Parabolic weighted norm inequalities

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December 16, 2018


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Outline of the talk

**Goal:** To develop a higher dimensional theory for Muckenhoupt weights and functions of bounded mean oscillation (BMO) related to certain nonlinear parabolic PDEs. This extends the existing one-dimensional theory to higher dimensions.

**Questions:** Characterization of the weighted norm inequalities for parabolic maximal functions through Muckenhoupt weights, Coifman-Rochberg type characterization of the parabolic BMO, Jones-Rubio de Francia type factorization of the parabolic Muckenhoupt weights, applications to PDEs.

**Tools:** Definitions that are compatible with the PDEs, Calderón-Zygmund type covering arguments, harmonic analysis techniques related to the weighted norm inequalities.
The Hardy–Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is

$$Mf(x) = \sup_{Q \ni x} \int_Q |f|,$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

Let $w \in L^1_{\text{loc}}(\mathbb{R}^n)$, $w \geq 0$, be a weight. The Muckenhoupt $A_p$ condition with $p > 1$ is

$$\sup_Q \int_Q w \left( \int_Q w^{1-p'} \right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$. 
The Muckenhoupt $A_1$ condition is

$$\sup_Q \int_Q w \left( \inf_Q w \right)^{-1} < \infty.$$ 

$A_\infty = \bigcup_{p \geq 1} A_p$.

Let $f \in L^1_{loc}(\mathbb{R}^n)$. $f \in \text{BMO}$, if

$$\sup_Q \int_Q |f - f_Q| < \infty.$$
The following statements are equivalent:

- $M : L^p(w) \to L^p(w)$, $p > 1$, is bounded, (maximal function theorem)
- $w \in A_p$, (Muckenhoupt’s theorem)
- $w = uv^{1-p}$ with $u, v \in A_1$. (Jones–Rubio de Francia factorization)

In addition:

- $\text{BMO} = \{ \lambda \log w : w \in A_p, \lambda > 0 \}$, (John–Nirenberg lemma)
- $f \in \text{BMO} \iff f = \alpha \log M\mu - \beta \log M\nu + b$ with $\mu, \nu$ positive Borel measures with almost everywhere finite maximal functions, $b \in L^\infty(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$. (Coifman–Rochberg characterization)
A nonnegative weak solution \( u \in W_{loc}^{1,p}(\mathbb{R}^n) \) to the elliptic \( p \)-Laplace equation

\[
\text{div}(|Du|^{p-2}Du) = 0, \quad p \in (1, \infty),
\]

satisfies the following properties:

- \( \log u \in \text{BMO} \), (logarithmic Caccioppoli’s estimate)
- \( u \in A_1 \), (weak Harnack’s inequality)
- \( \sup_Q u \leq C \inf_Q u \). (Harnack’s inequality)

These are the key points in Moser’s and Trudinger’s regularity theory in the 1960s.

**Question:** For which nonlinear parabolic PDEs is it possible to develop a similar theory? What are the correct definitions of the parabolic Muckenhoupt classes and the parabolic BMO?
The doubly nonlinear equation

A weak solution to the doubly nonlinear equation

\[(|u|^{p-2}u)_t - \text{div}(|Du|^{p-2}Du) = 0, \quad p \in (1, \infty),\]

is a function \(u = u(x, t) \in L^p_{loc}(-\infty, \infty; W^{1,p}_{loc}(\mathbb{R}^n))\) such that

\[
\int_\mathbb{R} \int_\mathbb{R}^n (|Du|^{p-2}Du \cdot D\phi - |u|^{p-2}u\phi_t) \, dx \, dt = 0
\]

for all \(\phi \in C^\infty_0(\mathbb{R}^{n+1}).\)

It is possible to consider more general equations if this type, but we only discuss the prototype equation here. From now on, the parameter \(p > 1\) will be fixed. When \(p = 2\), we have the heat equation.
The Barenblatt solution

Example

The function

\[ u(x, t) = t^{-\frac{n}{p(p-1)}} e^{-\frac{p-1}{p} \left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}} , \quad x \in \mathbb{R}^n, \quad t > 0, \]

is a solution of the doubly nonlinear equation in the upper half space \( \mathbb{R}^{n+1}_+ \).

