Self-improving results for Poincaré inequalities

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References

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Outline of the talk

- **Question:** Scale and location invariant uniform estimates in harmonic analysis and PDEs tend to improve themselves. We shall discuss self-improving phenomena related to a Poincaré inequality.
- **Goal:** Our goal is to give direct and transparent arguments with a special emphasis on the role of the relevant maximal function inequalities.
- **Tools:** Calderón–Zygmund type covering arguments, good lambda inequalities, harmonic analysis techniques related to weighted norm inequalities.

The doubling condition

We assume that

$$X = (X, d, \mu)$$

is a complete metric measure space, where μ is a doubling Borel regular measure on X.

The measure μ is doubling if there exists a constant $c_\mu \geq 1$ such that

$$0 < \mu(2B) \leq c_\mu \mu(B) < \infty$$

for all balls $B = B(x, r) = \{y \in X : d(y, x) < r\}$ with $x \in X$ and $r > 0$. Here $2B = B(x, 2r)$.

A dimension of a metric measure space

For all balls $B = B(x,r)$ that are centered at $x \in A \subset X$ with radius $r \le \operatorname{diam}(A)$, we have

$$\frac{\mu(B)}{\mu(A)} \ge 2^{-s}\left(\frac{r}{\operatorname{diam}(A)}\right)^s,$$

with $s = \log_2 c_\mu > 0$.

Note: The Euclidean space $\mathbb{R}^n$ is doubling with the doubling constant 2^n and $s = n$. This suggests that the exponent s is a generalization of the notion of dimension to metric measure spaces.

The upper gradient

Definition (Heinonen and Koskela 1998)

A nonnegative Borel function g on X is an upper gradient of a function u, if

$$|u(x) - u(y)| \leq \int_\gamma g \, ds.$$

for every $x, y \in X$ and for every path γ joining x and y in X.

Note: $|\nabla u|$ is an upper gradient of u in the Euclidean space $\mathbb{R}^n$. Thus upper gradient generalizes $|\nabla u|$ to metric measure spaces.

Remarks

- Upper gradient is not unique.
- If u has an upper gradient in $L^1(X)$, then there exists a unique minimal upper gradient g_u such that $g_u \leq g$ almost everywhere in X for all upper gradients $g \in L^1(X)$.
- Upper gradient is a local concept in the sense that the minimal upper gradient is zero almost everywhere in the set where the function is constant.
- Upper gradient also has some linear nature, but is not linear itself. The sum of the upper gradients of two functions is an upper gradient of the sum of the functions, but the analogous result does not hold for a difference of two functions.
- Using upper gradients it is possible to define first order Sobolev spaces (Shanmugalingam 2000) and functions of bounded variation (Ambrosio, Miranda Jr. and Pallara 2003) on a metric measure space.

The pointwise Lipschitz constant

Example

If u is a Lipschitz function on X, the pointwise Lipschitz constant of u at $x \in X$ is defined as

$$\operatorname{Lip} u(x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r}.$$

The Borel function $g = \operatorname{Lip} u$ is an upper gradient of u. In fact, it is the minimal upper gradient of u.

Poincaré inequality

A metric measure space X supports a (q, p)-Poincaré inequality, for exponents $1 \leq q, p < \infty$, if there are C and $\lambda \geq 1$ such that

- for all balls $B = B(x, r) \subset X$,
- for all locally integrable functions u on X,
- and for all upper gradients g of u

we have

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{\frac{1}{q}} \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{\frac{1}{p}}.$$

Here

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Takeaways

The Poincaré inequality creates a link between the metric, the measure and the gradient. It provides a way to pass from the infinitesimal information on the gradient to an oscillation estimate at larger scales. Roughly speaking, if the integral of the gradient is small, then the function does not oscillate much.

The doubling condition and the Poincaré inequality are relatively standard assumptions in analysis on metric measure spaces. There are several phenomena in harmonic analysis and PDEs for which a $(q, p - \varepsilon)$-Poincaré inequality for some $\varepsilon > 0$ would be a more natural assumption than a (q, p)-Poincaré inequality. This is related, among other issues, to the fact that the Hardy-Littlewood maximal function is not bounded on $L^1(X)$.

