Parabolic weighted norm inequalities

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References

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Outline of the talk

- **Goal:** To develop a higher dimensional theory for Muckenhoupt weights and functions of bounded mean oscillation (BMO) related to certain nonlinear parabolic PDEs. This extends the existing one-dimensional theory to higher dimensions.
- **Questions:** Characterization of the weighted norm inequalities for parabolic maximal functions through Muckenhoupt weights, Coifman-Rochberg type characterization of the parabolic BMO, Jones-Rubio de Francia type factorization of the parabolic Muckenhoupt weights, applications to PDEs.
- **Tools:** Definitions that are compatible with the PDEs, Calderón-Zygmund type covering arguments, harmonic analysis techniques related to the weighted norm inequalities.

Elliptic concepts I

- The Hardy–Littlewood maximal function of $f \in L^1_{loc}(\mathbb{R}^n)$ is

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f|,$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$ containing x.

- Let $w \in L^1_{loc}(\mathbb{R}^n)$, $w \geq 0$, be a weight. The Muckenhoupt A_p condition with $p > 1$ is

$$\sup_Q \fint_Q w \left(\fint_Q w^{1-p'} \right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$.

Elliptic concepts II

- The Muckenhoupt A_1 condition is

$$\sup_Q \fint_Q w \left(\inf_Q w\right)^{-1} < \infty.$$

- $A_\infty = \bigcup_{p\geq 1} A_p$.
- Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. $f \in \text{BMO}$, if

$$\sup_Q \fint_Q |f - f_Q| < \infty.$$

Classical results in Harmonic Analysis

The following statements are equivalent:

- $M : L^p(w) \to L^p(w)$, $p > 1$, is bounded, (maximal function theorem)
- $w \in A_p$, (Muckenhoupt's theorem)
- $w = uv^{1-p}$ with $u, v \in A_1$. (Jones–Rubio de Francia factorization)

In addition:

- $\mathrm{BMO} = \{\lambda \log w : w \in A_p, \lambda > 0\}$, (John–Nirenberg lemma)
- $f \in \mathrm{BMO} \iff f = \alpha \log M\mu - \beta \log M\nu + b$ with μ, ν positive Borel measures with almost everywhere finite maximal functions, $b \in L^\infty(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$. (Coifman–Rochberg characterization)

A PDE point of view

A nonnegative weak solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ to the elliptic p-Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \qquad p \in (1, \infty),$$

satisfies the following properties:

- $\log u \in \mathrm{BMO}$, (logarithmic Caccioppoli's estimate)
- $u \in A_1$, (weak Harnack's inequality)
- $\sup_Q u \leq C \inf_Q u$. (Harnack's inequality)

These are the key points in Moser's and Trudinger's regularity theory in the 1960s.

Question: For which nonlinear parabolic PDEs is it possible to develop a similar theory? What are the correct definitions of the parabolic Muckenhoupt classes and the parabolic BMO?

The doubly nonlinear equation

A weak solution to the doubly nonlinear equation

$$(|u|^{p-2}u)_t - \mathrm{div}(|Du|^{p-2}Du) = 0, \qquad p \in (1, \infty),$$

is a function $u = u(x,t) \in L^p_{\mathrm{loc}}(-\infty, \infty; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(|Du|^{p-2} Du \cdot D\phi - |u|^{p-2} u \phi_t \right) \,\mathrm{d}x \,\mathrm{d}t = 0$$

for all $\phi \in C_0^\infty(\mathbb{R}^{n+1})$.

It is possible to consider more general equations if this type, but we only discuss the prototype equation here. From now on, the parameter $p > 1$ will be fixed. When $p = 2$, we have the heat equation.

The Barenblatt solution

Example

The function

$$u(x,t) = t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}}, \; x \in \mathbb{R}^n, \; t > 0,$$

is a solution of the doubly nonlinear equation in the upper half space $\mathbb{R}^{n+1}_+$.

Observe: $u(x,t) > 0$ for every $x \in \mathbb{R}^n$ and $t > 0$. This indicates infinite speed of propagation of disturbances. When $p = 2$ we have the heat kernel.

