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Elliptic Partial Differential Equations

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Contents

1 Weak solutions

1.1 Euler-Lagrange equation

Second order elliptic equations of divergence type appear in the calculus of variations, which studies minimizers of certain integrals modeling, for example, the energy of a system. The underlying function space is usually assumed to be a Sobolev space, but we begin with a brief introduction under the assumption that minimizers exist and that all appearing functions are smooth. Later we consider the corresponding problems in Sobolev spaces and show that the minimizers are not necessarily smooth.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary and let

$$F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, F = F(x, \zeta, \xi)$$

be a smooth function. Smoothness in a compact set $\overline{\Omega}$ means that the function and its all partial derivatives have continuous extensions from Ω to $\overline{\Omega}$. Consider a variational integral

$$I(v) = \int_\Omega F(x, u(x), Dv(x))\,dx$$

for smooth functions $v : \overline{\Omega} \to \mathbb{R}$ satisfying the boundary condition

$$u = g \quad \text{on } \partial\Omega.$$

A function $u \in C^\infty(\overline{\Omega})$ is a minimizer of the variational integral $I(\cdot)$ above with the boundary values g, if

$$I(u) \leqslant I(v)$$

for every $v \in C^\infty(\overline{\Omega})$ with $v = g$ on $\partial\Omega$. In particular, we have

$$\int_\Omega F(x, u(x), Du(x))\,dx \leqslant \int_\Omega F(x, u(x) + \varphi(x), D(u(x) + \varphi(x))\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$. Let

$$i(\varepsilon) = I(u + \varepsilon\varphi), \quad \varepsilon \in \mathbb{R}.$$

If u is a minimizer, and since $u + \varepsilon\varphi = g$ on $\partial\Omega$, the function $i(\varepsilon)$ has a minimum at $\varepsilon = 0$, which implies that $i'(0) = 0$. Since

$$i(\varepsilon) = I(u + \varepsilon\varphi) = \int_\Omega F\left(x, u(x) + \varepsilon\varphi(x), Du(x) + \varepsilon D\varphi(x)\right) dx,$$

a direct computation by applying the chain rule and switching the order of differentiation and integration shows that

$$\begin{aligned} i'(\varepsilon) = \int_\Omega \Bigg(& \sum_{i=1}^n \frac{\partial}{\partial \xi_i} F\left(x, u(x) + \varepsilon\varphi(x), Du(x) + \varepsilon D\varphi\right) \frac{\partial \varphi}{\partial x_i}(x) \\ & + \frac{\partial}{\partial \zeta} F\left(x, u(x) + \varepsilon\varphi(x), Du(x) + \varepsilon D\varphi(x)\right) \varphi(x) \Bigg) dx. \end{aligned}$$

By setting $\varepsilon = 0$, we conclude that

$$0 = i'(0) = \int_\Omega \left(\sum_{i=1}^n \frac{\partial}{\partial \xi_i} F(x, u(x), Du(x)) \frac{\partial \varphi}{\partial x_i}(x) + \frac{\partial}{\partial \zeta} F(x, u(x), Du(x)) \varphi(x) \right) dx.$$

Since φ has a compact support in Ω, an integration by parts gives

$$\int_\Omega \left(-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \xi_i} F(x, u(x), Du(x)) \right) + \frac{\partial}{\partial \zeta} F(x, u(x), Du(x)) \right) \varphi(x)\, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. This implies that u is a solution to the partial differential equation

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \xi_i} F(x, u(x), Du(x)) \right) + \frac{\partial}{\partial \zeta} F(x, u(x), Du(x)) = 0 \quad \text{in} \quad \Omega,$$

or equivalently

$$-\operatorname{div} A(x, u(x), Du(x)) + B(x, u(x), Du(x)) = 0 \quad \text{in} \quad \Omega,$$

where

$$A = A(x, u(x), Du(x)) = \left(\frac{\partial}{\partial \xi_1} F(x, u(x), Du(x)), \dots, \frac{\partial}{\partial \xi_n} F(x, u(x), Du(x)) \right)$$

and

$$B = B(x, u(x), Du(x)) = \frac{\partial}{\partial \zeta} F(x, u(x), Du(x)).$$

Note that by the chain rule we have

$$-\sum_{i,j=1}^n \left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} F(x, u(x), Du(x)) \right) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \frac{\partial}{\partial \zeta} F(x, u(x), Du(x)) = 0 \quad \text{in} \quad \Omega.$$

This is the Euler-Lagrange equation associated with the variational integral $I(\cdot)$. Observe that this is a nonlinear second order partial differential equation of divergence form. Moreover, we have

$$\frac{\partial^2}{\partial \xi_i \partial \xi_j} F(x, u(x), Du(x)) = \frac{\partial^2}{\partial \xi_j \partial \xi_i} F(x, u(x), Du(x)), \quad i.j = 1, 2, \dots, n,$$

so that the coefficient matrix is symmetric. Convexity assumptions are needed to show existence and uniqueness of minimizers in the calculus of variation. More precisely, assume that there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial \xi_i \partial \xi_j} F(x,\zeta,\xi)\xi_i\xi_j \geqslant \theta|\xi|^2$$

for every $x \in \Omega$, $\zeta \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. This condition asserts that the mapping $\xi \mapsto F(x,\zeta,\xi)$ is uniformly convex for every $x \in \Omega$ and $\zeta \in \mathbb{R}$.

Example 1.1. Let

$$F(x,\zeta,\xi) = \frac{1}{2}|\xi|^2.$$

Then

$$\frac{\partial}{\partial \xi_i} F(x,\zeta,\xi) = \frac{1}{2}\frac{\partial}{\partial \xi_i}|\xi|^2 = \frac{1}{2}\frac{\partial}{\partial \xi_i}\sum_{i=1}^{\infty} \xi_i^2 = \xi_i, \quad i = 1,2,\dots,n,$$

and $\frac{\partial}{\partial \zeta} F(x,\zeta,\xi) = 0$. Thus the Euler-Lagrange equation associated with the variational integral

$$I(v) = \frac{1}{2}\int_\Omega |Dv(x)|^2 \, dx$$

is

$$\begin{aligned} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_i} F(x,u(x),Du(x))\right) &= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}(x)\right) \\ &= -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) = -\Delta u(x) = 0 \quad \text{in} \quad \Omega. \end{aligned}$$

In other words, a minimizer with the boundary values g is a solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

for the Laplace equation.

Example 1.2. The Euler-Lagrange equation associated with the variational integral

$$I(v) = \frac{1}{2}\int_\Omega |Dv(x)|^2 \, dx - \int_\Omega f(x)v(x)\, dx$$

is the Poisson equation $-\Delta u = f$ (exercise).

Example 1.3. Let

$$F(x,\zeta,\xi) = \frac{1}{2}\left(\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j + c\zeta^2\right) - \zeta f(x),$$

where $A = A(x) = (a_{ij}(x))$ is a symmetric $n \times n$ matrix and $c \in \mathbb{R}$. Then

$$\frac{\partial}{\partial \xi_i} F(x,\zeta,\xi) = \sum_{j=1}^{n} a_{ij}(x), \quad i = 1,2,\dots,n,$$

and

$$\frac{\partial}{\partial \zeta} F(x,\zeta,\xi) = c\zeta - f(x)$$

Thus the Euler-Lagrange equation associated with the variational integral

$$\begin{aligned} I(v) &= \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)\frac{\partial v}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x) + cv(x)^2 \right) dx - \int_\Omega f(x)v(x)\,dx \\ &= \frac{1}{2}\int_\Omega \left(ADv(x)\cdot Dv(x) + cv(x)^2 \right) dx - \int_\Omega f(x)v(x)\,dx, \end{aligned}$$

is

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + cu = f \quad \text{in} \quad \Omega.$$

These lectures discuss the divergence type partial differential equation in the example above. The uniform convexity condition on the variational integral leads to the uniform ellipticity condition on the coefficients a_{ij}, $i,j = 1,2,\dots,n$. This condition is applied in the proof of the existence of a solution and in the regularity theory for weak solutions to elliptic partial differential equations with bounded measurable coefficients.

1.2 Second order divergence type PDEs

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We consider the Dirichlet boundary value problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

where $u : \Omega \to \mathbb{R}$ is the unknown function. Here $f, g : \overline{\Omega} \to \mathbb{R}$ are given functions and L denotes a second order (linear) partial differential operator of the form

$$Lu(x) = -\sum_{i,j=1}^n D_j(a_{ij}(x)D_iu(x)) + \sum_{i=1}^n b_i(x)D_iu(x) + c(x)u(x) \tag{1.4}$$

for given coefficient functions a_{ij}, b_i and c, $i,j = 1,\dots,n$. Here we denote the partial derivatives as

$$D_iu(x) = \frac{\partial u}{\partial x_i}(x), \quad i = 1,\dots,n.$$

The operator can be written as

$$Lu(x) = -\operatorname{div}(A(x)Du(x)) + b(x)\cdot Du(x) + c(x)u(x)$$

where

$$A(x)=\begin{bmatrix} a_{11}(x) & \dots & a_{n1}(x) \\ a_{12}(x) & \dots & a_{n2}(x) \\ \vdots & \ddots & \vdots \\ a_{1n}(x) & \dots & a_{nn}(x) \end{bmatrix}$$

is an $n\times n$ matrix and $b(x)=(b_1(x),\dots,b_n(x))$ is a column vector. The negative sign in front of the second order terms disappears after integration by parts and in the definition of weak solutions later. We say that (1.4) is of divergence form and we assume the symmetry condition

$$a_{ij}(x)=a_{ji}(x) \quad \text{for almost every} \quad x\in\Omega, \quad i,j=1,\dots,n. \tag{1.5}$$

Under this assumption the eigenvalues of the symmetric $n\times n$ matrix $A(x)=(a_{ij}(x))$ are real numbers.

Remark 1.6. In the constant coefficient case when every a_{ij}, $i,j=1,\dots,n$, is constant, we may always assume that $a_{ij}=a_{ji}$. To see this observe that $D_jD_iu=D_iD_ju$ and we may replace both a_{ij} and a_{ji} by $\frac{1}{2}(a_{ij}+a_{ji})$, which does not change the operator (exercise).

Definition 1.7. We say that the operator L in (1.4) is uniformly elliptic, if there exists constants $0<\lambda\leqslant\Lambda<\infty$ such that

$$\lambda|\xi|^2\leqslant\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\leqslant\Lambda|\xi|^2$$

for almost every $x\in\Omega$ and every $\xi\in\mathbb{R}^n$.

THE MORAL: The uniform ellipticity condition gives uniform bounds for the speed of diffusion to each direction. In particular, the diffusion does not extinct or blow up.

Remark 1.8. The ellipticity condition implies that the coefficient functions a_{ij}, $i,j=1,\dots,n$, are nonnegative and essentially bounded. To see this, let $i\neq j$ and choose $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ such that $\xi_i=\xi_j=1$ and $\xi_k=0$ for $k\neq i,j$. Then

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j=a_{ij}(x)+a_{ji}(x)=2a_{ij}(x)$$

and thus $0<2a_{ij}(x)\leqslant\Lambda$ for almost every $x\in\Omega$. For the diagonal element, we choose $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ such that $\xi_i=1$ and $\xi_k=0$ for $k\neq i$. Then

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j=a_{ii}(x)$$

and thus $0<a_{ii}(x)\leqslant\Lambda$ for almost every $x\in\Omega$. It follows that

$$\|a_{ij}\|_{L^\infty(\Omega)}\leqslant\Lambda, \quad i,j=1,\dots,n.$$

Remark 1.9. The ellipticity condition can be written in the form

$$\lambda|\xi|^2 \leqslant A(x)\xi\cdot\xi \leqslant \Lambda|\xi|^2$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. In particular, this implies that for almost every point $x \in \Omega$ the symmetric matrix $A(x) = (a_{ij}(x))$ is strictly positive definite and the real eigenvalues $\lambda_i(x)$, $i = 1,\dots,n$, of $A(x)$ satisfy

$$\lambda \leqslant \lambda_i(x) \leqslant \Lambda, \quad \text{for every} \quad x \in \Omega, \quad i = 1,\dots,n.$$

Example 1.10. If $A(x) = I$, $b_i = 0$, and $c = 0$, we have the Poisson equation

$$Lu(x) = -\operatorname{div}(A(x)Du(x)) = -\operatorname{div} Du(x) = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) = -\Delta u(x) = f(x).$$

For $f = 0$, we have the Laplace equation $\Delta u = 0$.

Remark 1.11. It is rather standard in the PDE theory that the variables are not written down explicitly in functions unless there is a specific reason to do so. This makes expressions shorter and, hopefully, more readable.

Remark 1.12. We shall focus on the the case $p = 2$, but it is possible to consider nonlinear variational integrals

$$I(v) = \int_\Omega F(x,Dv)\,dx,$$

where $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies the structural conditions

(1) $F(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^n$,
(2) $F(x,\cdot)$ is strictly convex and differentiable for every $x \in \Omega$ and
(3) there exist constants $0 < \alpha \leqslant \beta < \infty$ such that

$$\alpha|\xi|^p \leqslant F(x,\xi) \leqslant \beta|\xi|^p$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$ with $1 < p < \infty$.

On the other hand, we may consider nonlinear PDEs of the form

$$-\operatorname{div} A(x,Du) = 0,$$

where $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the structural conditions

(1) $A(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^n$,
(2) $A(x,\cdot)$ is continuous for every $x \in \Omega$,
(3) $|A(x,\xi)| \leqslant a|\xi|^{p-1}$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
(4) $A(x,\xi)\cdot\xi \geqslant c\xi|^p$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$ and
(5) $(A(x,\xi') - A(x,\xi))\cdot(\xi' - \xi) > 0$ for every $x \in \Omega$ and $\xi' \neq \xi \in \mathbb{R}^n$.

Let F satisfy (1)–(3) above and $A = (A_1, \dots, , A_n)$,

$$A_i(x,\xi) = \frac{\partial}{\partial \xi_i} F(x,\xi), \quad i = 1, 2, \dots, n.$$

Then

$$-\operatorname{div} A(x, Du(x)) = -\sum_{i=1}^{n} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi_i} F(x, Du(x))\right) = 0$$

is the Eular-Lagrange equation associated with the variational integral

$$I(v) = \int_\Omega F(x, Dv)\, dx$$

and A satisfies (1)–(5) above with $a = 2^p \beta$ and $c = \alpha$, see [14, Lemma 2.95] and [14, Theorem 2.98].