Observe: \( u(x, t) > 0 \) for every \( x \in \mathbb{R}^n \) and \( t > 0 \). This indicates infinite speed of propagation of disturbances. When \( p = 2 \) we have the heat kernel.
Structural properties when $p \neq 2$

- Solutions can be scaled.
- Constants cannot be added to a solution.
- The sum of two solutions is not a solution.
If $u(x, t)$ is a solution, so does $u(\lambda x, \lambda^p t)$ with $\lambda > 0$.

This suggests that in the natural geometry for the doubly nonlinear equation the time variable scales as the modulus of the space variable raised to power $p$.

Consequently, the Euclidean balls and cubes have to be replaced by parabolic rectangles respecting this scaling in all estimates.
Parabolic rectangles

Definition

Let $Q = Q(x, l) \subset \mathbb{R}^n$ be a cube with center $x$ and side length $l$. Let $p \in [1, \infty)$, $\gamma \in [0, 1)$ and $t \in \mathbb{R}$. Denote

$$R = R(x, t, l) = Q(x, l) \times (t - l^p, t + l^p),$$
$$R^+(\gamma) = Q(x, l) \times (t + \gamma l^p, t + l^p) \quad \text{and}$$
$$R^-(\gamma) = Q(x, l) \times (t - l^p, t - \gamma l^p).$$

We say that $R$ is a parabolic rectangle with center at $(x, t)$ and sidelenath $l$. $R^{\pm}(\gamma)$ are the upper and lower parts of $R$. Parameter $\gamma$ is the time lag.
Remarks

- These rectangles respect the natural geometry of the doubly nonlinear equation.
- The time lag $\gamma > 0$ is an unavoidable feature of the theory rather than a mere technicality. This can be seen from the Barenblatt solution and the heat kernel already when $p = 2$. For example, Harnack’s inequality does not hold without a time lag.
Lemma (Moser 1964, Trudinger 1968, K.–Kuusi 2007)

If $u$ is a nonnegative weak solution of the doubly nonlinear equation

$$(u^{p-1})_t - \text{div}(|Du|^{p-2}Du) = 0, \quad p \in (1, \infty),$$

then we have scale and location invariant Harnack's inequality

$$\sup_{R(\gamma)^-} u \leq C(n, p, \gamma) \inf_{R(\gamma)^+} u$$

with $\gamma > 0$. 
Proof.

\[
\sup_{R(\gamma)^-} u \leq C \left( \int_{2R(\gamma)^-} u^\varepsilon \right)^{\frac{1}{\varepsilon}} \\
\leq C \left( \int_{2R(\gamma)^+} u^{-\varepsilon} \right)^{-\frac{1}{\varepsilon}} \\
\leq C \inf_{R(\gamma)^+} u.
\]

The second inequality follows from a logarithmic Caccioppoli estimate and a parabolic John-Nirenberg lemma (or a Bombieri lemma). We shall return to this later.
Parabolic Muckenhoupt condition

**Definition**

Let $\gamma \in (0, 1)$ and $q > 1$. $w \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$, $w > 0$, is in the parabolic Muckenhoupt class $A^+_q(\gamma)$, if

$$
\sup_R \int_{R(\gamma)^-} w \left( \int_{R(\gamma)^+} w^{1-q'} \right)^{q-1} < \infty,
$$

where the supremum is over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $w \in A^-_q(\gamma)$.

**Observe:** The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the upper and lower parts $R^\pm(\gamma)$ is essential for us.
Remarks

- Classical $A_q$ weights with a trivial extension in time belong to the parabolic $A_q^+(\gamma)$ class.
- If $w \in A_q^+(\gamma)$, then $e^t w \in A_q^+(\gamma)$.
- Parabolic $A_q^+(\gamma)$ weights are not necessarily doubling, because they can grow arbitrarily fast in time.
Harnack’s inequality implies that nonnegative solutions to the doubly nonlinear equation belong to $A^+_q(\gamma)$ for every $\gamma \in (0, 1)$ and $q > 1$.