Bi-Lipschitz invariance

A Poincaré inequality is invariant under a change of coordinates with a bi-Lipschitz mapping. This is a characteristic of all analysis in metric spaces, since, in general, we do not have any linear structure and thus the concepts should not depend on such a structure even if it exists in some special cases.

Not only the topological structure, but also the choice of the metric, is essential for the validity of a Poincaré inequality. For example, consider the snowflaking

$$(X, d, \mu) \to (X, d^\alpha, \mu) \quad \text{with} \quad 0 < \alpha < 1.$$

This is a quasisymmetric change of the metric, but snowflaked metric space (X, d^α, μ) has no rectifiable paths and consequently cannot support a Poincaré inequality.

It is enough to consider Lipschitz functions

Theorem (Keith 2003)

If X is complete, then X supports a Poincaré inequality if and only if X supports a Poincaré inequality for compactly supported Lipschitz functions.

Thus we may assume that functions are Lipschitz continuous.

Doubling and Poincaré: examples

There is a large variety of spaces equipped with a doubling measure and supporting a Poincaré inequality. These include Euclidean spaces with Lebesgue measure and weighted Euclidean spaces with Muckenhoupt weights, as well as graphs, complete Riemannian manifolds with nonnegative Ricci curvature, Heisenberg groups and more general Carnot–Carathéodory spaces.

On Riemannian manifolds, Grigoryan and Saloff–Coste observed that the doubling condition and the $(1,2)$-Poincaré inequality are not only sufficient, but also necessary, conditions for a scale-invariant parabolic Harnack principle for the heat equation. The corresponding question seems to be open for the $(1,p)$-Poincaré inequality with $p \neq 2$.

What do we mean by self-improvement?

Assume that X supports a (q, p)-Poincaré inequality. By Hölder's inequality, X supports

- $(q - \varepsilon, p)$-Poincaré inequality, for $0 < \varepsilon \leq q - 1$ and
- $(q, p + \varepsilon)$-Poincaré inequality, for $\varepsilon > 0$.

Questions:

- (Improvement on the left-hand side) Does there exist $\varepsilon > 0$ such that X supports a $(q + \varepsilon, p)$-Poincaré inequality?
- (Improvement on the right-hand side) Does there exist $0 < \varepsilon \leq p - 1$ such that X supports a $(q, p - \varepsilon)$-Poincaré inequality?

Answers: The answer to both questions is yes under very general conditions.

Self-improvement of a Poincaré inequality

Consider the case $q = 1$.

(a) $(1, p) \implies (p + \varepsilon, p)$: If X supports a $(1, p)$-Poincaré inequality with $p \geq 1$, then X supports a (p, p)-Poincaré inequality and, moreover, a $(p + \varepsilon, p)$-Poincaré inequality for some $\varepsilon > 0$. (Bakry–Coulhon–Ledoux–Saloff–Coste 1995, Hajłasz–Koskela 1995)

(b) $(1, p) \implies (p, p - \varepsilon)$: If X is complete and supports a $(1, p)$-Poincaré inequality with $p > 1$, then there exists $\varepsilon > 0$ such that X supports a $(1, p - \varepsilon)$-Poincaré inequality. (Keith–Zhong 2008)

Geodesic space

Definition

We say that X is a geodesic space, if every pair of points in X can be joined by a path, whose length is equal to the distance between the points.

Quasiconvex space

If a complete doubling metric measure space supports a Poincaré inequality, then the space is quasiconvex, that is, there exists a constant such that every pair of points can be connected with a path whose length is at most the constant times the distance between the points.

Since a Poincaré inequality is invariant under bi-Lipschitz mappings, it follows that every complete doubling metric measure space supporting a Poincaré inequality can be turned into a geodesic space by a bi-Lipschitz change of the metric

$$d_{new}(x, y) = \inf \text{length}(\gamma_{xy}),$$

where the infimum is taken over all rectifiable paths γ_{xy} joining x and y. (Semmes 1996)

Thus we may assume that X is geodesic.

The dilation constant in a Poincaré inequality

Theorem (Hajłasz–Koskela 1995)

If X is a geodesic space, then a Poincaré inequality implies that a Poincaré inequality holds with the dilation constant $\lambda = 1$.

Thus we may assume that the dilation constant in a Poincaré inequality is one.

The Keith–Zhong theorem

We consider the special case $B = X$.