Example 1.13. Remark 1.11 covers the p-Laplace equation

$$-\operatorname{div}(|Du|^{p-2} Du) = 0,$$

is the Euler-Lagrange equation associated with the p-Dirichlet integral

$$I(v) = \int_\Omega |Dv|^p\, dx.$$

For this we refer to [13] and [14].

Example 1.14. Let $A(x) = (a_{ij}(x))$ be a symmetric matrix of bounded measurable functions satisfying the ellipticity condition

$$\lambda |\xi|^2 \leqslant A(x)\xi \cdot \xi \leqslant \Lambda |\xi|^2$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$ with $0 < \lambda \leqslant \Lambda < \infty$. Let

$$F(x,\xi) = \tfrac{1}{p}\big(A(x)\xi \cdot \xi\big)^{\frac{p}{2}}, \quad 1 < p < \infty,$$

and

$$I(v) = \int_\Omega F(x, Dv)\, dx = \frac{1}{p}\int_\Omega \big(A(x)Dv(x) \cdot Dv(x)\big)^{\frac{p}{2}}\, dx.$$

Then F satisfies (1)–(3) in Remark 1.11 and the associated Euler–Lagrange equation is

$$-\operatorname{div} A(x, Du) = -\operatorname{div}\Big(\big(A(x)Du(x) \cdot Du(x)\big)^{\frac{p}{2}-1} A(x)Du(x)\Big) = 0$$

with

$$A(x,\xi) = \big(A(x)\xi \cdot \xi\big)^{\frac{p}{2}-1} A(x)\xi,$$

see [14, Example 2.101].

1.3 Physical interpretation

Consider a fluid moving with velocity $b = (b_1,\dots,b_n)$ in a domain in $\mathbb{R}^n$ and let $u = u(x,t)$ describe the concentration of a chemical in the fluid at point x at moment t. Observe that the concentration changes in time. Assume that the total amount of chemical in any subdomain $\Omega' \subset \Omega$ changes only because of inward or outward flux through the boundary $\partial\Omega'$. This gives

$$\frac{\partial}{\partial t}\int_{\Omega'} u\,dx = \int_{\partial\Omega'} aDu\cdot\nu\,dS - \int_{\partial\Omega'} ub\cdot\nu\,dS. \tag{1.15}$$

where $\nu = \nu(x) = (\nu_1(x),\dots,\nu_n(x))$ is the outward pointing unit normal vector on $\partial\Omega'$ and

$$Du(x)\cdot\nu(x) = \frac{\partial u}{\partial\nu}(x), \quad x \in \partial\Omega',$$

is the outward normal derivative of u and $a > 0$ is the diffusion constant. The first integral on the right-hand side describes how much chemical comes in through the boundary by diffusion by assuming that the flux is proportional to the gradient, but in the opposite direction, that is, the flow is from higher concentration to lower. Note that $Du(x)\cdot\nu(x) > 0$, $x \in \partial\Omega$, if the concentration outside is greater than inside. The second integral on the right-hand side describes the amount of chemical that moves through the boundary by advection, that is, is transported by the flux. The negative sign is explained by the fact that ν is an outward pointing unit normal.

By the Gauss-Green theorem

$$\int_{\Omega'} D_i u(x)\,dx = \int_{\partial\Omega'} u(x)\nu_i(x)\,dS(x), \quad i = 1,\dots,n,$$

By differentiating under integral and using the Gauss-Green theorem in (1.15) we obtain

$$\begin{aligned}
\int_{\Omega'} u_t\,dx &= \int_{\partial\Omega'} aDu\cdot\nu\,dS - \int_{\partial\Omega'} ub\cdot\nu\,dS \\
&= \int_{\partial\Omega'} a\sum_{i=1}^n (D_i u)\nu_i\,dS - \int_{\partial\Omega'}\sum_{i=1}^n ub_i\nu_i\,dS \\
&= \sum_{i=1}^n \left(\int_{\partial\Omega'} a(D_i u)\nu_i\,dS - \int_{\partial\Omega'} ub_i\nu_i\,dS\right) \\
&= \sum_{i=1}^n \left(\int_{\Omega'} D_i(aD_i u)\,dx - \int_{\Omega'} D_i(ub_i)\,dx\right) \\
&= \int_{\Omega'} a\sum_{i=1}^n D_i(D_i u)\,dx - \int_{\Omega'}\sum_{i=1}^n D_i(ub_i)\,dx \\
&= \int_{\Omega'} a\,\operatorname{div} Du\,dx - \int_{\Omega'} \operatorname{div}(ub)\,dx \\
&= \int_{\Omega'} a\Delta u\,dx - \int_{\Omega'} \operatorname{div}(ub)\,dx.
\end{aligned}$$

Since this holds for every $\Omega' \subset \Omega$, we conclude that u satisfies the parabolic PDE

$$u_t - \underbrace{a\Delta u}_{\text{diffusion}} + \underbrace{\operatorname{div}(ub)}_{\text{advection}} = 0.$$

The derivation of the PDE above was done in the case when a is constant, which means that the diffusion does not depend on the location of the point x in the domain Ω. If the diffusion is not uniform in the domain, that is, the coefficient a depends on the location $x \in \Omega$, then a is a function of x. If the diffusion is not isotropic in the sense that it is faster to some directions than others, then the constant diffusion matrix $A(x) = aI$ can be replaced with a more general symmetric matrix $A(x) = (a_{ij}(x))$. This leads to

$$\frac{\partial}{\partial t}\int_{\Omega'} u\,dx = \sum_{i,j=1}^{n}\int_{\partial\Omega'}(a_{ij}D_i u)\nu_j\,dS - \sum_{i=1}^{n}\int_{\partial\Omega'} ub_i\nu_i\,dS, \quad i,j = 1,\dots,n.$$

and the PDE becomes

$$u_t - \underbrace{\sum_{i,j=1}^{n} D_j(a_{ij}D_i u)}_{\text{diffusion}} + \underbrace{\sum_{i=1}^{n} D_i(b_i u)}_{\text{advection}} = 0.$$

If the total amount of u is not conserved, then additional term cu for a creation or depletion of chemical, for example, in chemical reactions, and external source f appear. Then we have the nonhomogeneous PDE

$$u_t - \underbrace{\sum_{i,j=1}^{n} D_j(a_{ij}D_i u)}_{\text{diffusion}} + \underbrace{\sum_{i=1}^{n} D_i(b_i u)}_{\text{advection}} = -\underbrace{cu}_{\text{decay}} + \underbrace{f}_{\text{source}}.$$

Here $a_{ij} = a_{ij}(x)$, $b_i = b_i(x)$, $c = c(x)$ and $f = f(x)$ are functions of x. This PDE can be used to model physical systems including chemical concentration, heat propagation and mass transport. If the system is in equilibrium in the sense that the solution does not depend on time, then $u_t = 0$ and we obtain the elliptic PDE

$$-\sum_{i,j=1}^{n} D_j(a_{ij}D_i u) + \sum_{i=1}^{n} D_i(b_i u) + cu = f,$$

where a_{ij}, b_i, c and f are smooth enough functions for $i,j = 1,\dots,n$. Observe that if we apply the Leibniz rule to the advection term we obtain

$$-\sum_{i,j=1}^{n} D_j(a_{ij}D_i u) + \sum_{i=1}^{n} b_i D_i u + \left(\sum_{i=1}^{n} D_i b_i + c\right)u = f.$$

and thus we have a PDE of type

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}D_i u) + \sum_{i=1}^{n} b_i D_i u + cu = f, \tag{1.16}$$

where L is a second order divergence type operator as in (1.4). A function $u \in C^2(\Omega)$ is a classical solution of (1.16), if it satisfies the PDE at every point $x \in \Omega$. In order to be able to show the existence of solutions for general coefficient functions a_{ij}, b_i, c and f, $i,j = 1,\dots,n$, we consider a weaker notion of solution.

THE MORAL: In order to understand the physical interpretation of a PDE it is better to consider an integrated version of a PDE instead of the pointwise version.

Remark 1.17. A nondivergence form operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_i D_i u + cu$$

can be written as

$$Lu = -\sum_{i,j=1}^{n} D_j (a_{ij} D_i u) + \sum_{i=1}^{n} \left(b_i + \sum_{j=1}^{n} D_j a_{ij} \right) D_i u + cu.$$

THE MORAL: A PDE in nondivergence form can be written in divergence form and vice versa. The main advantage of divergence form is in the arguments that are based on integration by parts.

1.4 Definition of weak solution

Sobolev space methods are important in existence results for PDEs. Let $u \in C^2(\Omega)$ be a classical solution to the Laplace equation

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0$$

and let $\varphi \in C_0^\infty(\Omega)$. An integration by parts gives

$$\begin{aligned} 0 &= \int_\Omega \varphi \Delta u \, dx = \int_\Omega \varphi \operatorname{div} Du \, dx = \int_\Omega \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \varphi \, dx \\ &= \sum_{j=1}^{n} \int_\Omega \frac{\partial^2 u}{\partial x_j^2} \varphi \, dx = -\sum_{j=1}^{n} \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_j} \, dx = -\int_\Omega Du \cdot D\varphi \, dx \end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. Conversely, if $u \in C^2(\Omega)$ and

$$\int_\Omega Du \cdot D\varphi \, dx = 0 \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega),$$

then by the computation above

$$\int_\Omega \varphi \Delta u \, dx = 0 \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega).$$

This implies that $\Delta u = 0$ in Ω. This shows that of $u \in C^2(\Omega)$, then $\Delta u = 0$ in Ω if and only if

$$\int_\Omega Du \cdot D\varphi \, dx = 0 \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega).$$

THE MORAL: There are second order derivatives in the definition of a classical solution to the Laplace equation, but in the definition above is enough to assume that only first order weak derivatives exist.

It is useful to define the meaning of a PDE even if $u \notin C^2(\Omega)$ and the coefficients $a_{ij} \notin C^1(\Omega)$. There are two main motivations for a definition of a weak solution to a PDE.

(1) Weak solutions are sometimes more accessible than classical solutions.

(2) In some cases the classical solution does not exist at all. Thus weak solutions may be the only solutions to the problem.

The general strategy in existence theory for PDEs is to weaken to the notion of a solution so that a problem has a solution. Regularity theory studies whether the PDE is strong enough to give extra regularity to a weak solution. It is natural to begin with existence theory so that we know that the PDE has enough solutions.

Assumption: We consider L is as in (1.4) and make a standing assumption that $\Omega \subset \mathbb{R}^n$ is a bounded open set,

$$a_{ij}, b_i, c \in L^\infty(\Omega), \quad i, j = 1, \dots, n$$

and

$$f \in L^2(\Omega).$$

Moreover, we assume that symmetry condition in (1.5) and the ellipticity condition in Definition 1.7 hold true. These assumptions will not be repeated at every occasion. Sometimes we assume more smoothess on the coefficients or on the domain or set some of coefficients to zero, but these will be specified case by case.

Motivation: If $u \in C^2(\Omega)$, $a_{ij} \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ then we can integrate by parts and $Lu = f$ gives

$$\begin{aligned}
\int_\Omega f\varphi \, dx &= \int_\Omega \left(-\sum_{i,j=1}^n D_j(a_{ij} D_i u) + \sum_{i=1}^n b_i D_i u + cu \right) \varphi \, dx \\
&= \int_\Omega \left(\sum_{i,j=1}^n (a_{ij} D_i u) D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi \right) dx
\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. Observe that there are only first order derivatives of u and no derivatives of the coefficients a_{ij} in the integral above.

On the other hand, if

$$\int_\Omega \left(\sum_{i,j=1}^n (a_{ij} D_i u) D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi \right) dx = \int_\Omega f\varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$, then

$$\int_\Omega \left(-\sum_{i,j=1}^n D_j(a_{ij}D_i u) + \sum_{i=1}^n b_i D_i u + cu - f \right) \varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$ and consequently $Lu(x) = f(x)$ for every $x \in \Omega$.

THE MORAL: A function $u \in C^2(\Omega)$ is a classical solution of (1.16) if and only if it is a weak solution of (1.16) in the sense of the definition below. Observe that the negative sign in front of the second order terms disappears after integration by parts.

Next we define a weak solution to the Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

so that the solution itself belongs to a Sobolev space and the boundary values are taken in the Sobolev sense.

Definition 1.18. A function $u \in W_0^{1,2}(\Omega)$ is a weak solution of $Lu = f$ in Ω, where L is as in (1.4), if

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi \right) dx = \int_\Omega f \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

THE MORAL: The definition of a weak solution is based on integration by parts. A classical solution satisfies the PDE pointwise, but a weak solution satisfies the PDE in integral sense. There are second order derivatives in the definition of a classical solution, but in the definition above is enough to assume that only first order weak derivatives exist. This is compatible with Sobolev spaces.

Remarks 1.19:

(1) Observe that it is enough to assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ in the definition of weak solution. This gives a local notion of solution without any boundary conditions, so that this definition applies to PDEs with Dirichlet, Neumann or other boundary conditions. This local definition is useful when we study regularity of solutions inside the domain. However, solutions are not unique without fixing the boundary values.

(2) A solution $u \in W^{1,2}(\Omega)$ to the Dirichlet problem with nonzero boundary values $g \in W^{1,2}(\Omega)$,

$$\begin{cases} Lu = f & \text{in} \quad \Omega, \\ u - g \in W_0^{1,2}(\Omega), \end{cases}$$

can be obtained by considering $w = u-g \in W_0^{1,2}(\Omega)$, which is a weak solution of the problem

$$\begin{cases} Lw = \overline{f} \quad \text{in} \quad \Omega, \\ w \in W_0^{1,2}(\Omega), \end{cases}$$

with $\overline{f} = f - Lg$. Both approaches lead to the same result (exercise).

Example 1.20. A function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to the Laplace equation $\Delta u = 0$ in Ω, if

$$\int_\Omega Du \cdot D\varphi \, dx = \int_\Omega \sum_{i=1}^n D_i u D_i \varphi \, dx = 0 \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega). \tag{1.21}$$

A function $u \in W^{1,2}(\Omega)$ is a weak solution to $\Delta u = 0$ in Ω with boundary values $g \in W^{1,2}(\Omega)$, if $u - g \in W_0^{1,2}(\Omega)$ and it satisfies (1.21).