**Observe:** This gives examples of nontrivial functions in the parabolic Muckenhoupt classes. For example, the Barenblatt solution belongs all parabolic Muckenhoupt classes.
Lemma (K.-Saari)

- **(Inclusion)** $1 < q < r < \infty \implies A_q^+(\gamma) \subset A_r^+(\gamma)$.
- **(Duality)** $w \in A_q^+(\gamma) \iff w^{1-q'} \in A_{q'}^-(\gamma)$.
- **(Forward in time doubling)** If $w \in A_q^+(\gamma)$ and $E \subset R^+(\gamma)$, then
  \[
  \frac{w(R^-(\gamma))}{w(E)} \leq C \left( \frac{|R^-(\gamma)|}{|E|} \right)^q.
  \]
- **(Equivalence)** If $w \in A_q^+(\gamma)$ for some $\gamma \in [0, 1)$, then $w \in A_q^+(\gamma')$ for all $\gamma' \in (0, 1)$. 
Proof.

The fact that the conditions $A_q^+(\gamma)$ are equivalent for all $\gamma \in (0, 1)$ follows from duality, forward in time doubling condition and a subdivision argument using the fact that the parabolic rectangles become flat at small scales for $p > 1$. 

$\square$
The parabolic maximal operator

**Definition**

Let $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ and $\gamma \in (0, 1)$. We define the parabolic forward in time maximal function

$$M^{\gamma+} f(x, t) = \sup \int_{R^+(\gamma)} |f|,$$

where the supremum is over all parabolic rectangles $R(x, t)$ centered at $(x, t)$. The parabolic backward in time operator $M^{\gamma-}$ is defined analogously.

**Observe:** The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the point $(x, t)$ and the rectangle $R^+(\gamma)$ is essential for us.
Our main result is

**Theorem (K.–Saari 2016)**

*Let $q > 1$. The following claims are equivalent:*

- $w \in A^+_q(\gamma)$ for some $\gamma \in (0, 1)$,
- $w \in A^+_q(\gamma)$ for all $\gamma \in (0, 1)$,
- *(Strong type estimate)* $M^{\gamma^+} : L^q(w) \to L^q(w)$ for all $\gamma \in (0, 1)$,
- *(Weak type estimate)* $M^{\gamma^+} : L^q(w) \to L^{q,\infty}(w)$ for all $\gamma \in (0, 1)$.
Proof.

- Equivalence of $A^+_q(\gamma)$ for all $\gamma \in (0, 1)$ is applied to prove that the $A^+_q(\gamma)$ condition is necessary.
- The sufficiency part of the weak type estimate uses a modification (parabolic rectangles $n \geq 2$) of a covering argument by Forzani, Martín–Reyes and Ombrosi.
- The strong type estimate follows from a reverse Hölder type inequality, equivalence of $A^+_q(\gamma)$ for all $\gamma \in (0, 1)$ and interpolation.
For the one-sided maximal operator \((\gamma = 0, \text{that is, without a lag})\)

\[
M^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_{x}^{x+h} |f|
\]

and the corresponding one-sided Muckenhoupt weights \(w \in A_p^+\),

\[
\sup_{x,h} \frac{1}{h} \int_{x-h}^{x} w \left( \frac{1}{h} \int_{x}^{x+h} w^{1-p'} \right)^{p-1} < \infty,
\]

it is known that \(M^+ : L^p(w) \to L^p(w) \iff w \in A_p^+\).
(Sawyer 1986)
There is a complete one-dimensional theory including $A^+_\infty$, one-sided reverse Hölder inequality and one-sided BMO. (Cruz–Uribe, Martín–Reyes, Neugebauer, Olesen, Pick, de la Torre, ...)

The time lag disappears in the one-dimensional case.

Higher dimensional case has turned out to be more challenging. Some partial results are known. (Berkovits, Forzani, Lerner, Martín–Reyes, Ombrosi 2010–2011)

Our approach gives a complete characterization in the higher dimensional case with a time lag.
Lemma (K.–Saari 2016)

Let $w \in A_q^+(\gamma)$ and $\gamma \in (0, 1)$. Then there is $\varepsilon > 0$ such that

$$
\left( \int_{R^- (0)} w^{1+\varepsilon} \right)^{1/(1+\varepsilon)} \leq C \int_{R^+ (0)} w
$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$.