Theorem (Keith–Zhong 2008)

Let X be a geodesic space such that

- $0 < \operatorname{diam}(X) < \infty$,
- *X supports a (p, p)-Poincaré inequality with $p > 1$ and $\lambda = 1$.*

Then there exists $0 < \varepsilon < p - 1$ such that

$$\left(\frac{1}{\operatorname{diam}(X)^p} \fint_X |u - u_X|^p \, d\mu \right)^{\frac{1}{p}} \lesssim \left(\fint_X g_u^{p-\varepsilon} \, d\mu \right)^{\frac{1}{p-\varepsilon}}$$

for every Lipschitz continuous function $u : X \to \mathbb{R}$.

The fractional sharp maximal function

Definition

The fractional sharp maximal function of $f \in L^1_{\mathrm{loc}}(X)$ is

$$M^\sharp f(x) = \sup_{x \in B} \left(\frac{1}{\operatorname{diam}(B)^p} \fint_B |f - f_B|^p \, d\mu \right)^{\frac{1}{p}},$$

where the supremum is taken over all balls $B \subset X$ containing x.

A pointwise estimate

Lemma

Assume that $u \in L^1_{\text{loc}}(X)$*. Then*

$$|u(x) - u(y)| \lesssim d(x,y)\big(M^\sharp u(x) + M^\sharp u(y)\big)$$

for almost every $x, y \in X$*.*

This gives a characterization for Lipschitz continuous functions $u : X \to \mathbb{R}$.

The idea of a proof.

Let $B_i = B(x, 2^{-i} r)$, $i = 0, 1, \dots$. By a telescoping argument

$$
\begin{aligned}
|u(x) - u_{B(x,r)}| &\le c_\mu \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}| \, d\mu \\
&\le c_\mu \sum_{i=0}^{\infty} 2^{1-i} r \left(\frac{1}{\operatorname{diam}(B_i)^p} \fint_{B_i} |u - u_{B_i}|^p \, d\mu \right)^{\frac{1}{p}} \\
&\le c_\mu M^\sharp u(x) \sum_{i=0}^{\infty} 2^{1-i} r \\
&\le C(c_\mu) r \, M^\sharp u(x).
\end{aligned}
$$

□

Takeaway

Roughly speaking this gives

$$g_u(x) \lesssim \operatorname{Lip} u(x) \lesssim M^\sharp u(x)$$

for almost every $x \in X$. This holds for every Lipschitz continuous function $u : X \to \mathbb{R}$.

The Hardy–Littlewood maximal function

Definition

The Hardy–Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(X)$ is

$$Mf(x) = \sup_{x \in B} \fint_B |f| \, d\mu,$$

A (p,p)-Poincaré inequality implies

$$M^\sharp u(x) \lesssim \big(Mg_u^p(x)\big)^{\frac{1}{p}}$$

for every $x \in X$. This holds for functions satisfying a (p,p)-Poincaré inequality.

A first attempt for the proof.

$$\left(\frac{1}{\operatorname{diam}(X)^p}\fint_X |u-u_X|^p\,d\mu\right)^{\frac{p-\varepsilon}{p}} \le \inf_{x\in X}\left(M^\sharp u(x)\right)^{p-\varepsilon}$$

$$\le \fint_X \left(M^\sharp u\right)^{p-\varepsilon} d\mu$$

$$\lesssim \fint_X \left(Mg_u^p\right)^{\frac{p-\varepsilon}{p}} d\mu$$

$$\lesssim \fint_X g_u^{p-\varepsilon}\,d\mu \quad (?)$$

□

A problem: The last inequality is not true.

Reason: Mg_u^p is not necessarily integrable, if $g_u \in L^p(X)$.

A maximal function inequality

Theorem (K.–Lehrbäck–Vähäkangas–Zhong 2017)

Let X be a geodesic space such that

- $0 < \operatorname{diam}(X) < \infty$,
- *X supports a (p,p)-Poincaré inequality for some $1 < p < \infty$ and with the dilatation constant $\lambda = 1$.*

There exists $0 < \delta < 1$ such that

$$\fint_X \left(M^\sharp u\right)^{p-\varepsilon} d\mu \le (1-\delta) \fint_X \left(M^\sharp u\right)^{p-\varepsilon} d\mu + C \fint_X g_u^p \left(M^\sharp u\right)^{-\varepsilon} d\mu$$

for every Lipschitz function $u : X \to \mathbb{R}$.