Example 1.22. Let $f \in L^2(\Omega)$. A function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to the Poisson equation $-\Delta u = f$ in Ω, if

$$\int_\Omega Du \cdot D\varphi \, dx = \int_\Omega f \varphi \, dx \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega). \tag{1.23}$$

A function $u \in W_0^{1,2}(\Omega)$ is a weak solution to $-\Delta u = f$ in Ω with zero boundary values, if it satisfies (1.23).

Example 1.24. Let $n = 1$, $\Omega = (0,2)$, $b = 0 = c$, $a = 1$ and

$$f(x) = \begin{cases} 1, & x \in (0,1], \\ 2, & x \in (1,2). \end{cases}$$

Consider the problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(0) = 0 = u(2), \end{cases}$$

with $Lu(x) = -(au(x))' = -u''(x)$. By solving

$$Lu(x) = -u''(x) = f(x) = \begin{cases} 1, & x \in (0,1], \\ 2, & x \in (1,2), \end{cases}$$

in the subintervals $(0,1)$ and $(1,2)$ respectively, and requiring that the solution u belongs to $C^1(\Omega)$, we obtain

$$u(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{5}{4}x, & x \in (0,1], \\ -x^2 + \frac{9}{4}x - \frac{1}{2}, & x \in (1,2). \end{cases}$$

We observe that $u \in C^1(\Omega)$, but $u \notin C^2(\Omega)$. In particular, u is not a classical solution to the problem above.

Claim: u is a weak solution.

Reason. The first task is to show that $u \in W^{1,2}(\Omega)$ and $u \in W_0^{1,2}(\Omega)$, which is left as an exercise. Let $0 < \varepsilon < 1$ and $\varphi \in C_0^\infty(\Omega)$. Since u is a classical solution to

$$Lu(x) = -u''(x) = f(x)$$

when $x \in (0, 1-\varepsilon) \cup (1+\varepsilon, 2)$, using integration by parts, we have

$$\begin{aligned}\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} f(x)\varphi(x)\,dx &= -\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} u''(x)\varphi(x)\,dx \\ &= \int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} u'(x)\varphi'(x)\,dx \\ &\quad - \big(u'(1-\varepsilon)\varphi(1-\varepsilon) - 0 + 0 - u'(1+\varepsilon)\varphi(1+\varepsilon)\big).\end{aligned}$$

By the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0}\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} f(x)\varphi(x)\,dx = \int_{(0,2)} f(x)\varphi(x)\,dx$$

and

$$\lim_{\varepsilon\to 0}\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} u'(x)\varphi'(x)\,dx = \int_{(0,2)} u'(x)\varphi'(x)\,dx.$$

Moreover, since $u \in C^1(\Omega)$, we have

$$u'(1-\varepsilon)\varphi(1-\varepsilon) - u'(1+\varepsilon)\varphi(1+\varepsilon) \to 0$$

as $\varepsilon \to 0$. Thus

$$\int_{(0,2)} u'(x)\varphi'(x)\,dx = \int_{(0,2)} f(x)\varphi(x)\,dx \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega).$$

■

THE MORAL: Even if the coefficients are smooth and the operator is uniformly elliptic, the weak solution does not necessarily belong to $C^2(\Omega)$. In particular, the problem does not necessarily have a classical solution.

Example 1.25. Let $n = 1$, $\Omega = (0,2)$, $f = 1$, $b = 0 = c$,

$$a(x) = \begin{cases} 1, & x \in (0,1], \\ 2, & x \in (1,2). \end{cases}$$

Consider the problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(0) = 0 = u(2), \end{cases}$$

where $Lu(x) = -(a(x)u'(x))'$. By solving the equation in the subintervals $(0,1)$ and $(1,2)$ respectively, as well as requiring suitable conditions at $x = 1$, we obtain

$$u(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{5}{6}x, & x \in (0,1], \\ -\frac{1}{4}x^2 + \frac{5}{12}x + \frac{1}{6}, & x \in (1,2). \end{cases}$$

We observe that $u \notin C^1(\Omega)$. However, u is a weak solution to the above problem (exercise).

THE MORAL: If the coefficients are not smooth, the weak solution does not necessarily belong to $C^1(\Omega)$. In particular, the problem does not have a classical solution and the weak solution does not even have the first order derivatives in the classical sense.

1.5 Serrin's example

We begin with reconsidering Serrin's example of a pathological weak solution, see [17]. See also Meyers [15], Chen and Wu [1, p. 189] and Giaquinta [5, p. 157]. This example shows that under the assumption that $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,\dots,n$, the best result we can hope for is that weak solutions are locally Hölder continuous. See also Remark 4.23, Remark 4.37 and Remark 5.36 below.

$\boxed{(1)}$ Let $n \geqslant 2$ and $0 < \alpha < 1$. We claim that the function $u : \mathbb{R}^n \to \mathbb{R}$,

$$u(x) = u(x_1,\dots,x_n) = x_1|x|^{-\alpha} \tag{1.26}$$

is a classical solution to

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)) = 0 \quad \text{for every} \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1.27}$$

where

$$a_{ij}(x) = \delta_{ij} + \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}\frac{x_ix_j}{|x|^2}, \quad i,j = 1,\dots,n. \tag{1.28}$$

Here δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

By the chain rule, we have

$$\begin{aligned} D_i(|x|^r) &= D_i\left(\sqrt{x_1^2+x_2^2+\dots+x_n^2}\right)^r = D_i\left(x_1^2+x_2^2+\dots+x_n^2\right)^{\frac{r}{2}} \\ &= \frac{r}{2}\left(x_1^2+x_2^2+\dots+x_n^2\right)^{\frac{r}{2}-1}\cdot 2x_i = rx_i\left(x_1^2+x_2^2+\dots+x_n^2\right)^{\frac{r-2}{2}} \\ &= rx_i\left(\sqrt{x_1^2+x_2^2+\dots+x_n^2}\right)^{r-2} = rx_i|x|^{r-2}, \quad i = 1,\dots,n, \end{aligned}$$

for every $r \in \mathbb{R}$ and $x \neq 0$. For $i \neq 1$, this implies

$$D_iu(x) = D_i(x_1|x|^{-\alpha}) = x_1\cdot(-\alpha)x_i|x|^{-\alpha-2} = -\alpha x_1x_i|x|^{-\alpha-2}.$$

For $i = 1$, we have

$$D_1u(x) = D_1(x_1|x|^{-\alpha}) = x_1\cdot(-\alpha)x_1|x|^{-\alpha-2} + |x|^{-\alpha} = |x|^{-\alpha} - \alpha x_1^2|x|^{-\alpha-2}.$$

This gives

$$D_iu(x) = \delta_{i1}|x|^{-\alpha} - \alpha x_1x_i|x|^{-\alpha-2}, \quad i = 1,\dots,n. \tag{1.29}$$

Let

$$b = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}.$$

A direct computation gives

$$\begin{aligned}
a_{ij}D_i u(x) &= \left(\delta_{ij} + b\frac{x_i x_j}{|x|^2}\right)\left(\delta_{i1}|x|^{-\alpha} - \alpha x_1 x_i |x|^{-\alpha-2}\right) \\
&= \delta_{ij}\delta_{i1}|x|^{-\alpha} + \delta_{i1} b x_i x_j |x|^{-\alpha-2} - \delta_{ij}\alpha x_1 x_i |x|^{-\alpha-2} - \alpha b x_1 x_i^2 x_j |x|^{-\alpha-4},
\end{aligned}$$

for $i, j = 1, \dots, n$. We observe that

$$\sum_{i,j=1}^n D_j(a_{ij}(x)D_i u(x)) = \sum_{j=1}^n D_j\left(\sum_{i=1}^n a_{ij}(x)D_i u(x)\right),$$

where

$$\begin{aligned}
&\sum_{i=1}^n a_{ij}(x)D_i u(x) \\
&= \sum_{i=1}^n \left(\delta_{ij}\delta_{i1}|x|^{-\alpha} + \delta_{i1} b x_i x_j |x|^{-\alpha-2} - \delta_{ij}\alpha x_1 x_i |x|^{-\alpha-2} - \alpha b x_1 x_i^2 x_j |x|^{-\alpha-4}\right) \\
&= \delta_{1j}|x|^{-\alpha} + b x_1 x_j |x|^{-\alpha-2} - \alpha x_1 x_j |x|^{-\alpha-2} - \alpha b x_1 x_j |x|^{-\alpha-4}\sum_{i=1}^n x_i^2 \\
&= \delta_{1j}|x|^{-\alpha} + b x_1 x_j |x|^{-\alpha-2} - \alpha x_1 x_j |x|^{-\alpha-2} - \alpha b x_1 x_j |x|^{-\alpha-4}|x|^2 \\
&= \delta_{1j}|x|^{-\alpha} + (b-\alpha-\alpha b) x_1 x_j |x|^{-\alpha-2}, \quad j = 1, \dots, n.
\end{aligned}$$

For $j \neq 1$, this implies

$$\begin{aligned}
D_j\left(\sum_{i=1}^n a_{ij}(x)D_i u(x)\right) &= D_j\left(\delta_{1j}|x|^{-\alpha} + (b-\alpha-\alpha b) x_1 x_j |x|^{-\alpha-2}\right) \\
&= (b-\alpha-\alpha b)x_1\left(|x|^{-\alpha-2} + x_j(-\alpha-2)x_j|x|^{-\alpha-4}\right) \\
&= (b-\alpha-\alpha b)x_1\left(|x|^{-\alpha-2} - (\alpha+2)x_j^2|x|^{-\alpha-4}\right).
\end{aligned}$$

For $j = 1$, we have

$$\begin{aligned}
&D_1\left(\sum_{i=1}^n a_{i1}(x)D_i u(x)\right) = D_1\left(\delta_{11}|x|^{-\alpha} + (b-\alpha-\alpha b)x_1^2|x|^{-\alpha-2}\right) \\
&= -\alpha x_1|x|^{-\alpha-2} + (b-\alpha-\alpha b)\left(2x_1|x|^{-\alpha-2} + x_1^2(-\alpha-2)x_1|x|^{-\alpha-4}\right) \\
&= -\alpha x_1|x|^{-\alpha-2} + (b-\alpha-\alpha b)x_1\left(2|x|^{-\alpha-2} - (\alpha+2)x_1^2|x|^{-\alpha-4}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&D_j\left(\sum_{i=1}^n a_{ij}(x)D_i u(x)\right) \\
&= -\delta_{1j}\alpha x_1|x|^{-\alpha-2} + (b-\alpha-\alpha b)x_1\left((1+\delta_{1j})|x|^{-\alpha-2} - (\alpha+2)x_j^2|x|^{-\alpha-4}\right)
\end{aligned}$$

for $j = 1,\dots,n$. By summing up, we obtain

$$\begin{aligned}
&\sum_{j=1}^{n} D_j\left(\sum_{i=1}^{n} a_{ij}(x) D_i u(x)\right)\\
&= \sum_{j=1}^{n}(-\delta_{1j}\alpha x_1|x|^{-\alpha-2} + (b-\alpha-\alpha b)x_1((1+\delta_{1j})|x|^{-\alpha-2} - (\alpha+2)x_j^2|x|^{-\alpha-4}))\\
&= \sum_{j=1}^{n} -\delta_{1j}\alpha x_1|x|^{-\alpha-2} + \sum_{j=1}^{n}(b-\alpha-\alpha b)x_1(1+\delta_{1j})|x|^{-\alpha-2}\\
&\qquad - \sum_{j=1}^{n}(b-\alpha-\alpha b)x_1(\alpha+2)x_j^2|x|^{-\alpha-4}\\
&= -\alpha x_1|x|^{-\alpha-2} + (b-\alpha-\alpha b)x_1|x|^{-\alpha-2}\sum_{j=1}^{n}(1+\delta_{1j})\\
&\qquad - (b-\alpha-\alpha b)x_1(\alpha+2)|x|^{-\alpha-4}\sum_{j=1}^{n} x_j^2\\
&= -\alpha x_1|x|^{-\alpha-2} + (b-\alpha-\alpha b)x_1|x|^{-\alpha-2}(n+1) - (b-\alpha-\alpha b)x_1(\alpha+2)|x|^{-\alpha-2}\\
&= (-\alpha + (b-\alpha-\alpha b)(n+1) - (b-\alpha-\alpha b)(\alpha+2))x_1|x|^{-\alpha-2}.
\end{aligned}$$

By (1.27), this expression should be equal to 0 for every $x \neq 0$. This is possible if

$$\begin{aligned}
&-\alpha + (b-\alpha-\alpha b)(n+1) - (b-\alpha-\alpha b)(\alpha+2) = 0\\
\iff &-\alpha + bn + b - \alpha n - \alpha - \alpha bn - \alpha b - b\alpha - 2b + \alpha^2 + 2\alpha + \alpha^2 b + 2\alpha b = 0\\
\iff &bn + b - \alpha bn - \alpha b - b\alpha - 2b + \alpha^2 b + 2\alpha b = \alpha + \alpha n + \alpha - \alpha^2 - 2\alpha\\
\iff &b(n+1-\alpha n - 2 + \alpha^2) = \alpha n - \alpha^2.
\end{aligned}$$

This implies

$$b = \frac{\alpha n - \alpha^2}{n-1-\alpha n+\alpha^2} = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)},$$

and this is precisely how b was defined.

$\boxed{(2)}$ The coefficients a_{ij}, $i,j = 1,\dots,n$, in (1.28) can be represented as a symmetric matrix

$$A(x) = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x)\\ a_{21}(x) & \cdots & a_{2n}(x)\\ \vdots & \ddots & \vdots\\ a_{n1}(x) & \cdots & a_{nn}(x)\end{bmatrix} = \begin{bmatrix} 1+b\frac{x_1^2}{|x|^2} & b\frac{x_1x_2}{|x|^2} & \cdots & b\frac{x_1x_n}{|x|^2}\\ b\frac{x_1x_2}{|x|^2} & 1+b\frac{x_2^2}{|x|^2} & \cdots & b\frac{x_2x_n}{|x|^2}\\ \vdots & \vdots & \ddots & \vdots\\ b\frac{x_1x_n}{|x|^2} & b\frac{x_2x_n}{|x|^2} & \cdots & 1+b\frac{x_n^2}{|x|^2}\end{bmatrix}. \tag{1.30}$$

We observe that $b = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}$ is positive, since $0 < \alpha < 1$ and $n \geqslant 2$.