Structural properties when $p \neq 2$

- Solutions can be scaled.
- Constants cannot be added to a solution.
- The sum of two solutions is not a solution.

Parabolic geometry

- If $u(x,t)$ is a solution, so does $u(\lambda x, \lambda^p t)$ with $\lambda > 0$.
- This suggests that in the natural geometry for the doubly nonlinear equation the time variable scales as the modulus of the space variable raised to power p.
- Consequently, the Euclidean balls and cubes have to be replaced by parabolic rectangles respecting this scaling in all estimates.

Parabolic rectangles

Definition

Let $Q = Q(x, l) \subset \mathbb{R}^n$ be a cube with center x and side length l. Let $p \in [1, \infty)$, $\gamma \in [0, 1)$ and $t \in \mathbb{R}$. Denote

$$\begin{aligned} R = R(x, t, l) &= Q(x, l) \times (t - l^p, t + l^p), \\ R^+(\gamma) &= Q(x, l) \times (t + \gamma l^p, t + l^p) \quad \text{and} \\ R^-(\gamma) &= Q(x, l) \times (t - l^p, t - \gamma l^p). \end{aligned}$$

We say that R is a parabolic rectangle with center at (x, t) and sidelength l. $R^{\pm}(\gamma)$ are the upper and lower parts of R. Parameter γ is the time lag.

Remarks

- These rectangles respect the natural geometry of the doubly nonlinear equation.
- The time lag $\gamma > 0$ is an unavoidable feature of the theory rather than a mere technicality. This can be seen from the Barenblatt solution and the heat kernel already when $p = 2$. For example, Harnack's inequality does not hold without a time lag.

Harnack's inequality

Lemma (Moser 1964, Trudinger 1968, K.–Kuusi 2007)

If u is a nonnegative weak solution of the doubly nonlinear equation

$$(u^{p-1})_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \qquad p \in (1,\infty),$$

then we have scale and location invariant Harnack's inequality

$$\sup_{R(\gamma)^-} u \le C(n,p,\gamma) \inf_{R(\gamma)^+} u$$

with $\gamma > 0$*.*

Proof.

$$
\begin{aligned}
\sup_{R(\gamma)^-} u &\le C \left(\fint_{2R(\gamma)^-} u^{\varepsilon} \right)^{\frac{1}{\varepsilon}} \\
&\le C \left(\fint_{2R(\gamma)^+} u^{-\varepsilon} \right)^{-\frac{1}{\varepsilon}} \\
&\le C \inf_{R(\gamma)^+} u.
\end{aligned}
$$

The second inequality follows from a logarithmic Caccioppoli estimate and a parabolic John-Nirenberg lemma (or a Bombieri lemma). We shall return to this later. □

Parabolic Muckenhoupt condition

Definition

Let $\gamma \in (0,1)$ and $q > 1$. $w \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $w > 0$, is in the parabolic Muckenhoupt class $A^+_q(\gamma)$, if

$$\sup_R \fint_{R(\gamma)^-} w \left(\fint_{R(\gamma)^+} w^{1-q'} \right)^{q-1} < \infty,$$

where the supremum is over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $w \in A^-_q(\gamma)$.

Observe: The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the upper and lower parts $R^\pm(\gamma)$ is essential for us.

Remarks

- Classical A_q weights with a trivial extension in time belong to the parabolic $A_q^+(\gamma)$ class.
- If $w \in A_q^+(\gamma)$, then $e^t w \in A_q^+(\gamma)$.
- Parabolic $A_q^+(\gamma)$ weights are not necessarily doubling, because they can grow arbitrarily fast in time.

PDE connections

Harnack's inequality implies that nonnegative solutions to the doubly nonlinear equation belong to $A_q^+(\gamma)$ for every $\gamma \in (0,1)$ and $q > 1$.

Observe: This gives examples of nontrivial functions in the parabolic Muckenhoupt classes. For example, the Barenblatt solution belongs all parabolic Muckenhoupt classes.