Absorption

Thus

$$A \le (1-\delta)A + B, \quad 0 < \delta < 1,$$

with

$$A = \fint_X \big(M^\sharp u\big)^{p-\varepsilon}\, d\mu \quad \text{and} \quad B = C \fint_X g_u^p \big(M^\sharp u\big)^{-\varepsilon}\, d\mu.$$

The proof is based on the absorption

$$A \le (1-\delta)A + B \quad \Longrightarrow \quad A \le \frac{B}{\delta}.$$

Note that $A < \infty$, since u is Lipschitz continuous and $0 < \mu(X) < \infty$.

Proof of the self-improving property

$$\begin{aligned}\left(\frac{1}{\operatorname{diam}(X)^p}\fint_X |u-u_X|^p\,d\mu\right)^{\frac{p-\varepsilon}{p}} &\le \inf_{x\in X}\left(M^\sharp u(x)\right)^{p-\varepsilon}\\ &\le \fint_X \left(M^\sharp u\right)^{p-\varepsilon}\,d\mu\\ &= A \le \frac{B}{\delta}\\ &= \frac{C}{\delta}\fint_X g_u^p\left(M^\sharp u\right)^{-\varepsilon}\,d\mu\\ &\le \frac{C}{\delta}\fint_X g_u^{p-\varepsilon}\,d\mu,\end{aligned}$$

since $M^\sharp u \in L^p(X)$ is an upper gradient of u and thus

$$\left(M^\sharp u\right)^{-\varepsilon} \le g_u^{-\varepsilon}$$

almost everywhere. Here we use the assumption that $X = B(x,R)$ for large $R > 0$.

Proof of the maximal function inequality

$$\int_X (M^\sharp u)^{p-\varepsilon}\,d\mu = (p-\varepsilon)\int_0^\infty t^{p-\varepsilon-1}\mu(E_t)\,dt$$
$$= (p-\varepsilon)\left(\int_0^{\frac{t_0}{2}} t^{p-\varepsilon-1}\mu(E_t)\,dt + \int_{\frac{t_0}{2}}^\infty t^{p-\varepsilon-1}\mu(E_t)\,dt\right),$$

where $E_t = \{M^\sharp u > t\}$ and

$$t_0 = \left(\frac{1}{\operatorname{diam}(X)^p} \fint_X |u-u_X|^p\,d\mu\right)^{\frac{1}{p}}.$$

Proof of the maximal function inequality

Since $E_t = X$ for $0 < t < t_0$, we have

$$\begin{aligned}
(p-\varepsilon)\int_0^{\frac{t_0}{2}} t^{p-\varepsilon-1}\mu(E_t)\,dt &= \frac{p-\varepsilon}{2^{p-\varepsilon-1}}\int_0^{\frac{t_0}{2}} (2t)^{p-\varepsilon-1}\mu(E_{2t})\,dt \\
&\le \frac{p-\varepsilon}{2^{p-\varepsilon}}\int_0^{\infty} s^{p-\varepsilon-1}\mu(E_s)\,ds \\
&= \frac{1}{2^{p-\varepsilon}}\int_X \left(M^\sharp u\right)^{p-\varepsilon}\,d\mu.
\end{aligned}$$

Since $p-\varepsilon > 1$, we have

$$\int_X \left(M^\sharp u\right)^{p-\varepsilon}\,d\mu \le 2(p-\varepsilon)\int_{\frac{t_0}{2}}^{\infty} t^{p-\varepsilon-1}\mu(E_t)\,dt.$$

A good lambda inequality

Lemma

There exists $\alpha > 0$ *such that*

$$t^p\mu(E_t) \lesssim \frac{(t2^k)^p}{2^{k\alpha}}\mu(E_{t2^k}) + \frac{1}{k^p}\sum_{j=k}^{2k-1}(t2^j)^p\mu(E_{t2^j}) + \int_{E_t\setminus E_{t4^k}} g_u^p\,d\mu$$

for every $t > \frac{t_0}{2}$ *and* $k \in \mathbb{N}$*. Here the constants are independent of both* t *and* k

The integration of the first two terms will produce the absorption term A for k sufficiently large. Moreover, an integration of the last term will produce the term B.

The red term is the bottleneck!