We show that the coefficients a_{ij} are bounded for every $i,j = 1,\dots,n$. By (1.28) and the triangle inequality

$$\begin{aligned}
|a_{ij}(x)| &= \left|\delta_{ij} + \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}\frac{x_ix_j}{|x|^2}\right| \leqslant 1 + \left|b\frac{x_ix_j}{|x|^2}\right|\\
&= 1 + b\frac{|x_i||x_j|}{|x|^2} \leqslant 1 + b\frac{|x||x|}{|x|^2} = 1+b, \quad i,j = 1,\dots,n,
\end{aligned}$$

for every $x \in \mathbb{R}^n \setminus \{0\}$. This implies that

$$\max_{i,j} \|a_{ij}\|_{L^\infty(\mathbb{R}^n)} \leqslant 1+b. \tag{1.31}$$

$\boxed{(3)}$ Next we show that (1.27) is uniformly elliptic. We claim that

$$|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leqslant (1+b)|\xi|^2 \tag{1.32}$$

for every $x \in \mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R}^n$. This implies that the uniform ellipticity condition in Definition 1.7 is satisfied with

$$\lambda = 1 \quad \text{and} \quad \Lambda = 1 + \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}.$$

Observe that $\Lambda > 1$ can be made arbitrarily close to 1 by choosing $\alpha > 0$ small enough.

We begin with the lower bound. To this end, we observe that

$$\begin{aligned}
\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j &= \sum_{j=1}^n \xi_j \sum_{i=1}^n (a_{ij}(x)\xi_i) = \sum_{j=1}^n \xi_j \sum_{i=1}^n \left(\delta_{ij}\xi_i + b\frac{x_i x_j}{|x|^2}\xi_i\right) \\
&= \sum_{j=1}^n \xi_j \left(\xi_j + \frac{b x_j}{|x|^2}\sum_{i=1}^n x_i\xi_i\right) = \sum_{j=1}^n \xi_j^2 + \frac{b}{|x|^2}\sum_{j=1}^n x_j\xi_j \sum_{i=1}^n x_i\xi_i \\
&= |\xi|^2 + \frac{b}{|x|^2}(x\cdot\xi)^2 \geqslant |\xi|^2,
\end{aligned}$$

since $b > 0$ and $(x\cdot\xi)^2 \geqslant 0$. This proves the left-hand inequality in (1.32). The right-hand side inequality in (1.32) follows from the Cauchy-Schwarz inequality, since

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j = |\xi|^2 + \frac{b}{|x|^2}(x\cdot\xi)^2 \leqslant |\xi|^2 + \frac{b}{|x|^2}|x|^2|\xi|^2 = |\xi|^2 + b|\xi|^2 = (1+b)|\xi|^2.$$

We discuss a matrix version of (1.32). Since

$$\xi^T A(x)\xi = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j,$$

we have

$$|\xi|^2 \leqslant \xi^T A(x)\xi \leqslant (1+b)|\xi|^2 \quad \text{for every } x \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n.$$

This implies

$$\xi^T A(x)\xi \geqslant |\xi|^2 > 0 \quad \text{for every } \xi \neq 0$$

and thus the matrix $A(x)$ is positive definite. This implies that the the matrix $A(x)$ has n positive eigenvalues, with multiplicities. Let $v \neq 0$ be an eigenvector of $A(x)$ corresponding to the eigenvalue $\lambda > 0$. Then

$$\begin{gathered}
v^T A(x)v = v^T\lambda v = \lambda v^T v = \lambda|v|^2 \\
\Longrightarrow |v|^2 \leqslant v^T A(x)v = \lambda|v|^2 \leqslant (1+b)|v|^2 \\
\Longrightarrow 1 \leqslant \lambda \leqslant 1+b.
\end{gathered}$$

This shows that all eigenvalues of $A(x)$ belong to the interval $[1,1+b]$.

Since equality occurs in the Cauchy-Schwarz inequality, if the vectors are linearly dependent, we have

$$x^T A(x)x = \sum_{i,j=1}^n a_{ij}(x)x_i x_j = |x|^2 + \frac{b}{|x|^2}(x\cdot x)^2 = (1+b)|x|^2.$$

We claim that $x \neq 0$ is an eigenvector of $A(x)$ corresponding to the eigenvalue $1+b$. By a direct computation, we show that

$$A(x)x = (1+b)x,$$

that is,

$$\begin{bmatrix} 1+b\frac{x_1^2}{|x|^2} & b\frac{x_1x_2}{|x|^2} & \cdots & b\frac{x_1x_n}{|x|^2} \\ b\frac{x_1x_2}{|x|^2} & 1+b\frac{x_2^2}{|x|^2} & \cdots & b\frac{x_2x_n}{|x|^2} \\ \vdots & \vdots & \ddots & \vdots \\ b\frac{x_1x_n}{|x|^2} & b\frac{x_2x_n}{|x|^2} & \cdots & 1+b\frac{x_n^2}{|x|^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (1+b) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The jth row of the product on the left-hand side is

$$\begin{aligned} x_j + b\frac{x_1^2x_j}{|x|^2} + b\frac{x_2^2x_j}{|x|^2} + \dots + b\frac{x_n^2x_j}{|x|^2} &= x_j + x_j\frac{b}{|x|^2}\sum_{i=1}^n x_i^2 \\ &= x_j\left(1+\frac{b}{|x|^2}\cdot|x|^2\right) = (1+b)x_j, \quad j=1,\dots,n, \end{aligned}$$

which clearly is the same as the corresponding row on the right-hand side.

The trace, that is the sum of the diagonal elements, of a square matrix equals to the sum of the eigenvalues. In this case

$$\operatorname{tr}(A(x)) = \sum_{i=1}^n 1 + b\frac{x_i^2}{|x|^2} = n + b\frac{\sum_{i=1}^n x_i^2}{|x|^2} = n+b.$$

Since $1+b$ is an eigenvalue of $A(x)$, the sum of all other eigenvalues

$$\sum_{i=1}^{n-1}\lambda_i = n+b-(1+b) = n-1.$$

We proved above that all eigenvalues of $A(x)$ belong to the interval $[1,1+b]$. This implies that all other $n-1$ eigenvalues of $A(x)$, except $1+b$, are equal to 1. Thus the characteristic polynomial of the matrix $A(x)$ is

$$\det(A-\lambda I) = (-1)^n(\lambda-(1+b))(\lambda-1)^{n-1}.$$

$\boxed{(4)}$ Let u be as in (1.26). We claim that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. First we show that $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. For every $r>0$, we have

$$\begin{aligned} \int_{B(0,r)} |u(x)|^2\,dx &= \int_{B(0,r)} |x_1|^2|x|^{-2\alpha}\,dx \leqslant \int_{B(0,r)} |x|^2|x|^{-2\alpha}\,dx \\ &= \int_{B(0,r)} |x|^{2(1-\alpha)}\,dx = \omega_{n-1}\int_0^r \rho^{2(1-\alpha)}\rho^{n-1}\,d\rho \\ &= \frac{1}{2(1-\alpha)+n}\rho^{2(1-\alpha)+n}\Big|_0^r = \frac{1}{2(1-\alpha)+n}r^{2(1-\alpha)+n} < \infty, \end{aligned}$$

since $2(1-\alpha)+n>0$.

Next we show that $D_i u \in L^2_{\text{loc}}(\mathbb{R}^n)$, $i=1,\dots,n$. By (1.29), we have

$$D_i u(x) = \delta_{i1}|x|^{-\alpha} - \alpha x_1 x_i |x|^{-\alpha-2}, \quad i=1,\dots,n,$$

for every $x \neq 0$. It is an exercise to show that $D_i u$, $i=1,\dots,n$, is the weak partial derivative of u in $\mathbb{R}^n$. For every $r>0$, we have

$$\begin{aligned}
\int_{B(0,r)} |D_i u(x)|^2\,dx &= \int_{B(0,r)} \left||x|^{-\alpha} - \alpha x_1^2 |x|^{-\alpha-2}\right|^2 dx \\
&= \int_{B(0,r)} \left(|x|^{-\alpha}\left|1-\alpha x_1^2|x|^{-2}\right|\right)^2 dx \\
&= \int_{B(0,r)} |x|^{-2\alpha}\left|1-\alpha x_1^2|x|^{-2}\right|^2 dx.
\end{aligned}$$

We note that

$$\begin{aligned}
1 \geqslant 1-\alpha x_1^2|x|^{-2} &\geqslant 1-\alpha|x|^2|x|^{-2} = 1-\alpha > 0 \\
&\Longrightarrow \left|1-\alpha x_1^2|x|^{-2}\right| \leqslant 1 \\
&\Longrightarrow \left|1-\alpha x_1^2|x|^{-2}\right|^2 \leqslant 1.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{B(0,r)} |D_i u(x)|^2\,dx &\leqslant \int_{B(0,r)} |x|^{-2\alpha}\,dx = \omega_{n-1}\int_0^r \rho^{-2\alpha}\rho^{n-1}\,d\rho \\
&= \frac{1}{n-2\alpha}\rho^{n-2\alpha}\Big|_0^r = \frac{1}{n-2\alpha} r^{n-2\alpha} < \infty.
\end{aligned}$$

Since $D_i u$ is bounded in $\mathbb{R}^n \setminus B(0,r)$, we conclude that $D_i u \in L^2_{\text{loc}}(\mathbb{R}^n)$, $i=1,\dots,n$.

$\boxed{(5)}$ Since u is a classical solution to

$$-\sum_{i,j=1}^n D_j(a_{ij}(x) D_i u(x)) = 0 \quad \text{for every} \quad x \in \mathbb{R}^n \setminus \{0\},$$

we have

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi\,dx = \int_{\mathbb{R}^n\setminus\{0\}} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi\,dx = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

Assume then that $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let $0<r<\frac{1}{2}$ and let $\eta \in C_0^\infty(B(0,2r))$ be a cutoff function with

$$0 \leqslant \eta \leqslant 1, \quad \eta = 1 \quad \text{in} \quad B(0,r) \quad \text{and} \quad |D\eta| \leqslant \frac{2}{r}.$$

Then $(1-\eta)\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and thus

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij} D_i u D_j((1-\eta)\varphi)\,dx \\
&= \int_{\mathbb{R}^n} \sum_{i,j=1}^n (1-\eta) a_{ij} D_i u D_j \varphi\,dx - \int_{\mathbb{R}^n} \sum_{i,j=1}^n \varphi a_{ij} D_i u D_j \eta\,dx.
\end{aligned}$$

We observe that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} \sum_{i,j=1}^n \varphi a_{ij} D_i u D_j \eta \, dx\right| \leqslant \sum_{i,j=1}^n \int_{\mathbb{R}^n} |\varphi||a_{ij}|D_i u||D_j \eta| \, dx \\
&\leqslant \|\varphi\|_{L^\infty(\mathbb{R}^n)} \max_{i,j} \|a_{ij}\|_{L^\infty(\mathbb{R}^n)} \sum_{i,j=1}^n \int_{\mathbb{R}^n} |D_i u||D_j \eta| \, dx \\
&\leqslant \|\varphi\|_{L^\infty(\mathbb{R}^n)} \max_{i,j} \|a_{ij}\|_{L^\infty(\mathbb{R}^n)} \frac{2}{r} \sum_{i,j=1}^n \int_{B(0,2r)} |D_i u| \, dx \\
&\leqslant \|\varphi\|_{L^\infty(\mathbb{R}^n)} \max_{i,j} \|a_{ij}\|_{L^\infty(\mathbb{R}^n)} \frac{2}{r} \sum_{i,j=1}^n \left(\int_{B(0,2r)} |D_i u|^2 \, dx\right)^{\frac{1}{2}} |B(0,2r)|^{\frac{1}{2}} \\
&\leqslant c r^{\frac{n-2}{2}} \left(\int_{B(0,2r)} |Du|^2 \, dx\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad r \to 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
0 &= \lim_{r\to 0}\left(\int_{\mathbb{R}^n} \sum_{i,j=1}^n (1-\eta) a_{ij} D_i u D_j \varphi \, dx - \int_{\mathbb{R}^n} \sum_{i,j=1}^n \varphi a_{ij} D_i u D_j \eta \, dx\right) \\
&= \lim_{r\to 0} \int_{\mathbb{R}^n} \sum_{i,j=1}^n (1-\eta) a_{ij} D_i u D_j \varphi \, dx \\
&= \int_{\mathbb{R}^n} \sum_{i,j=1}^n \lim_{r\to 0} (1-\eta) a_{ij} D_i u D_j \varphi \, dx \\
&= \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx
\end{aligned}
$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$. Here we used the fact that

$$\lim_{r\to 0}(1-\eta(x)) = 1 \quad \text{for every} \quad x \in \mathbb{R}^n \setminus \{0\}$$

and the dominated convergence theorem with the integrable majorant

$$
\begin{aligned}
&|(1-\eta) a_{ij} D_i u D_j \varphi| \leqslant |a_{ij} D_i u D_j \varphi| \\
&\leqslant \|D\varphi\|_{L^\infty(\mathbb{R}^n)} \max_{i,j} \|a_{ij}\|_{L^\infty(\mathbb{R}^n)} |Du| \in L^1(\mathbb{R}^n).
\end{aligned}
$$

THE MORAL: A weak solution in (1.26) to a uniformly elliptic equation with bounded coefficients in (1.27) is locally Hölder continuous with the exponent $1-\alpha$ (exercise), but the weak gradient is unbounded in every neighbourhood of the origin. In particular, the gradient is not continuous. Thus a weak solution is not smoother than locally Hölder continuous without further assumptions on the coefficients.