**Observe:** This is weaker than the standard RHI, because there is a time lag between the rectangles $R^- (0)$ and $R^+ (0)$. Otherwise, we would have the standard $A_\infty$ condition, which implies that the weight is doubling.
Remarks

- A self improving property:

  \[ w \in A_q^+(\gamma) \implies w \in A_{q-\epsilon}^+(\gamma) \text{ for some } \epsilon > 0. \]

  An application of the RHI makes the lag bigger, but this does not matter.

- The lag appears even if we begin with a parabolic Muckenhoupt condition without lag:

  \[ A_p^+(0) \implies A_{p-\epsilon}^+(\gamma) \text{ for some } \epsilon > 0 \text{ and } \gamma > 0. \]

  The same phenomenon was encountered by Lerner and Ombrosi (2010, \( n = 2 \)) and Berkovits (2011, \( n \geq 2 \)).
Proof.

First we prove a distribution set estimate

\[ w(\hat{R} \cap \{w > \lambda\}) \leq C\lambda |\tilde{R} \cap \{w > \beta \lambda\}|, \]

where \(\hat{R}\) and \(\tilde{R}\) are certain parabolic rectangles.

It is not clear how to apply dyadic structures for parabolic rectangles. However, certain Calderón-Zygmund type covering arguments can be used.

Once the distribution set estimate is done, the claim follows from Cavalieri’s principle.
Except for the one-dimensional case, an extra time lag seems to appear in the arguments. Roughly speaking a condition without lag implies strong type estimates for a parabolic maximal operator with a time lag. This means that a complete characterization without a lag seems to be out of reach. We do not know whether this is possible or not.

In our case both the maximal operator and the Muckenhoupt condition have a time lag $\gamma > 0$. Moreover $p > 1$. This allows us to prove necessity and sufficiency of the parabolic Muckenhoupt condition for the weak and strong type weighted norm inequalities for the corresponding maximal function.
Let $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ and $\gamma \in (0, 1)$. We say that $f \in \text{PBMO}^+$ if and for each parabolic rectangle $R$ there is a constant $a_R$ such that

$$
\sup_R \left( \int_{R(\gamma)^+} (f - a_R)^+ + \int_{R(\gamma)^-} (f - a_R)^- \right) < \infty,
$$

where the supremum is taken over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $f \in \text{PBMO}^-$. 

**Observe:** The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ is essential for us. The definitions with different lags are equivalent as in the case of the Muckenhoupt condition.
Remark

The original condition in the papers by Moser and Garofalo–Fabes is

$$\sup_{R} \left( \int_{R(0)^+} \sqrt{(f - a_R)_+} + \int_{R(0)^-} \sqrt{(f - a_R)_-} \right) < \infty.$$ 

By the John–Nirenberg lemma, these functions belong to $\text{PBMO}^+$. We shall return to this question. Thus our approach extends the classical theory.
The parabolic John–Nirenberg lemma: Let $u \in \text{PBMO}^+$ and $\gamma \in (0, 1)$. Then there are constants $A, B > 0$ such that

$$|R^+(\gamma) \cap \{(u - a_R)_+ > \lambda\}| \leq Ae^{-B\lambda}|R^+(\gamma)|$$

and

$$|R^-(\gamma) \cap \{(u - a_R)_- > \lambda\}| \leq Ae^{-B\lambda}|R^-(\gamma)|.$$
Theorem (Moser 1964, Trudinger 1968, K.–Saari 2016)

Let $u$ be a nonnegative weak solution of the doubly nonlinear equation

$$(u^{p-1})_t - \text{div}(|Du|^{p-2}Du) = 0, \quad p \in (1, \infty).$$

Then $-\log u \in \text{PBMO}^+$. 

Observe: This gives examples of parabolic BMO functions.
Proof.