Proof of the good lambda inequality

Fix a level $t > \frac{t_0}{2}$. Let $x \in E_t = \{M^\sharp u > t\}$. If $\frac{t_0}{2} < t < t_0$, choose

$$B_x = B(x, 2\,\mathrm{diam}(X)) = X.$$

If $t \geq t_0$, then choose a ball $B_x \subset X$ that contains the point x, and satisfies the stopping time conditions

$$\begin{cases} t < \left(\dfrac{1}{\mathrm{diam}(B_x)^p} \displaystyle\fint_{B_x} |u - u_{B_x}|^p \, d\mu \right)^{\frac{1}{p}}, \\ \left(\dfrac{1}{\mathrm{diam}(2B_x)^p} \displaystyle\fint_{2B_x} |u - u_{2B_x}|^p \, d\mu \right)^{\frac{1}{p}} \leq t. \end{cases}$$

Let $\mathcal{F}_t \subset \{B_x : x \in E_t\}$ be a countable collection of disjoint pairwise stopping time balls with $\cup_{B \in \mathcal{F}_t} B \subset E_t$ and $E_t \subset \cup_{B \in \mathcal{F}_t} 5B$.

Proof of the good lambda inequality

By the doubling condition

$$t^p \mu(E_t) \le t^p \sum_{B\in\mathcal{F}_t} \mu(5B) \lesssim \sum_{B\in\mathcal{F}_t} t^p \mu(B).$$

Since $\mathcal{F}_t$ is a collection of pairwise disjoint balls, it suffices to prove that the following localized inequality

$$t^p \mu(B) \lesssim \frac{(t2^k)^p}{2^{k\alpha}} \mu(E_{t2^k} \cap B) + \frac{1}{k^p} \sum_{j=k}^{2k-1} (t2^j)^p \mu(E_{t2^j} \cap B) + \int_{B\setminus E_{t4^k}} g_u^p \, d\mu$$

holds for every $B \in \mathcal{F}_t$.

Only the case $\mu(E_{t2^k} \cap B) < \frac{1}{2}\mu(B)$ is non-trivial. Hence in the sequel we will focus on this case.

Proof of the good lambda inequality

By the stopping time conditions and an iteration of the doubling condition,

$$
\begin{aligned}
t^p \mu(B) &\le \frac{1}{\operatorname{diam}(B)^p} \int_B |u - u_B|^p \, d\mu \\
&\lesssim \frac{(t2^k)^p}{2^{k\alpha}} \mu(E_{t2^k} \cap B) \\
&\qquad + \frac{1}{\operatorname{diam}(B)^p} \int_{B \setminus E_{t2^k}} |u - u_{B \setminus E_{t2^k}}|^p \, d\mu.
\end{aligned}
$$

(Obtaining this is not entirely trivial.)

The first term in the right is one of the absorption terms. Hence, our focus will be on estimating the last integral.

We have obtained freedom to modify u outside of the set $X \setminus E_{t2^k}$.

Lipschitz truncation

Lemma

Let $t > 0$ and $j \in \mathbb{N}$. Then

$$|u(x) - u(y)| \lesssim d(x,y)\big(M^\sharp u(x) + M^\sharp u(y)\big) \lesssim t2^j d(x,y)$$

for every $x, y \in X \setminus E_{t2^j} = \{M^\sharp u \le t2^j\}$.

As a consequence, the restriction

$$u|_{X \setminus E_{t2^j}} : X \setminus E_{t2^j} \to \mathbb{R}$$

is $Ct2^j$-Lipschitz continuous. By using McShane extension, we can extend it to a function $u_j : X \to \mathbb{R}$ that is $Ct2^j$-Lipschitz continuous on X and satisfies

$$u_j|_{X \setminus E_{t2^j}} = u|_{X \setminus E_{t2^j}}.$$

Proof of the good lambda inequality

Define

$$h = \frac{1}{k} \sum_{j=k}^{2k-1} u_j.$$

Then h coincides with u on $B \setminus E_{t2^k}$. Furthermore,

$$g_h = \frac{1}{k} \sum_{j=k}^{2k-1} \left\{ Ct2^j \mathbf{1}_{E_{t2^j}} + g_u \mathbf{1}_{X \setminus E_{t2^j}} \right\}$$

is an upper gradient of h. By straightforward estimates,

$$\mathbf{1}_B g_h^p \lesssim \frac{1}{k^p} \sum_{j=k}^{2k-1} (t2^j)^p \mathbf{1}_{E_{2^j t} \cap B} + g_u^p \mathbf{1}_{B \setminus E_{4^k t}}.$$

This convexity trick by Keith–Zhong is the core of the proof. The factor k^{-p}, instead of k^{-1}, is essential.