$\boxed{(6)}$ Next we modify the example to justify the assumption that $u \in W^{1,2}_{\text{loc}}(\Omega)$ from the point of view of regularity theory. Let $n \geqslant 2$ and $0 < \varepsilon < 1$. The function $u : B(0,1) \to \mathbb{R}$,

$$u(x) = u(x_1, \dots, x_n) = x_1 |x|^{1-n-\varepsilon} \tag{1.33}$$

is a classical solution to

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(x)) = 0 \quad \text{for every} \quad x \in B(0,1)\setminus\{0\},$$

where

$$a_{ij}(x) = \delta_{ij} + (a-1)\frac{x_i x_j}{|x|^2}, \quad i,j = 1,\dots,n, \tag{1.34}$$

and

$$a = \frac{n-1}{\varepsilon(\varepsilon+n-2)}.$$

This problem is essentially the same as in (1.26), (1.27) and (1.28) with α replaced by $1-\varepsilon$.

$$\begin{aligned} a-1 &= \frac{n-1}{\varepsilon(\varepsilon+n-2)} - 1 = \frac{n-1-\varepsilon(\varepsilon+n-2)}{\varepsilon(\varepsilon+n-2)} \\ &= \frac{n-1-\varepsilon^2-\varepsilon n+2\varepsilon}{\varepsilon(\varepsilon+n-2)} = \frac{(1-\varepsilon)(n-1+\varepsilon)}{\varepsilon(\varepsilon+n-2)}. \end{aligned}$$

By inserting $\alpha = 1-\varepsilon$ we have $a-1 = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}$ which equals b in the step (1).

It is an exercise to show that the coefficients are bounded and that the uniform ellipticity condition in Definition 1.7 is satisfied with $\lambda = 1$ and $\Lambda = a$.

We have (exercise)

$$u \in W^{1,p}(\Omega), \quad p < \frac{n}{n+\varepsilon-1}.$$

Observe that $p < 2$, when $n \geqslant 2$, and thus

$$u \notin W^{1,2}(\Omega) \quad \text{for every} \quad 0 < \varepsilon < 1.$$

However, as in the step (5), we see that

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(B(0,1))$ (exercise). In this sense u is a weak solution to

$$-\sum_{i,j=1}^{n} D_j(a_{ij} D_i u) = 0 \quad \text{in} \quad B(0,1),$$

but $u \notin W^{1,2}(\Omega)$ for every $0 < \varepsilon < 1$. Clearly the function u is neither locally bounded nor has a local maximum principle. See also Remark 4.23 below.

This example can be used, moreover, to show that the Dirichlet problem need not have a unique solution. In fact, let $v \in W^{1,2}(\Omega)$ be the unique weak solution with the same boundary values on $\partial\Omega$ as u. Then $u - v = 0$ on $\partial\Omega$, but $u - v$ is not identically zero in Ω. This shows that the identically zero function and $v - u$ are weak solutions to the Dirichlet problem with zero boundary values. Thus the problem has two solutions corresponding to the same data, provided we give up the requirement that these solutions belong to $W^{1,2}(\Omega)$.

THE MORAL: Local boundedness, uniqueness and maximum principle may not hold without the assumption that a weak solution belongs to $W^{1,2}_{\text{loc}}(\Omega)$. Thus the usual requirement that a weak solution belongs to $W^{1,2}_{\text{loc}}(\Omega)$ is an essential part of the theory.

1.6 Sobolev-Poincaré inequalities

We recall several versions of the Sobolev inequality. These results will be applied throughout. We begin with the Gagliardo-Nirenberg-Sobolev inequality.

Theorem 1.35. Let $1 \leqslant p < n$ and $p^* = \frac{np}{n-p}$. There exists $c = c(n,p)$ such that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leqslant c\left(\int_{\mathbb{R}^n} |Du|^p\,dx\right)^{\frac{1}{p}}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$.

THE MORAL: The Sobolev-Gagliardo-Nirenberg inequality implies that $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, when $1 \leqslant p < n$. More precisely, $W^{1,p}(\mathbb{R}^n)$ is continuously imbedded in $L^{p^*}(\mathbb{R}^n)$, when $1 \leqslant p < n$. Observe that $p^* > p$. This is the Sobolev embedding theorem for $1 \leqslant p < n$.

Remark 1.36. Let $1 \leqslant p < n$ and let $\Omega \subset \mathbb{R}^n$ be an open set. By considering the zero extension of u to the complement of Ω, Theorem 1.35 implies that

$$\left(\int_{\Omega} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leqslant c(n,p)\left(\int_{\Omega} |Du|^p\,dx\right)^{\frac{1}{p}}$$

for every $u \in W^{1,p}_0(\Omega)$. This is a version of the Sobolev-Gagliardo-Nirenberg inequality for Sobolev spaces with zero boundary values.

Next we discuss a version of the Sobolev-Gagliardo-Nirenberg inequality for the full range $1 \leqslant p < \infty$.

Theorem 1.37. Let $1 \leqslant p < \infty$, let $\Omega \subset \mathbb{R}^n$ be an open set with $|\Omega| < \infty$, and assume that $u \in W^{1,p}_0(\Omega)$. Let $1 \leqslant q \leqslant p^* = \frac{np}{n-p}$, for $1 \leqslant p < n$, and $1 \leqslant q < \infty$ for $n \leqslant p < \infty$. There exists a constant $c = c(n,p,q)$ such that

$$\left(\int_{\Omega} |u|^q\,dx\right)^{\frac{1}{q}} \leqslant c|\Omega|^{\frac{1}{n}-\frac{1}{p}+\frac{1}{q}}\left(\int_{\Omega} |Du|^p\,dx\right)^{\frac{1}{p}}.$$

THE MORAL: Let $\Omega \subset \mathbb{R}^n$ be an open set with $|\Omega| < \infty$. If $u \in W^{1,p}_0(\Omega)$, then $u \in L^q(\Omega)$ for some $q > p$.

Proof. Extend u as zero outside Ω. Then $Du(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. Assume first that $1 \leqslant p < n$. Hölder's inequality and the Gagliardo-Nirenberg-Sobolev inequality imply

$$\begin{aligned}\Big(\int_\Omega |u|^q\,dx\Big)^{\frac{1}{q}} &\leqslant |\Omega|^{\frac{1}{n}-\frac{1}{p}+\frac{1}{q}}\Big(\int_\Omega |u|^{\frac{np}{n-p}}\,dx\Big)^{\frac{n-p}{np}}\\ &\leqslant c(n,p)|\Omega|^{\frac{1}{n}-\frac{1}{p}+\frac{1}{q}}\Big(\int_\Omega |Du|^p\,dx\Big)^{\frac{1}{p}}.\end{aligned}$$

Assume then that $n \leqslant p < \infty$. If $q > p$, choose $1 < \widetilde{p} < n$ satisfying $q = \frac{n\widetilde{p}}{n-\widetilde{p}}$. By the first part of the proof and Hölder's inequality, we obtain

$$\begin{aligned}\Big(\int_\Omega |u|^q\,dx\Big)^{\frac{1}{q}} &\leqslant c(n,p,q)|\Omega|^{\frac{1}{n}-\frac{1}{\widetilde{p}}+\frac{1}{q}}\Big(\int_\Omega |Du|^{\widetilde{p}}\,dx\Big)^{\frac{1}{\widetilde{p}}}\\ &\leqslant c(n,p,q)|\Omega|^{\frac{1}{n}-\frac{1}{p}+\frac{1}{q}}\Big(\int_\Omega |Du|^p\,dx\Big)^{\frac{1}{p}}.\end{aligned}$$

Finally, if $q \leqslant p$, the claim follows from the previous case for some $\widetilde{q} > q$ and Hölder's inequality on the left-hand side. □

Remark 1.38. A Poincaré inequality for Sobolev functions with zero boundary values follows from Theorem 1.37 by choosing $q = p$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and let $1 \leqslant p < \infty$. There is a constant $c = c(n,p)$ such that

$$\int_\Omega |u|^p\,dx \leqslant c|\Omega|^{\frac{p}{n}}\int_\Omega |Du|^p\,dx \leqslant c\,\mathrm{diam}(\Omega)^p\int_\Omega |Du|^p\,dx$$

for every $u \in W_0^{1,p}(\Omega)$.

Next we discuss a Sobolev-Poincaré inequality on balls.

Theorem 1.39. Let $1 \leqslant p < n$, let $\Omega \subset \mathbb{R}^n$ be an open set and assume that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. There exists a constant $c = c(n,p)$ such that

$$\Big(\fint_{B(x,r)} |u-u_{B(x,r)}|^{p^*}\,dy\Big)^{\frac{1}{p^*}} \leqslant cr\Big(\fint_{B(x,r)} |Du|^p\,dy\Big)^{\frac{1}{p}}$$

for every ball $B(x,r) \Subset \Omega$.

The next theorem gives a general Sobolev-Poincaré inequality for Sobolev functions.

Theorem 1.40. Let $1 \leqslant p < \infty$, let $\Omega \subset \mathbb{R}^n$ be an open set, and assume that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Let $1 \leqslant q \leqslant p^* = \frac{np}{n-p}$ for $1 \leqslant p < n$ and $1 \leqslant q < \infty$ for $n \leqslant p < \infty$. There exists a constant $c = c(n,p,q)$ such that

$$\Big(\fint_{B(x,r)} |u-u_{B(x,r)}|^{q}\,dy\Big)^{\frac{1}{q}} \leqslant cr\Big(\fint_{B(x,r)} |Du|^p\,dy\Big)^{\frac{1}{p}} \tag{1.41}$$

for every ball $B(x,r) \Subset \Omega$.

T H E M O R A L : The Sobolev-Poincaré inequality asserts that the mean oscillation of a function in a Sobolev space is uniformly bounded by the mean value of the gradient over balls. In other words, if the gradient is small in average, the function does not oscillate too much. Moreover, there is a gain in the sense that the exponent q on the left-hand side is bigger than the exponent p on the right-hand side. The result holds for the full range $1 \leqslant p < \infty$ and not only for the Sobolev exponent p^*.

Proof. By the Soblev-Poincaré inequality with the Sobolev conjugate exponent, for $1 \leqslant p < n$, there exists a constant $c = c(n,p)$ such that

$$\begin{aligned}\left(\fint_{B(x,r)} |u-u_{B(x,r)}|^{\frac{np}{n-p}}\,dy\right)^{\frac{n-p}{np}} &= cr^{-\frac{n-p}{p}}\left(\int_{B(x,r)} |u-u_{B(x,r)}|^{\frac{np}{n-p}}\,dy\right)^{\frac{n-p}{np}}\\ &\leqslant cr^{1-\frac{n}{p}}\left(\int_{B(x,r)}|Du|^p\,dy\right)^{\frac{1}{p}}\\ &= cr\left(\fint_{B(x,r)}|Du|^p\,dy\right)^{\frac{1}{p}}.\end{aligned}\tag{1.42}$$

For $1 \leqslant p < n$, inequality (1.41) follows from (1.42) and Hölder's inequality on the left-hand side.

In the case $p \geqslant n$ we proceed as in the proof of Theorem 1.37. For $q > p \geqslant n$, there exists $1 < s < n$ such that $q = \frac{ns}{n-s}$, and (1.41) follows from (1.42) with exponent s and an application of Hölder's inequality on the right-hand side. For $q \leqslant p$, the claim follows from the previous case and Hölder's inequality on the left-hand side. □

Remark 1.43. By choosing $q = p$ in Theorem 1.40, we obtain a Poincaré inequality on balls. Let $1 \leqslant p < \infty$, let $\Omega \subset \mathbb{R}^n$ be an open set and assume that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. There exists $c = c(n,p)$ such that

$$\left(\fint_{B(x,r)} |u-u_{B(x,r)}|^p\,dy\right)^{\frac{1}{p}} \leqslant cr\left(\fint_{B(x,r)}|Du|^p\,dy\right)^{\frac{1}{p}}$$

for every $B(x,r) \Subset \Omega$.

It is enough to consider constant functions to see that it is not possible to replace the mean oscillation by the mean value on the right-hand side of (1.41) for a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, that is, in general we cannot replace

$$\left(\fint_{B(x,r)} |u-u_{B(z,r)}|^q\,dy\right)^{\frac{1}{q}}$$

by

$$\left(\fint_{B(x,r)} |u(x)|^q\,dy\right)^{\frac{1}{q}}$$

in (1.41). However, this is possible for functions $u \in W_0^{1,p}(B(x,r))$. This result follows from Corollary 1.37, but we give an alternative proof which is based on the Sobolev-Poincaré inequality (1.41). This technique can be adapted to other situations as well, see Theorem 1.47 below.

Theorem 1.44. Let $B(x,r) \subset \mathbb{R}^n$. Let $1 \leqslant q \leqslant p^* = \frac{np}{n-p}$ for $1 \leqslant p < n$ and $1 \leqslant q < \infty$ for $n \leqslant p < \infty$. There exists a constant $c = c(n,p,q)$ such that

$$\left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}} \leqslant cr \left(\fint_{B(x,r)} |Du|^p \, dy \right)^{\frac{1}{p}} \tag{1.45}$$

for every $u \in W_0^{1,p}(B(x,r))$.

Proof. We may assume that $q > 1$ since the claim for $q = 1$ follows from Hölder's inequality. Consider first the case. Then $u = 0$ in $B(x,2r) \setminus B(x,r)$. By Hölder's inequality

$$\begin{aligned} |u_{B(x,2r)}| &\leqslant \fint_{B(x,2r)} |u| \chi_{B(x,r)}(x) \, dy \\ &\leqslant \left(\frac{|B(x,r)|}{|B(x,2r)|} \right)^{1-\frac{1}{q}} \left(\fint_{B(x,2r)} |u|^q \, dy \right)^{\frac{1}{q}} \\ &= (2^{-n})^{1-\frac{1}{q}} \left(\fint_{B(x,2r)} |u|^q \, dy \right)^{\frac{1}{q}}. \end{aligned} \tag{1.46}$$

Using Minkowski's inequality, the Sobolev-Poincaré inequality from Theorem 1.40 for $B(x,2r)$ and (1.46), we obtain

$$\begin{aligned} \left(\fint_{B(x,2r)} |u|^q \, dy \right)^{\frac{1}{q}} &\leqslant \left(\fint_{B(x,2r)} |u - u_{B(x,2r)}|^q \, dy \right)^{\frac{1}{q}} + |u_{B(x,2r)}| \\ &\leqslant c(n,p,q) r \left(\fint_{B(x,2r)} |Du|^p \, dy \right)^{\frac{1}{p}} + (2^{-n})^{1-\frac{1}{q}} \left(\fint_{B(x,2r)} |u|^q \, dy \right)^{\frac{1}{q}} < \infty. \end{aligned}$$

Since $(2^{-n})^{1-\frac{1}{q}} < 1$, the second term on the right-hand side can be absorbed to the left-hand side, and thus

$$\left(\fint_{B(x,2r)} |u|^q \, dy \right)^{\frac{1}{q}} \leqslant c(n,p,q) r \left(\fint_{B(x,2r)} |Du|^p \, dy \right)^{\frac{1}{p}}.$$

Finally, the mean value integrals on both sides can be taken with respect to $B(x,r)$ since $u = 0$ and $Du = 0$ in $B(x,2r) \setminus B(x,r)$. □

It is also possible to replace the mean oscillation by the mean value on the left-hand side of (1.41) for a function $u \in W_{\text{loc}}^{1,p}(\Omega)$ do not necessary have zero boundary values but that vanish in a large subset.