- \( f = -\log u \).
- Cavalieri’s principle and a logarithmic Caccippoli inequality imply
  \[
  \sup_{R} \left( \int_{R^+} (f - a_R)^\beta + \int_{R^-} (f - a_R)^\beta \right) < \infty
  \]
  with \( \beta = \min\{\frac{p-1}{2}, 1\} \leq 1 \).
- The John-Nirenberg machinery gives
  \[
  \sup_{R} \left( \int_{R^+(\gamma)} (f - a_R)^\gamma + \int_{R^-(\gamma)} (f - a_R)^\gamma \right) < \infty
  \]
  for \( \gamma > 0 \).
- \( f \in \text{PBMO}^+ \).
Theorem (K.–Saari 2016)

Let \( f \in \text{PBMO}^+ \) and \( \gamma \in (0, 1) \). Then there are positive Borel measures \( \mu, \nu \) satisfying

\[
M^{\gamma^+} \mu < \infty \quad \text{and} \quad M^{\gamma^-} \nu < \infty
\]

almost everywhere in \( \mathbb{R}^{n+1} \), a bounded function \( b \) and constants \( \alpha, \beta \geq 0 \) such that

\[
f = -\alpha \log M^{\gamma^-} \mu + \beta \log M^{\gamma^+} \nu + b.
\]

Conversely, if the above holds with \( \gamma = 0 \), then \( f \in \text{PBMO}^+ \).

Observe: This gives a method to produce examples of parabolic BMO functions.
Proof.

\[ \text{PBMO}^+ = \{-\lambda \log w : w \in A^+_q(\gamma), \lambda > 0\}. \text{ (The John-Nirenberg lemma)} \]

Let \( \delta \in (0, 1) \) and \( \gamma \in (0, \delta 2^{1-p}) \). Then

\[ w \in A^+_q(\delta) \iff w = u v^{1-p}, \]

where \( u \in A^+_1(\gamma) \) and \( v \in A^-_1(\gamma) \). A weight \( w \) belongs to the parabolic Muckenhoupt \( A^+_1(\gamma) \) class, if

\[ M^{\gamma^-} w \leq Cw \]

almost everywhere in \( \mathbb{R}^{n+1} \). The class \( A^-_1(\gamma) \) is defined by reversing the direction of time. (Jones factorization)

The rest follows from a similar reasoning as in the classical case. (Coifman–Rochberg, Coifman–Jones–Rubio de Francia)
If $f \in \text{BMO}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain satisfying a suitable chaining condition, then John–Nirenberg inequality holds not only locally over cubes but also globally over whole $\Omega$. (Reimann–Rychener, Smith-Stegenga, Staples)

- Olli Saari has obtained similar parabolic local to global results in space-time cylinders.
- The proofs are based on delicate chaining arguments.
Theorem (Saari 2016)

Let $u$ be a positive weak solution to the doubly nonlinear equation on $\Omega \times (0, T)$, where $\Omega$ is a nice domain (satisfying a quasihyperbolic boundary condition). Then there exists $\epsilon > 0$ such that

$$u^\epsilon \in L^1(\Omega \times (0, T - \epsilon)).$$

Proof.

Follows from a global John–Nirenberg inequality.

Remark: This result seems to be new even for the heat equation.
• It is possible to develop theory for parabolic Muckenhoupt weights related to the doubly nonlinear parabolic PDE. The results are new even for the heat equation.

• A complete Muckenhoupt type characterization of the weighted norm inequalities can be obtained with connections to the parabolic BMO.

• The time lag is both a challenge and an opportunity.

• There is a rather complete one-dimensional theory without the lag.

• The proofs are based on delicate Calderón–Zygmund type covering arguments.

• The results and methods can be applied in nonlinear PDEs.
Open problems

- $A_\infty^+(\gamma)$? Partial results by K.-Saari.
- Strong type inequalities with $p = 1$ in the geometry?
- The case $\gamma = 0$?
- Mapping properties of the forward in time maximal operator? Partial results by Saari.
- Similar theory for other nonlinear parabolic PDEs?
- Metric measure spaces (Aimar)?