Proof of the good lambda inequality

By the (p,p)-Poincaré inequality,

$$\begin{aligned}
\frac{1}{\operatorname{diam}(B)^p} & \int_{B\setminus E_{t2^k}} |u - u_{B\setminus E_{t2^k}}|^p \, d\mu \\
& \lesssim \frac{1}{\operatorname{diam}(B)^p} \int_B |h - h_B|^p \, d\mu \\
& \lesssim \int_B g_h^p \, d\mu \\
& \lesssim \frac{1}{k^p} \sum_{j=k}^{2k-1} (t2^j)^p \mu(E_{2^j t} \cap B) + \int_{B\setminus E_{4^k t}} g_u^p \, d\mu.
\end{aligned}$$

The desired local inequality follows.

A two measure Poincaré inequality

Definition

Let $1 \leq q, p < \infty$. A pair (ν, μ) of Borel regular measures on X is called (q, p)-admissible, if

- the measures ν and μ are doubling,
- there exists a constant C such that a two measure (q, p)-Poincaré inequality

$$\left(\fint_B |u - u_{B;\nu}|^q \, d\nu \right)^{\frac{1}{q}} \leq Cr \left(\fint_B g_u^p \, d\mu \right)^{\frac{1}{p}}$$

holds for every ball $B = B(x, r)$ and a Lipschitz function $u \colon X \to \mathbb{R}$.

What do we mean by the self-improvement?

Let (ν, μ) be a pair of (q, p)-admissible measures. By Hölder's inequality, the pair (ν, μ) is

- $(q - \varepsilon, p)$-admissible, if $0 < \varepsilon \leq q - 1$ and
- $(q, p + \varepsilon)$-admissible, if $\varepsilon > 0$.

Questions:

- (Improvement on the left-hand side) Does there exist $\varepsilon > 0$ such that (ν, μ) is $(q + \varepsilon, p)$-admissible?
- (Improvement on the right-hand side) Does there exist $0 < \varepsilon \leq p - 1$ such that (ν, μ) is $(q, p - \varepsilon)$-admissible?

Answers: The answer to both of these questions is no, in general. Moreover, for the same reason.

The balance condition

Definition

Assume that $1 \leq q, p < \infty$. We say that a pair (ν, μ) of locally finite Borel measures satisfies a (q, p)-balance condition, if

$$\frac{r'}{r}\left(\frac{\nu(B')}{\nu(B)}\right)^{\frac{1}{q}} \leq C\left(\frac{\mu(B')}{\mu(B)}\right)^{\frac{1}{p}}$$

whenever

$$B = B(x, r) \quad \text{and} \quad B' = B(x', r')$$

are balls in X such that $x' \in B$ and $0 < r' \leq r$.

The case $\nu = \mu$

Example

Let $X = \mathbb{R}^n$ is equipped with the Euclidean metric and $\mu = \nu = \mathcal{L}_n$. Then (ν, μ) satisfies a (q, p)-balance condition with $q = np/(n-p) > p$ for $1 \le p < n$.

Example

More generally, let $1 \le p < \infty$ and let X be a two measure metric space such that $\nu = \mu$. Then there exists $q > p$ such that X satisfies a (q, p)-balance condition. Furthermore, if $1 < p < \infty$, then X satisfies a $(p, p - \varepsilon)$-balance condition for some $\varepsilon > 0$. This explains why the balance condition is not visible in the self-improvement results for the one measure Poincaré inequalities.

Poincaré implies balance

Lemma (Chanillo–Wheeden 1985)

Let $1 \le q, p < \infty$ and assume that (ν, μ) is (q, p)-admissible. Then the pair (ν, μ) satisfies the (q, p)-balance condition.

The idea of a proof.