Theorem 1.47. Let $1 \leqslant p < \infty$, let $\Omega \subset \mathbb{R}^n$ be an open set, and assume that $u \in W_{\text{loc}}^{1,p}(\Omega)$. Let $B(x,r) \Subset \Omega$ be a ball. Assume that $u = 0$ in a set $E \subset B(x,r)$

satisfying $|E| \geqslant \gamma |B(x,r)|$ with $0 < \gamma \leqslant 1$. Let $1 \leqslant q \leqslant p^* = \frac{np}{n-p}$ for $1 \leqslant p < n$, and $1 \leqslant q < \infty$ for $n \leqslant p < \infty$. There exists a constant $c = c(n,p,q,\gamma)$ such that

$$\left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}} \leqslant cr \left(\fint_{B(x,r)} |Du|^p \, dy \right)^{\frac{1}{p}}. \tag{1.48}$$

Proof. We may assume that $q > 1$ since the claim for $q = 1$ follows from Hölder's inequality. By Hölder's inequality

$$\begin{aligned} |u_{B(x,r)}| &= \left| \fint_{B(x,r)} u \, dy \right| \leqslant \fint_{B(x,r)} |u| \, dy = \fint_{B(x,r)} |u| \chi_{B(x,r)\setminus E} \, dy \\ &\leqslant \left(\frac{|B(x,r) \setminus E|}{|B(x,r)|} \right)^{1-\frac{1}{q}} \left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}} \\ &\leqslant (1-\gamma)^{1-\frac{1}{q}} \left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}}. \end{aligned} \tag{1.49}$$

Since $0 \leqslant (1-\gamma)^{1-\frac{1}{q}} < 1$. Using Minkowski's inequality, the Sobolev-Poincaré inequality in Theorem 1.40 and (1.49), we obtain

$$\begin{aligned} \left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}} &\leqslant \left(\fint_{B(x,r)} |u - u_{B(x,r)}|^q \, dy \right)^{\frac{1}{q}} + |u_{B(x,r)}| \\ &\leqslant c(n,p,q) r \left(\fint_{B(x,r)} |Du|^p \, dy \right)^{\frac{1}{p}} + (1-\gamma)^{1-\frac{1}{q}} \left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}} < \infty. \end{aligned}$$

Since $0 \leqslant (1-\gamma)^{1-\frac{1}{q}} < 1$, the second term on the right-hand side can be absorbed to the left-hand side, and we conclude that there exists a constant $c = c(n,p,q,\gamma)$ such that

$$\left(\fint_{B(x,r)} |u|^q \, dy \right)^{\frac{1}{q}} \leqslant cr \left(\fint_{B(x,r)} |Du|^p \, dy \right)^{\frac{1}{p}}. \qquad \square$$

1.7 Young's inequality

Before stating the main results of this chapter, we recall two useful versions of Young's inequality.

Lemma 1.50 (Young's inequality). Let $1 < p < \infty$ and $a, b \geqslant 0$, then

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ or equivalently $p' = \frac{p}{p-1}$.

Remark 1.51. Young's inequality for $p = 2$ follows immediately from

$$(a-b)^2 \geqslant 0 \Longleftrightarrow a^2 - 2ab + b^2 \geqslant 0 \Longleftrightarrow \frac{a^2}{2} + \frac{b^2}{2} \geqslant ab \geqslant 0.$$

Lemma 1.52 (Young's inequality with ε). Let $1 < p < \infty$, $a, b \geqslant 0$ and $\varepsilon > 0$. Then

$$ab \leqslant \varepsilon a^p + c b^{p'},$$

where

$$c = c(\varepsilon, p) = (p\varepsilon)^{-\frac{1}{p-1}} \frac{p-1}{p}$$

Proof. We apply Young's inequality to $\widetilde{a} = (p\varepsilon)^{\frac{1}{p}} a$ and $\widetilde{b} = (p\varepsilon)^{-\frac{1}{p}} b$. This gives

$$\begin{aligned} ab &= (p\varepsilon)^{\frac{1}{p}} a (p\varepsilon)^{-\frac{1}{p}} b \\ &\leqslant \frac{p\varepsilon a^p}{p} + (p\varepsilon)^{-\frac{p'}{p}} \frac{b^{p'}}{p'} \\ &= \varepsilon a^p + (p\varepsilon)^{-\frac{1}{p-1}} \frac{p-1}{p} b^{p'}. \end{aligned}$$ □

Remark 1.53. For $p = 2$, $a, b \geqslant 0$ and $\varepsilon > 0$, we have

$$ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.$$

Remark 1.54. It is essential that ε can be chosen as small as we please. We shall use the inequality in the following context. Suppose that $f \in L^p(A)$ and $g \in L^p(A)$ and that

$$\int_A |f|^p \, dx \leqslant c \int_A |f|^{p-1} |g| \, dx$$

for some constant $c > 0$.

Then by applying Young's inequality with ε we obtain

$$\begin{aligned} \int_A |f|^p \, dx &\leqslant c \int_A |f|^{p-1} |g| \, dx \\ &\leqslant c\varepsilon \int_A |f|^{(p-1)\cdot\frac{p}{p-1}} \, dx + c(\varepsilon, p) \int_A |g|^p \, dx. \end{aligned}$$

Now we can move the L^p-integral of f to the left-hand-side and obtain

$$(1 - c\varepsilon) \int_A |f|^p \, dx \leqslant c(\varepsilon, p) \int_A |g|^p \, dx.$$

If $1 - c\varepsilon > 0$ or equivalently $\varepsilon < \frac{1}{c}$, then the estimate above implies that

$$\int_A |f|^p \, dx \leqslant \frac{c(\varepsilon, p)}{1 - c\varepsilon} \int_A |g|^p \, dx.$$

2 Existence results

In this chapter we discuss two methods to show that a weak solution to a PDE exists under very general conditions. The first method is a Hilbert space approach which applies to linear PDEs only. Then we consider direct methods in the calculus of variations, which is a Banach space approach and applies to nonlinear PDEs as well.

2.1 Hilbert space approach for the Laplace equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $g \in W^{1,2}(\Omega)$. Consider the Dirichlet problem

$$\begin{cases} -\Delta u = 0 \quad \text{in} \quad \Omega, \\ u - g \in W_0^{1,2}(\Omega). \end{cases}$$

Recall trom Example 1.20 that a function $u \in W^{1,2}(\Omega)$ is a weak solution to the problem above, if $u - g \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega Du \cdot D\varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$.

Claim: $\langle u, v \rangle = \displaystyle\int_\Omega Du \cdot Dv \, dx$ is an inner product in $W_0^{1,2}(\Omega)$.

Reason. We show that $\langle u, u \rangle = 0$ implies that $u = 0$ almost everywhere in Ω. We note that

$$\langle u, u \rangle = \int_\Omega Du \cdot Du \, dx = \int_\Omega |Du|^2 \, dx,$$

which shows that $\langle u,u\rangle = 0$ implies that $Du = 0$ almost everywhere in Ω. We apply the Poincaré inequality, see Remark 1.38, to obtain

$$\int_\Omega |u|^2\,dx \leqslant c(\operatorname{diam}\Omega)^2 \int_\Omega |Du|^2\,dx = 0,$$

for every $u \in W_0^{1,2}(\Omega)$. This show that that $u = 0$ almost everywhere in Ω. The other properties of inner product are clear (exercise). ■

Claim: $F(v) = -\int_\Omega Dg\cdot Dv\,dx$ is a bounded linear functional on $W_0^{1,2}(\Omega)$.

Reason. It is clear that F is a linear operator. By Hölder's inequality we have

$$\begin{aligned}
|F(v)| &= \left|\int_\Omega Dg\cdot Dv\,dx\right| \leqslant \int_\Omega |Dg\cdot Dv|\,dx \\
&\leqslant \int_\Omega |Dg||Dv|\,dx \leqslant \left(\int_\Omega |Dg|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |Dv|^2\,dx\right)^{\frac{1}{2}} \\
&\leqslant \|g\|_{W^{1,2}(\Omega)}\|v\|_{W_0^{1,2}(\Omega)}
\end{aligned}$$

for every $v \in W_0^{1,2}(\Omega)$. ■

By the Riesz representation theorem, there exists a unique $w \in W_0^{1,2}(\Omega)$ such that

$$F(v) = \langle w,v\rangle = \int_\Omega Dw\cdot Dv\,dx$$

for every $v \in W_0^{1,2}(\Omega)$. Thus

$$\int_\Omega Dw\cdot Dv\,dx = -\int_\Omega Dg\cdot Dv\,dx$$

and consequently

$$\int_\Omega (Dw\cdot Dv + Dg\cdot Dv)\,dx = \int_\Omega (Dw+Dg)\cdot Dv\,dx = 0$$

for every $v \in W_0^{1,2}(\Omega)$. Let $u = w+g$. Then $u-g = w \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega Du\cdot Dv\,dx = \int_\Omega (Dw+Dg)\cdot Dv\,dx = 0$$

for every $v \in W_0^{1,2}(\Omega)$. In particular, this holds for every $v \in C_0^\infty(\Omega)$. This show that u is a unique weak solution to the problem

$$\begin{cases} -\Delta u = 0 \quad \text{in} \quad \Omega, \\ u-g \in W_0^{1,2}(\Omega). \end{cases}$$

2.2 Hilbert space approach for more general elliptic PDEs

Assume that $b_i = 0$ for $i = 1,\dots,n$. The Riesz representation theorem can be used to prove the existence of a weak solution to the Dirichlet problem

$$\begin{cases} -\sum_{i,j=1}^n D_j(a_{ij}D_iu) + cu = f \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{2.1}$$

in any bounded open subset Ω of $\mathbb{R}^n$. More general boundary values can be considered as in Remark 1.19 (2). To this end, we define a candidate for an inner product in $W_0^{1,2}(\Omega)$ as

$$\langle u,v\rangle = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv \right) dx. \tag{2.2}$$

Recall that the standard inner product in $W_0^{1,2}(\Omega)$ is obtained by choosing $a_{ij} = 1$, if $i = j$, and $a_{ij} = 0$, if $i \neq j$, and $c = 1$.

Remark 2.3. By Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega \left| \sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv \right| dx \\
&\quad \leqslant \sum_{i,j=1}^n \int_\Omega |a_{ij}D_iuD_jv|\,dx + \int_\Omega |cuv|\,dx \\
&\quad \leqslant \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)}\|D_iu\|_{L^2(\Omega)}\|D_iv\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\quad \leqslant \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)}\|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)} + \|c\|_{L^\infty(\Omega)}\|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)} \\
&\quad = \left(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \right) \|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)} < \infty.
\end{aligned}$$

This shows that the integrand in (2.2) is an integrable function with finite integral.

Thus $\langle u,v\rangle$ in (2.2) is a finite number whenever $u, v \in W_0^{1,2}(\Omega)$. Next we show that (2.2) really is an inner product under a certain condition on function c.

Lemma 2.4. There exists a constant $c_0 = c_0(\lambda, n) \leqslant 0$ such that if $c \geqslant c_0$, then (2.2) defines an inner product in $W_0^{1,2}(\Omega)$.

THE MORAL: It is important to have $c_0 \leqslant 0$ so that the case $c = 0$ is included in the theory. The proof below shows that $c \geqslant 0$ is immediate, but the point is that we can do better than that.

Proof. We show that $\langle u,u\rangle = 0$ implies $u = 0$ when $c \geqslant c_0$. To prove this, we recall the Poincaré inequality

$$\int_\Omega |u|^2\,dx \leqslant \mu \int_\Omega |Du|^2\,dx, \quad \mu = c(\operatorname{diam}\Omega)^2,$$

which holds true for every $u \in W_0^{1,2}(\Omega)$, see Remark 1.38. By the ellipticity condition, see Definition 1.7, we have

$$\begin{aligned}
\langle u,u\rangle &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j u + c|u|^2\right) dx \\
&\geqslant \lambda \int_\Omega |Du|^2\,dx + c_0 \int_\Omega |u|^2\,dx \\
&= \frac{\lambda}{2}\int_\Omega |Du|^2\,dx + \frac{\lambda}{2}\int_\Omega |Du|^2\,dx + c_0 \int_\Omega |u|^2\,dx \\
&\geqslant \frac{\lambda}{2}\int_\Omega |Du|^2\,dx + \left(\frac{\lambda}{2\mu} + c_0\right)\int_\Omega |u|^2\,dx \\
&\geqslant \alpha \|u\|^2_{W_0^{1,p}(\Omega)},
\end{aligned}$$

where

$$\alpha = \min\left\{\tfrac{\lambda}{2}, \tfrac{\lambda}{2\mu} + c_0\right\}. \tag{2.5}$$

In particular, this shows that $\langle u,u\rangle \geqslant 0$. If $c \geqslant c_0 > -\frac{\lambda}{2\mu}$, then $\alpha > 0$ and it follows that $\langle u,u\rangle = 0$ implies $\|u\|_{W_0^{1,p}(\Omega)} = 0$ and thus $u = 0$. The other properties of an inner product are clear (exercise). □

Remark 2.6. For the norm induced by the inner product (2.2) we have

$$\begin{aligned}
\|u\|^2 = \langle u,u\rangle &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j u + c|u|^2\right) dx \\
&\leqslant \Lambda \int_\Omega |Du|^2\,dx + \|c\|_\infty \int_\Omega |u|^2\,dx \leqslant \beta \|u\|^2_{W_0^{1,2}(\Omega)},
\end{aligned}$$

with $\beta = \max\{\Lambda, \|c\|_\infty\}$. Thus

$$\sqrt{\alpha}\|u\|_{W_0^{1,2}(\Omega)} \leqslant \|u\| \leqslant \sqrt{\beta}\|u\|_{W_0^{1,2}(\Omega)},$$

for every $u \in W_0^{1,p}(\Omega)$, where α is as in (2.5). This shows that $\|\cdot\|_{W_0^{1,2}(\Omega)}$ and $\|\cdot\|$ are equivalent norms in $W_0^{1,2}(\Omega)$ if $c \geqslant c_0$.