Structural properties

Lemma (K.-Saari)

- *(Inclusion)* $1 < q < r < \infty \Longrightarrow A_q^+(\gamma) \subset A_r^+(\gamma)$.
- *(Duality)* $w \in A_q^+(\gamma) \Longleftrightarrow w^{1-q'} \in A_{q'}^-(\gamma)$.
- *(Forward in time doubling) If* $w \in A_q^+(\gamma)$ *and* $E \subset R^+(\gamma)$*, then*
$$\frac{w(R^-(\gamma))}{w(E)} \le C \left(\frac{|R^-(\gamma)|}{|E|} \right)^q .$$
- *(Equivalence) If* $w \in A_q^+(\gamma)$ *for some* $\gamma \in [0,1)$*, then* $w \in A_q^+(\gamma')$ *for all* $\gamma' \in (0,1)$.

Proof.

The fact that the conditions $A_q^+(\gamma)$ are equivalent for all $\gamma \in (0,1)$ follows from duality, forward in time doubling condition and a subdivision argument using the fact that the parabolic rectangles become flat at small scales for $p > 1$. □

The parabolic maximal operator

Definition

Let $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ and $\gamma \in (0,1)$. We define the parabolic forward in time maximal function

$$M^{\gamma+} f(x,t) = \sup \fint_{R^+(\gamma)} |f|,$$

where the supremum is over all parabolic rectangles $R(x,t)$ centered at (x,t). The parabolic backward in time operator $M^{\gamma-}$ is defined analogously.

Observe: The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the point (x,t) and the rectangle $R^+(\gamma)$ is essential for us.

Characterization of parabolic Muckenhoupt weights

Our main result is

Theorem (K.–Saari 2016)

Let $q > 1$. The following claims are equivalent:

- $w \in A_q^+(\gamma)$ *for some* $\gamma \in (0,1)$,
- $w \in A_q^+(\gamma)$ *for all* $\gamma \in (0,1)$,
- *(Strong type estimate)* $M^{\gamma+} : L^q(w) \to L^q(w)$ *for all* $\gamma \in (0,1)$,
- *(Weak type estimate)* $M^{\gamma+} : L^q(w) \to L^{q,\infty}(w)$ *for all* $\gamma \in (0,1)$.

Proof.

- Equivalence of $A_q^+(\gamma)$ for all $\gamma \in (0,1)$ is applied to prove that the $A_q^+(\gamma)$ condition is necessary.
- The sufficiency part of the weak type estimate uses a modification (parabolic rectangles $n \geq 2$) of a covering argument by Forzani, Martín–Reyes and Ombrosi.
- The strong type estimate follows from a reverse Hölder type inequality, equivalence of $A_q^+(\gamma)$ for all $\gamma \in (0,1)$ and interpolation.

□

One-dimensional theory

For the one-sided maximal operator ($\gamma = 0$, that is, without a lag)

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|$$

and the corresponding one-sided Muckenhoupt weights $w \in A_p^+$,

$$\sup_{x,h} \frac{1}{h} \int_{x-h}^{x} w \left(\frac{1}{h} \int_x^{x+h} w^{1-p'} \right)^{p-1} < \infty,$$

it is known that $M^+ : L^p(w) \to L^p(w) \iff w \in A_p^+$.
(Sawyer 1986)

Takeaways

- There is a complete one-dimensional theory including A_∞^+, one-sided reverse Hölder inequality and one-sided BMO . (Cruz–Uribe, Martín–Reyes, Neugebauer, Olesen, Pick, de la Torre,. . .)
- The time lag disappears in the one-dimensional case.
- Higher dimensional case has turned out to be more challenging. Some partial results are known. (Berkovits, Forzani, Lerner, Martín–Reyes, Ombrosi 2010–2011)
- Our approach gives a complete characterization in the higher dimensional case with a time lag.

Reverse Hölder inequality

Lemma (K.–Saari 2016)

Let $w \in A_q^+(\gamma)$ and $\gamma \in (0,1)$. Then there is $\varepsilon > 0$ such that

$$\left(\fint_{R^-(0)} w^{1+\varepsilon} \right)^{1/(1+\varepsilon)} \le C \fint_{R^+(0)} w$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$.