Test the two measure Poincaré inequality by using the Lipschitz function

$$u(y) = \operatorname{dist}(y, x') \max\left\{1 - \frac{\operatorname{dist}(y, B')}{r'}, 0\right\}, \qquad y \in X.$$

The estimates are a bit tedious. □

A (q, p)-balance condition is not sufficient for (q, p)-admissibility, but it turns out to be a very useful necessary condition.

Poincaré and balance imply Sobolev

Theorem (Franchi–Pérez–Wheeden 1998)

Let $1 \le p \le q < \infty$ *and let* (ν, μ) *be a pair of locally finite Borel measures. Assume that*

- *the pair* (ν, μ) *is* $(1, p)$*-admissible and*
- *the pair* (ν, μ) *satisfies a* (q, p)*-balance condition.*

Then the pair (ν, μ) *is* (q, p)*-admissible.*

Example

Let μ be the n-dimensional Lebesgue measure on $\mathbb{R}^n$.
Let $1 < p < n$ and $\nu = w\mu$, where

$$w(x) = |x|^{-p}, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

- w belongs to the Muckenhoupt class $A_1(\mu) \subset A_\infty(\mu)$.
- The pair (ν, μ) is (p, p)-admissible.
- The (p, p)-balance condition.
- The pair (ν, μ) is not $(p + \varepsilon, p)$-admissible, since it does not satisfy a $(p + \varepsilon, p)$-balance condition for any $\varepsilon > 0$.
- The pair (ν, μ) is not $(p, p - \varepsilon)$-admissible, since it does not satisfy a $(p, p - \varepsilon)$-balance condition for any $0 < \varepsilon \leq p - 1$.

Takeaways

The previous example illustrates a case where a (p, p)-Poincaré inequality does not improve to a $(p, p-\varepsilon)$-Poincaré inequality for any $0 < \varepsilon \leq p-1$. The reason is that there is no $(p-\varepsilon)$-balance condition.

This does not occur in a metric space X equipped with a single measure μ. Therefore, in such a geodesic space X we have Keth–Zhong theorem.

Theorem (K.–Korte–Lehrbäck–Vähäkangas 2018)

Let $1 < p < \infty$ and $0 < \tau, \vartheta < p - 1$. Assume that X is a geodesic two-measure space satisfying the following assumptions:

- *the pair (ν, μ) is (p, p)-admissible,*
- *the pair (ν, μ) satisfies a $(p, p - \tau)$-balance condition for some $0 < \tau < p - 1$,*
- *(A_∞ condition) there are $c_{\nu,\mu} > 0$ and $\delta > 0$ such that*

$$\frac{\nu(A)}{\nu(B)} \leq c_{\nu,\mu} \left(\frac{\mu(A)}{\mu(B)} \right)^{\delta}$$

  whenever $B \subset X$ is a ball and $A \subset B$ is a Borel set.

Then the pair (ν, μ) is $(p, p - \varepsilon)$-admissible for some $0 < \varepsilon < p - 1$.

Remarks

- Self-improvement on the left-hand side can be understood in terms of the balance condition. These results are already available in the existing literature.
- We made a contribution to self-improvement on the right-hand side.
- It is unknown, whether the A_∞ condition can be avoided.

Takeaways

- A variant of single measure is originally proved by Keith–Zhong (Annals of Mathematics 2008). The proof is by contradiction and technical.
- A more straightforward proof has been discovered by Eriksson–Bique (2016). These proof is also complicated.
- A direct proof of the one measure case is provided by K.–Lehrbäck–Vähäkangas–Zhong (2017) and the two measure case by K.–Korte–Lehrbäck–Vähäkangas (2018).
- The proofs are based on maximal function arguments.

An open question

Let $1 < p < \infty$ Is it possible to characterize weights w for which

$$\int_X (M^\sharp u)^p w \, d\mu \lesssim \int_X g^p w \, d\mu$$

for every Lipschitz continuous $u : X \to \mathbb{R}$ and every upper gradient g of u? To our knowledge, this is an open problem even when $X = \mathbb{R}^n$ equipped with the Lebesgue measure.

Recall that

$$M^\sharp u(x) = \sup_{x \in B} \left(\frac{1}{\operatorname{diam}(B)^p} \fint_B |u - u_B|^p \, d\mu \right)^{\frac{1}{p}},$$

where the supremum is taken over all balls $B \subset X$ containing x. In our result, we had the estimate above with $w = (M^\sharp u)^{-\varepsilon}$.