Lemma 2.7. Let $\widehat{W}_0^{1,2}(\Omega)$ be $W_0^{1,2}(\Omega)$ with the inner product given by (2.2). Then

$$F(v) = \int_\Omega f v\,dx$$

is a bounded linear functional on $\widehat{W}_0^{1,2}(\Omega)$.

Remark 2.8. Note that $F(v) = \langle f, v\rangle_{L^2(\Omega)}$, where $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ is the standard inner product in $L^2(\Omega)$.

Proof. Hölder's inequality and the proof of Lemma 2.4 imply

$$\begin{aligned}|F(v)| = \left|\int_\Omega f v\,dx\right| &\leqslant \left(\int_\Omega |f|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant \|f\|_{L^2(\Omega)}\|v\|_{W_0^{1,2}(\Omega)} \leqslant \frac{1}{\sqrt{\alpha}}\|f\|_{L^2(\Omega)}\|v\|,\end{aligned}$$

where α is given by (2.5). □

Theorem 2.9. Assume that Ω is a bounded and open subset of $\mathbb{R}^n$ and $f \in L^2(\Omega)$. There exists $c_0 \leqslant 0$ such that (2.1) has a unique weak solution for every $c \geqslant c_0$.

THE MORAL: There exists a unique solution to the Dirichlet problem with zero boundary values in the Sobolev sense in any bounded set.

Proof. By Definition 1.18, a function $u \in W_0^{1,2}(\Omega)$ is a weak solution to (2.1) if

$$\langle u,v\rangle = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv\right)dx = \int_\Omega f v\,dx$$

for every $v \in C_0^\infty(\Omega)$. Here we used the inner product defined by (2.2). By Lemma 2.7

$$F(v) = \int_\Omega f v\,dx = \langle f,v\rangle_{L^2(\Omega)}$$

is a bounded linear functional on $\widehat{W}_0^{1,2}(\Omega)$. Note that $\widehat{W}_0^{1,2}(\Omega)$ is a Banach space, because $\|\cdot\|_{W_0^{1,2}(\Omega)}$ and $\|\cdot\|$ are equivalent norms in $W_0^{1,2}(\Omega)$, see Remark 2.6. Thus $\widehat{W}_0^{1,2}(\Omega)$ is a Hilbert space when $c \geqslant c_0$ given by Lemma 2.4. By the Riesz representation theorem, there exists a unique $u \in \widehat{W}_0^{1,2}(\Omega)$ such that

$$F(v) = \langle u,v\rangle = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv\right)dx$$

for every $v \in \widehat{W}_0^{1,2}(\Omega)$. By Remark 2.6, we have $\widehat{W}_0^{1,2}(\Omega) \subset W_0^{1,2}(\Omega)$. Thus $u \in W_0^{1,2}(\Omega)$. By Remark 2.6 again, we have $C_0^\infty(\Omega) \subset W_0^{1,2}(\Omega) \subset \widehat{W}_0^{1,2}(\Omega)$ and thus

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv\right)dx = \int_\Omega f v\,dx$$

for every $v \in C_0^\infty(\Omega)$. □

Example 2.10. Let $\Omega \subset \mathbb{R}^n$ be any bounded open set and $f \in L^2(\Omega)$. By Theorem 2.9 there exists a unique weak solution to the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega,\\ u \in W_0^{1,2}(\Omega), \end{cases}$$

that is, $u \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega Du\cdot D\varphi\,dx = \int_\Omega f\varphi\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Example 2.11. Let $\Omega \subset \mathbb{R}^n$ be any bounded open set, $Lu = -\Delta u$, $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$. By Remark 1.19 (2) a weak solution to the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u - g \in W_0^{1,2}(\Omega), \end{cases} \tag{2.12}$$

can be obtained by considering $w = u - g \in W_0^{1,2}(\Omega)$ and the problem

$$\begin{cases} -\Delta w = \overline{f} & \text{in} \quad \Omega, \\ w \in W_0^{1,2}(\Omega), \end{cases} \tag{2.13}$$

with $\overline{f} = f - Lg = f + \Delta g$. By Theorem 2.9 there exists a unique weak solution $w \in W_0^{1,2}(\Omega)$ to (2.13), that is,

$$\int_\Omega Dw \cdot D\varphi \, dx = \int_\Omega f\varphi \, dx - \int_\Omega Dg \cdot D\varphi \, dx$$

or equivalently

$$\int_\Omega (Dw + Dg) \cdot D\varphi \, dx = \int_\Omega f\varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. This means that $u = w + g$ is the unique solution of (2.12). Thus $u \in W^{1,2}(\Omega)$ is a weak solution of (2.12) if and only if $-\Delta u = f$ in weak sense and $u - g \in W_0^{1,2}(\Omega)$.

Example 2.14. Let $\Omega \subset \mathbb{R}^n$ be any bounded open set, $f \in L^2(\Omega)$ and $c \geqslant c_0$. By Theorem 2.9 there exists a unique weak solution to the problem

$$\begin{cases} -\Delta u + cu = f & \text{in} \quad \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

that is, $u \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega Du \cdot D\varphi \, dx + \int_\Omega cu\varphi \, dx = \int_\Omega f\varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Example 2.15. Let $n = 1$, $\Omega = (0,2)$, $c = 0 = b$, $f = 1$ and

$$a(x) = \begin{cases} x, & x \in (0,1], \\ 1, & x \in (1,2). \end{cases}$$

Consider the problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(0) = u(2) = 0. \end{cases}$$

Observe that L is not uniformly elliptic.

By solving

$$Lu(x) = -(a(x)u'(x))' = f(x) = 1$$

in $(0,1)$ and $(1,2)$ respectively, we obtain

$$u(x) = \begin{cases} -x + c_1 \ln x + c_2, & x \in (0,1], \\ -\frac{1}{2}x^2 + c_3 x + c_4, & x \in (1,2). \end{cases}$$

By the boundary conditions and requiring continuity at $x = 1$, we obtain

$$u(x) = \begin{cases} -x, & x \in (0,1], \\ -\frac{1}{2}x^2 + \frac{5}{2}x - 3, & x \in (1,2). \end{cases}$$

However, this is not a weak solution of the problem (exercise).

Example 2.16. Let $\Omega = (0,2)$ and

$$f(x) = a(x) = \begin{cases} 1, & x \in (0,1], \\ 0, & x \in [1,2). \end{cases}$$

Consider the problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(0) = u(2) = 0. \end{cases}$$

Observe that L is not uniformly elliptic. Then

$$u_1(x) = \begin{cases} -\frac{1}{2}x^2 + x, & x \in (0,1], \\ -x^2 + \frac{5}{2}x - 1, & x \in [1,2), \end{cases}$$

and

$$u_2(x) = \begin{cases} -\frac{1}{2}x^2 + x, & x \in (0,1], \\ 1 - \frac{1}{2}x, & x \in [1,2), \end{cases}$$

are weak solutions to $Lu = f$ (exercise).

THE MORAL: If the operator is not uniformly elliptic, a weak solution of a boundary value problem is not necessarily unique.

Example 2.17. Let $n = 1$, $\Omega = (0,\pi)$, $a = 1$, $b = 0$, $c = -4$. The operator L is uniformly elliptic, but the corresponding bilinear form

$$B[u,v] = \int_0^\pi \left(u'(x)v'(x) - 4u(x)v(x)\right) dx$$

is not positive definite on $W_0^{1,2}(\Omega)$. For example, if $u(x) = \sin x$, then

$$B[u,u] = \int_0^\pi \left((\cos x)^2 - 4(\sin x)^2\right) dx = -\frac{3\pi}{2}.$$

In particular, the bilinear form $B[u,v]$ is not an inner product on $W^{1,2}(\Omega)$.

Claim: Let $f(x) = \sin(2x)$. Then the problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(0) = 0 = u(\pi), \end{cases}$$

does not have any solutions.

Reason. Let $u \in W_0^{1,2}(\Omega)$. An integration by parts gives

$$\begin{aligned} B[u,v] &= \int_0^\pi \big(u'(x)v'(x) - 4u(x)v(x)\big)\,dx \\ &= \int_0^\pi \big(2u'(x)\cos(2x) - 4u(x)\sin(2x)\big)\,dx \\ &= \int_0^\pi (2u(x)\cos(2x))'\,dx \\ &= 0 \neq \int_0^\pi (\sin(2x))^2\,dx \\ &= \int_0^\pi f(x)v(x)\,dx, \end{aligned}$$

when $v(x) = \sin(2x) \in W_0^{1,2}(\Omega)$. Thus there does not exist a function $u \in W_0^{1,2}(\Omega)$ for which

$$B[u,v] = \int_\Omega f(x)v(x)\,dx \quad \text{for every} \quad v \in W_0^{1,2}(\Omega).$$

■

Observe that the corresponding homogeneous problem

$$\begin{cases} Lu(x) = 0, & x \in \Omega, \\ u(0) = 0 = u(\pi), \end{cases}$$

has infinitely many solutions $u(x) = a\sin(2x)$, $a \in \mathbb{R}$.

Remark 2.18. For a general operator L defined by (1.4), there is a bilinear form

$$B[u,v] = \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j v + \sum_{i=1}^n b_i D_i u v + cuv \right) dx,$$

where $u, v \in W_0^{1,2}(\Omega)$. If the functions b_i, $i = 1, \dots, n$ are not all equal to zero, then the bilinear form is not symmetric, that is, $B[u,v] \neq B[v,u]$ and the Riesz representation theorem cannot be applied as such, since $B[\cdot,\cdot]$ is not an inner product. In this case we may apply the Lax-Milgram theorem, which is a slightly extended version of the Riesz representation theorem, see [1, Theorem 2.3], [2, Theorem 3, p. 321], [8, Theorem 8.3] and [16, Theorem 2.3.2]. These results cover the case $c \geqslant c_0 \geqslant 0$. The general case can be investigated by the Fredholm alternative, see [1, Theorem 3.2], [2, Theorem 4, p. 323], [8, Theorem 8.6] and [16, Theorem 2.3.3].. Some features are visible in the following one-dimensional example.

Example 2.19. Let $n = 1$, $\Omega = (0,l)$, with $l > 0$, and consider the problem

$$\begin{cases} Lu(x) = -u''(x) + cu(x) = 0, & x \in \Omega, \\ u(0) = u(l) = 0. \end{cases}$$

This is a particular case of the so-called Sturm-Liouville problem, which arises, for example, in the separation of variables technique. We solve this problem by finding the constants c (eigenvalues) for which the problem has nontrivial solutions (eigenfunctions). We consider three cases.

$\boxed{c > 0}$ Then $c = \mu^2$ for some $\mu > 0$ and the general solution of the equation $u'' = cu$ is

$$u(x) = c_1 \sinh(\mu x) + c_2 \cosh(\mu x), \quad c_1, c_2 \in \mathbb{R}.$$

Recall that $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Since $u(0) = 0$ gives $c_2 = 0$ and $u(l) = 0$ gives $c_1 \sinh(\mu x) = 0$, we conclude that $c_1 = c_2 = 0$. In this case we only have the trivial solution $u = 0$.

$\boxed{c = 0}$ Then the equation reduces to $u'' = 0$ with the general solution

$$u(x) = c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions $u(0) = u(l) = 0$ imply $c_1 = c_2 = 0$ and $u = 0$.

$\boxed{c < 0}$ Then $c = -\mu^2$ for some $\mu > 0$ and the general solution of the equation $u'' = cu$ is

$$u(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x), \quad c_1, c_2 \in \mathbb{R}.$$

Hence, $u(0) = 0$ implies $c_2 = 0$ and $u(l) = 0$ implies $c_1 \sin(\mu x) = 0$. If we assume $c_1 \neq 0$, we obtain $\sin(\mu x) = 0$, and the possible values of $\mu > 0$ are

$$\mu_k = \frac{k\pi}{l}, \quad k \in \mathbb{Z} \setminus \{0\}.$$

Thus for every $k \in \mathbb{Z} \setminus \{0\}$ we may choose $c = -\mu_k^2 = -(\frac{k\pi}{l})^2$ and have a nontrivial solution

$$u_k(x) = \sin\left(\frac{k\pi x}{l}\right).$$

Note that the trivial solution $u = 0$ is also a solution to the problem.

2.3 Direct methods in the calculus of variations for the Laplace equation

Recall from Example 1.20 that a function $u \in W^{1,2}(\Omega)$ is a weak solution to $\Delta u = 0$ in Ω, if

$$\int_\Omega Du \cdot D\varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$.

The next lemma shows that, in the definition of a weak solution, the class of test functions can be taken to be the Sobolev space with zero boundary values.

Lemma 2.20. If $u \in W^{1,2}(\Omega)$ is a weak solution to the Laplace equation, then

$$\int_\Omega Du \cdot Dv\, dx = 0$$

for every $v \in W_0^{1,2}(\Omega)$.

Proof. Let $v_i \in C_0^\infty(\Omega)$, $i = 1,2,\dots$, be such that $v_i \to v$ in $W^{1,2}(\Omega)$. Then by the Cauchy-Schwarz inequality and Hölder's inequality, we have

$$\begin{aligned}\left|\int_\Omega Du \cdot Dv\, dx - \int_\Omega Du \cdot Dv_i\, dx\right| &= \left|\int_\Omega Du \cdot (Dv - Dv_i)\, dx\right| \\ &\leqslant \int_\Omega |Du||Dv - Dv_i|\, dx \\ &\leqslant \left(\int_\Omega |Du|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |Dv - Dv_i|^2\, dx\right)^{\frac{1}{2}} \to 0\end{aligned}$$

as $i \to \infty$. Thus

$$\int_\Omega Du \cdot Dv\, dx = \lim_{i\to\infty} \int_\Omega Du \cdot Dv_i\, dx = 0.$$

□

Remark 2.21. Assume that $\Omega \subset \mathbb{R}^n$ is bounded and $g \in W^{1,2}(\Omega)$. If there exists a weak solution $u \in W^{1,2}(\Omega)$ to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u - g \in W_0^{1,2}(\Omega), \end{cases}$$

then the solution is unique. Observe that the boundary values are taken in the Sobolev sense.