Observe: This is weaker than the standard RHI, because there is a time lag between the rectangles $R^-(0)$ and $R^+(0)$. Otherwise, we would have the standard A_∞ condition, which implies that the weight is doubling.

Remarks

- A self improving property:

$$w \in A_q^+(\gamma) \Longrightarrow w \in A_{q-\epsilon}^+(\gamma) \quad \text{for some } \varepsilon > 0.$$

  An application of the RHI makes the lag bigger, but this does not matter.

- The lag appears even if we begin with a parabolic Muckenhoupt condition without lag:

$$A_p^+(0) \Longrightarrow A_{p-\epsilon}^+(\gamma) \quad \text{for some } \varepsilon > 0 \text{ and } \gamma > 0.$$

  The same phenomenon was encountered by Lerner and Ombrosi (2010, $n = 2$) and Berkovits (2011, $n \geq 2$).

Proof.

- First we prove a distribution set estimate

$$w(\widehat{R} \cap \{w > \lambda\}) \le C\lambda|\widetilde{R} \cap \{w > \beta\lambda\}|,$$

  where $\widehat{R}$ and $\widetilde{R}$ are certain parabolic recangles.
- It is not clear how to apply dyadic structures for parabolic rectangles. However, certain Calderón-Zygmund type covering arguments can be used.
- Once the distribution set estimate is done, the claim follows from Cavalieri's principle.

□

Takeaways

Except for the one-dimensional case, an extra time lag seems to appear in the arguments. Roughly speaking a condition without lag implies strong type estimates for a parabolic maximal operator with a time lag. This means that a complete characterization without a lag seems to be out of reach. We do not know whether this is possible or not.

In our case both the maximal operator and the Muckenhoupt condition have a time lag $\gamma > 0$. Moreover $p > 1$. This allows us to prove necessity and sufficiency of the parabolic Muckenhoupt condition for the weak and strong type weighted norm inequalities for the corresponding maximal function.

Parabolic BMO

Definition

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $\gamma \in (0,1)$. We say that $f \in \mathrm{PBMO}^+$ if and for each parabolic rectangle R there is a constant a_R such that

$$\sup_R \left(\fint_{R(\gamma)^+} (f - a_R)_+ + \fint_{R(\gamma)^-} (f - a_R)_- \right) < \infty,$$

where the supremum is taken over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $f \in \mathrm{PBMO}^-$.

Observe: The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ is essential for us. The definitions with different lags are equivalent as in the case of the Muckenhoupt condition.

Remark

The original condition in the papers by Moser and Garofalo–Fabes is

$$\sup_R \left(\fint_{R(0)^+} \sqrt{(f - a_R)_+} + \fint_{R(0)^-} \sqrt{(f - a_R)_-} \right) < \infty.$$

By the John–Nirenberg lemma, these functions belong to PBMO^+. We shall return to this question. Thus our approach extends the classical theory.

Parabolic John-Nirenberg lemma

Theorem (Moser, Garofalo–Fabes, Aimar)

The parabolic John–Nirenberg lemma: Let $u \in \mathrm{PBMO}^+$ *and* $\gamma \in (0,1)$. *Then there are constants* $A, B > 0$ *such that*

$$|R^+(\gamma) \cap \{(u - a_R)_+ > \lambda\}| \le Ae^{-B\lambda}|R^+(\gamma)|$$

and

$$|R^-(\gamma) \cap \{(u - a_R)_- > \lambda\}| \le Ae^{-B\lambda}|R^-(\gamma)|.$$

Remark: The lag $\gamma > 0$ in the definitions allows us to characterize PBMO^+ with the John–Nirenberg lemma. The John–Nirenberg lemma cannot hold with $\gamma = 0$, because this would imply parabolic Harnack's estimates without a lag.

PDE connections

Theorem (Moser 1964, Trudinger 1968, K.–Saari 2016)

Let u be a nonnegative weak solution of the doubly nonlinear equation

$$(u^{p-1})_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \qquad p \in (1,\infty).$$

Then $-\log u \in \mathrm{PBMO}^+$.

