (K.-Lindqvist-Lukkari 2016)

Assume that Ω_T *is a bounded space-time cylinder and* $g \in C(\partial_p \Omega_T)$*. Then* $\overline{H}_g = \underline{H}_g$ *in* Ω_T*.*

Observe: No regularity assumptions on the base of the space-time cylinder are needed.

Outline of the proof 1(2)

- Reduce to smooth boundary function g by approximating with positive smooth functions.
- For g smooth, there exists a unique weak solution u to the problem with Sobolev boundary values.
- Show that $u = H_g$.
- In general, u does not belong to the upper class $\mathfrak{U}_g$.

Outline of the proof 2(2)

- For smooth boundary values, we need to construct functions belonging to the upper class $\mathfrak{U}_g$ that are sufficiently smooth in time and attain the correct boundary and initial values.
- This is done by a penalized problem, see V. Bögelein, T. Lukkari, and C. Scheven, *The obstacle problem for the porous medium equation*, Math. Ann. 363 (2015), no. 1-2, 455–499.
- Energy estimates for the time derivatives in the smooth case play a central role in the argument.

Remarks on noncylindrical domains

- For the heat equation, there is a resolutivity theorem in general space time domains in $\mathbb{R}^{N+1}$.
- For the PME resolutivity in general space-time domains remains open.
- For the PME in general space time domains it is not even known whether $\underline{H}_g \le \overline{H}_g$.

The elliptic comparison principle

The missing tool is the following "elliptic" comparison principle for the PME in general space-time domains: *Assume that O be a general open set in $\mathbb{R}^{N+1}$ and that v is m-supercaloric in O. If $u \in C(\overline{D})$ be a weak solution to the PME in $D \Subset O$ with $v \geq u$ on the whole topological boundary ∂D, does it follow that $v \geq u$ in D?*

For the p-parabolic equation we may add constants to solutions and a comparison principle for general open sets then follows from the space-time cylinder case by a straightforward exhaustion argument, see Tero Kilpeläinen and Peter Lindqvist, *On the Dirichlet boundary value problem for a degenerate parabolic equation*, SIAM J. Math. Anal. 27 (1996), no. 3, 661–683. For the PME adding constants is no longer possible.

For partial results, see Benny Avelin and Teemu Lukkari, *A comparison principle for the porous medium equation and its consequences*, *Rev. Mat. Iberoam.* 33 (2017), no. 2, 573–594.

Takeaways

- It is possible to develop a theory of supercaloric functions for the PME.
- Differences between several classes of supersolutions to the PME are relatively well understood.
- The results and methods can be applied in nonlinear potential theory for the PME.

Open problems 1(4)

- What is the corresponding theory of m-supercaloric functions and in the fast diffusion case $0 < m < 1$?
  - The question is also open for the p-parabolic equation when $1 < p < 2$.
- Is it possible to develop the Perron-Wiener-Brelot method for the PME in general space time domains?
- In particular, is it possible to prove the elliptic comparison principle for the PME?
  - Partial results: Benny Avelin and Teemu Lukkari, *A comparison principle for the porous medium equation and its consequences*, Rev. Mat. Iberoam. 33 (2017), no. 2, 573–594.

Open problems 2(4)

- Is it possible to develop capacity theory for the PME?
  - For the p-parabolic equation with $p \geq 2$: K., Riikka Korte, Tuomo Kuusi, Mikko Parviainen, *Nonlinear parabolic capacity and polar sets of superparabolic functions*, Math. Ann. 355 (2013), no. 4, 1349–1381.
  - Partial results for the PME: Benny Avelin and Teemu Lukkari, *A comparison principle for the porous medium equation and its consequences*, Rev. Mat. Iberoam. 33 (2017), no. 2, 573–594.
- Are polar sets for m-supercaloric functions sets of capacity zero?
- Are sets of capacity zero removable for bounded m-supercaloric functions?
  - For the p-parabolic equation with $p \geq 2$: Benny Avelin and Olli Saari, *Characterizations of interior polar sets for the degenerate p-parabolic equation*, arXiv 2015.

Open problems 3(4)

- Do the classes of viscosity supersolutions and *m*-supercaloric functions coincide?
  - For the *p*-parabolic equation: Petri Juutinen, Peter Lindqvist and Juan Manfredi, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, SIAM J. Math. Anal. 33 (2001), no. 3, 699–717.
  - Vesa Julin and Petri Juutinen, *A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation*, Comm. Partial Differential Equations 37 (2012), no. 5, 934–946.
  - Luis Caffarelli and Juan Luis Vázquez, *Viscosity solutions for the porous medium equation*, Differential equations: La Pietra 1996 (Florence), 13–26, Proc. Sympos. Pure Math., 65, Amer. Math. Soc., Providence, RI, 1999.
  - Cristina Brändle and Juan Luis Vázquez, *Viscosity solutions for quasilinear degenerate parabolic equations of porous medium type*, Indiana Univ. Math. J. 54 (2005), no. 3, 817–860.

Open problems 4(4)

- While uniqueness with sufficiently regular data and fixed boundary and initial values is also standard, uniqueness questions related to nonlinear equations with general measure data are rather delicate. For instance, the question whether the Barenblatt solution is the only solution of the PME with Dirac's delta seems to be open.
- What is the Wiener criterion for boundary regularity for the PME?