Observe: This gives examples of parabolic BMO functions.

Proof.

- $f = -\log u$.
- Cavalieri's principle and a logarithmic Caccippoli inequality imply

$$\sup_R \left(\fint_{R^+} (f - a_R)_+^\beta + \fint_{R^-} (f - a_R)_-^\beta \right) < \infty$$

with $\beta = \min\{\frac{p-1}{2}, 1\} \leq 1$.

- The John-Nirenberg machinery gives

$$\sup_R \left(\fint_{R^+(\gamma)} (f - a_R)_+ + \fint_{R^-(\gamma)} (f - a_R)_- \right) < \infty$$

for $\gamma > 0$.

- $f \in \mathrm{PBMO}^+$.

□

Coifman–Rochberg theorem

Theorem (K.–Saari 2016)

Let $f \in \mathrm{PBMO}^+$ and $\gamma \in (0,1)$. Then there are positive Borel measures μ, ν satisfying

$$M^{\gamma+}\mu < \infty \quad \text{and} \quad M^{\gamma-}\nu < \infty$$

almost everywhere in $\mathbb{R}^{n+1}$, a bounded function b and constants $\alpha, \beta \geq 0$ such that

$$f = -\alpha \log M^{\gamma-}\mu + \beta \log M^{\gamma+}\nu + b.$$

Conversely, if the above holds with $\gamma = 0$, then $f \in \mathrm{PBMO}^+$.

Observe: This gives a method to produce examples of parabolic BMO functions.

Proof.

- $\mathrm{PBMO}^+ = \{-\lambda \log w : w \in A_q^+(\gamma), \lambda > 0\}$. (The John-Nirenberg lemma)
- Let $\delta \in (0,1)$ and $\gamma \in (0, \delta 2^{1-p})$. Then

$$w \in A_q^+(\delta) \iff w = uv^{1-p},$$

where $u \in A_1^+(\gamma)$ and $v \in A_1^-(\gamma)$. A weight w belongs to the parabolic Muckenhoupt $A_1^+(\gamma)$ class, if

$$M^{\gamma-} w \le Cw$$

almost everywhere in $\mathbb{R}^{n+1}$. The class $A_1^-(\gamma)$ is defined by reversing the direction of time. (Jones factorization)

- The rest follows from a similar reasoning as in the classical case. (Coifman–Rochberg, Coifman–Jones–Rubio de Francia)

□

A local to global property

- If $f \in \mathrm{BMO}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain satisfying a suitable chaining condition, then John–Nirenberg inequality holds not only locally over cubes but also globally over whole Ω. (Reimann–Rychener, Smith-Stegenga, Staples)
- Olli Saari has obtained similar parabolic local to global results in space-time cylinders.
- The proofs are based on delicate chaining arguments.

A global integrability result

Theorem (Saari 2016)

Let u be a positive weak solution to the doubly nonlinear equation on $\Omega \times (0, T)$, where Ω is a nice domain (satisfying a quasihyperbolic boundary condition). Then there exists $\epsilon > 0$ such that

$$u^{\epsilon} \in L^1(\Omega \times (0, T - \epsilon)).$$

Proof.

Follows from a global John–Nirenberg inequality. □

Remark: This result seems to be new even for the heat equation.

Takeaways

- It is possible to develop theory for parabolic Muckenhoupt weights related to the doubly nonlinear parabolic PDE. The results are new even for the heat equation.
- A complete Muckenhoupt type characterization of the weighted norm inequalities can be obtained with connections to the parabolic BMO.
- The time lag is both a challenge and an opportunity.
- There is a rather complete one-dimensional theory without the lag.
- The proofs are based on delicate Calderón–Zygmund type covering arguments.
- The results and methods can be applied in nonlinear PDEs.

Open problems

- $A_\infty^+(\gamma)$? Partial results by K.–Saari.
- Strong type inequalities with $p = 1$ in the geometry?
- The case $\gamma = 0$?
- Mapping properties of the forward in time maximal operator? Partial results by Saari.
- Similar theory for other nonlinear parabolic PDEs?
- Metric measure spaces (Aimar)?