Reason. Let $u_1 \in W^{1,2}(\Omega)$, with $u_1 - g \in W_0^{1,2}(\Omega)$, and $u_2 \in W^{1,2}(\Omega)$, with $u_2 - g \in W_0^{1,2}(\Omega)$, be solutions to the Dirichlet problem above. By Lemma 2.20

$$\int_\Omega Du_1 \cdot Dv\, dx = 0 \quad \text{and} \quad \int_\Omega Du_2 \cdot Dv\, dx = 0$$

for every $v \in W_0^{1,2}(\Omega)$ and thus

$$\int_\Omega (Du_1 - Du_2) \cdot Dv\, dx = 0 \quad \text{for every} \quad v \in W_0^{1,p}(\Omega).$$

Since

$$u_1 - u_2 = \underbrace{(u_1 - g)}_{\in W_0^{1,2}(\Omega)} - \underbrace{(u_2 - g)}_{\in W_0^{1,2}(\Omega)} \in W_0^{1,2}(\Omega),$$

we may choose $v = u_1 - u_2$ and conclude

$$\int_\Omega |Du_1 - Du_2|^2\, dx = \int_\Omega (Du_1 - Du_2) \cdot (Du_1 - Du_2)\, dx = 0.$$

This implies $Du_1 - Du_2 = 0$ almost everywhere in Ω. By the Poincaré inequality, see Remark 1.38, we have

$$\int_\Omega |u_1 - u_2|^2\,dx \leqslant c\,\mathrm{diam}(\Omega)^2 \int_\Omega |Du_1 - Du_2|^2\,dx = 0.$$

This implies $u_1 - u_2 = 0 \iff u_1 = u_2$ almost everywhere in Ω. This is a PDE proof of uniqueness and in the proof of Theorem 2.35 we shall see a variational argument for the same result. ■

Next we consider a variational approach to the Dirichlet problem for the Laplace equation.

Definition 2.22. Assume that $g \in W^{1,2}(\Omega)$. A function $u \in W^{1,2}(\Omega)$ with $u - g \in W_0^{1,2}(\Omega)$ is a minimizer of the variational integral

$$I(u) = \int_\Omega |Du|^2\,dx$$

with boundary values g, if

$$\int_\Omega |Du|^2\,dx \leqslant \int_\Omega |Dv|^2\,dx$$

for every $v \in W^{1,2}(\Omega)$ with $v - g \in W_0^{1,2}(\Omega)$.

T H E M O R A L : A minimizer u minimizes the variational integral $I(u)$ in the class of functions with given boundary values, that is,

$$\int_\Omega |Du|^2\,dx = \inf\left\{\int_\Omega |Dv|^2\,dx : v \in W^{1,2}(\Omega),\, v - g \in W_0^{1,2}(\Omega)\right\}.$$

If there is a minimizer, then infimum can be replaced by minimum.

Theorem 2.23. Assume that $g \in W^{1,2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ with $u - g \in W_0^{1,2}(\Omega)$. Then

$$\int_\Omega |Du|^2\,dx = \inf\left\{\int_\Omega |Dv|^2\,dx : v \in W^{1,2}(\Omega),\, v - g \in W_0^{1,2}(\Omega)\right\}$$

if and only if u is a weak solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u - g \in W_0^{1,2}(\Omega). \end{cases}$$

T H E M O R A L : A function is a weak solution to the Dirichlet problem if and only if it is a minimizer of the corresponding variational integral with the given boundary values in the Sobolev sense.

Proof. $\boxed{\Longrightarrow}$ Assume that $u \in W^{1,2}(\Omega)$ is a minimizer with boundary values $g \in W^{1,2}(\Omega)$. We use the method of variations by Lagrange. Let $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon \in \mathbb{R}$. Then $(u+\varepsilon\varphi)-g \in W_0^{1,2}(\Omega)$ and

$$\begin{aligned}\int_\Omega |D(u+\varepsilon\varphi)|^2\,dx &= \int_\Omega (Du+\varepsilon D\varphi)\cdot(Du+\varepsilon D\varphi)\,dx\\ &= \int_\Omega |Du|^2\,dx + 2\varepsilon\int_\Omega Du\cdot D\varphi\,dx + \varepsilon^2\int_\Omega |D\varphi|^2\,dx\\ &= i(\varepsilon).\end{aligned}$$

Since u is a minimizer, $i(\varepsilon)$ has minimum at $\varepsilon = 0$, which implies that $i'(0) = 0$. Clearly

$$i'(\varepsilon) = 2\int_\Omega Du\cdot D\varphi\,dx + 2\varepsilon\int_\Omega |D\varphi|^2\,dx$$

and thus

$$i'(0) = 2\int_\Omega Du\cdot D\varphi\,dx = 0.$$

This shows that

$$\int_\Omega Du\cdot D\varphi\,dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$.

$\boxed{\Longleftarrow}$ Assume that $u \in W^{1,2}(\Omega)$ is a weak solution to $\Delta u = 0$ with $u - g \in W_0^{1,2}(\Omega)$ and let $v \in W^{1,2}(\Omega)$ with $v - g \in W_0^{1,2}(\Omega)$. Then

$$\begin{aligned}\int_\Omega |Dv|^2\,dx &= \int_\Omega |D(v-u)+Du|^2\,dx\\ &= \int_\Omega (D(v-u)+Du)\cdot(D(v-u)+Du)\,dx\\ &= \int_\Omega |D(v-u)|^2\,dx + 2\int_\Omega D(v-u)\cdot Du\,dx + \int_\Omega |Du|^2\,dx.\end{aligned}$$

Since

$$v-u = \underbrace{(v-g)}_{\in W_0^{1,2}(\Omega)} - \underbrace{(u-g)}_{\in W_0^{1,2}(\Omega)} \in W_0^{1,2}(\Omega),$$

by Lemma 2.20 we have

$$\int_\Omega Du\cdot D(v-u)\,dx = 0$$

and thus

$$\int_\Omega |Dv|^2\,dx = \int_\Omega |D(v-u)|^2\,dx + \int_\Omega |Du|^2\,dx \geqslant \int_\Omega |Du|^2\,dx$$

for every $v \in W^{1,2}(\Omega)$ with $v - g \in W_0^{1,2}(\Omega)$. Thus u is a minimizer. $\square$

Next we give an existence proof using the direct methods in the calculus variations. This means that, instead of the PDE, the argument uses the variational integral.

Theorem 2.24. Assume that Ω is a bounded open subset of $\mathbb{R}^n$. Then for every $g \in W^{1,2}(\Omega)$ there exists a unique minimizer $u \in W^{1,2}(\Omega)$ with $u-g \in W_0^{1,2}(\Omega)$, which satisfies

$$\int_\Omega |Du|^2\,dx = \inf\left\{\int_\Omega |Dv|^2\,dx : v \in W^{1,2}(\Omega),\, v-g \in W_0^{1,2}(\Omega)\right\}.$$

THE MORAL: The Dirichlet problem for the Laplace equation has a unique solution with Sobolev boundary values in any bounded open set.

WARNING: It is not clear whether the solution to the variational problem attains the boundary values pointwise.

Proof. $\boxed{(1)}$ Since $I(u) \geqslant 0$, in particular, it is bounded from below in $W^{1,2}(\Omega)$ and since u is a minimizer, $g \in W^{1,2}(\Omega)$ and $g-g=0 \in W_0^{1,2}(\Omega)$, we note that

$$0 \leqslant m = \inf\left\{\int_\Omega |Du|^2\,dx : u \in W^{1,2}(\Omega),\, u-g \in W_0^{1,2}(\Omega)\right\} \leqslant \int_\Omega |Dg|^2\,dx < \infty.$$

The definition of infimum then implies that there exists a minimizing sequence $u_i \in W^{1,2}(\Omega)$ with $u_i - g \in W_0^{1,2}(\Omega)$, $i = 1,2,\dots$, such that

$$\lim_{i\to\infty} \int_\Omega |Du_i|^2\,dx = m.$$

The existence of the limit implies the sequence $(I(u_i))$ is bounded. Thus there exists a constant $M < \infty$ such that

$$I(u_i) = \int_\Omega |Du_i|^2\,dx \leqslant M \quad \text{for every} \quad i = 1,2,\dots,$$

$\boxed{(2)}$ By the Poincaré inequality, see Remark 1.38, we obtain

$$\begin{aligned}
&\int_\Omega |u_i - g|^2\,dx + \int_\Omega |D(u_i-g)|^2\,dx \\
&\leqslant c\,\mathrm{diam}(\Omega)^2 \int_\Omega |D(u_i-g)|^2\,dx + \int_\Omega |D(u_i-g)|^2\,dx \\
&\leqslant (c\,\mathrm{diam}(\Omega)^2+1) \int_\Omega |Du_i - Dg|^2\,dx \\
&\leqslant (c\,\mathrm{diam}(\Omega)^2+1)\left(2\int_\Omega |Du_i|^2\,dx + 2\int_\Omega |Dg|^2\,dx\right) \\
&\leqslant c(\mathrm{diam}(\Omega)^2+1)\left(M + \int_\Omega |Dg|^2\,dx\right) < \infty
\end{aligned}$$

for every $i = 1,2,\dots$ This shows that $(u_i - g)$ is a bounded sequence in $W_0^{1,2}(\Omega)$.

$\boxed{(3)}$ By sequential weak compactness of $W_0^{1,2}(\Omega)$ there is a subsequence $(u_{i_k}-g)$ and a function $u \in W^{1,2}(\Omega)$, with $u-g \in W_0^{1,2}(\Omega)$, such that $u_{i_k} \to u$ weakly in $L^2(\Omega)$

and $\frac{\partial u_{i_k}}{\partial x_j} \to \frac{\partial u}{\partial x_j}$, $j = 1,\dots,n$, weakly in $L^2(\Omega)$ as $k \to \infty$. By lower semicontinuity of L^2-norm with respect to weak convergence, we have

$$\int_\Omega |Du|^2\,dx \leqslant \liminf_{k\to\infty} \int_\Omega |Du_{i_k}|^2\,dx = \lim_{i\to\infty} \int_\Omega |Du_i|^2\,dx.$$

Since $u \in W^{1,2}(\Omega)$, with $u - g \in W_0^{1,2}(\Omega)$, we have

$$m \leqslant \int_\Omega |Du|^2\,dx \leqslant \lim_{i\to\infty} \int_\Omega |Du_i|^2\,dx = m$$

which implies

$$\int_\Omega |Du|^2\,dx = m.$$

Thus u is a minimizer.

$\boxed{(4)}$ To show uniqueness, let $u_1 \in W^{1,2}(\Omega)$, with $u_1 - g \in W_0^{1,2}(\Omega)$ and $u_2 \in W^{1,2}(\Omega)$, with $u_2 - g \in W_0^{1,2}(\Omega)$ be minimizers of $I(u)$ with the same boundary function $g \in W^{1,2}(\Omega)$. Assume that $u_1 \neq u_2$, that is, $|\{x \in \Omega : u_1(x) \neq u_2(x)\}| > 0$. By the Poincaré inequality, Remark 1.38, we have

$$0 < \int_\Omega |u_1 - u_2|^2\,dx \leqslant c\,\mathrm{diam}(\Omega)^2 \int_\Omega |Du_1 - Du_2|^2\,dx$$

and thus $|\{x \in \Omega : Du_1(x) \neq Du_2(x)\}| > 0$. Let $v = \frac{u_1+u_2}{2}$. Then $v \in W^{1,2}(\Omega)$ and

$$v - g = \frac{1}{2}\underbrace{(u_1 - g)}_{\in W_0^{1,2}(\Omega)} + \frac{1}{2}\underbrace{(u_2 - g)}_{\in W_0^{1,2}(\Omega)} \in W_0^{1,2}(\Omega).$$

By strict convexity of $\xi \mapsto |\xi|^2$ we conclude that

$$|Dv|^2 < \frac{1}{2}|Du_1|^2 + \frac{1}{2}|Du_2|^2 \quad \text{on} \quad \{x \in \Omega : Du_1(x) \neq Du_2(x)\}.$$

Since $|\{x \in \Omega : Du_1(x) \neq Du_2(x)\}| > 0$ and using the fact that both u_1 and u_2 are minimizers, we obtain

$$\int_\Omega |Dv|^2\,dx < \frac{1}{2}\int_\Omega |Du_1|^2\,dx + \frac{1}{2}\int_\Omega |Du_2|^2\,dx = \frac{1}{2}m + \frac{1}{2}m = m.$$

Thus $I(v) < m$. This is a contradiction with the fact that u_1 and u_2 are minimizers. $\square$

Remarks 2.25:

(1) This approach generalizes to other variational integrals as well. Indeed, the proof above is based on the following steps:

(a) Choose a minimizing sequence.

(b) Use coercivity

$$\|u_i\|_{W^{1,2}(\Omega)} \to \infty \Longrightarrow I(u_i) \to \infty.$$

to show that the minimizing sequence is bounded in the Sobolev space.

(c) Use reflexivity to show that there is a weakly converging subsequence.

(d) Use lower semicontinuity of the variational integral to show that the limit is a minimizer.

(e) Use strict convexity of the variational integral to show uniqueness.

(2) If we consider $C^2(\Omega)$ instead of $W^{1,2}(\Omega)$ in the Dirichlet problem above, then we end up having the following problems. If we equip $C^2(\Omega)$ with the supremum norm

$$\|u\|_{C^2(\Omega)} = \|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)} + \|D^2u\|_{L^\infty(\Omega)},$$

where D^2u is the Hessian matrix of second order partial derivatives, then the variational integral is not coercive nor the space is reflexive. Indeed, when $n \geqslant 2$ it is possible to construct a sequence of functions for which the supremum tends to infinity, but the L^2 norm of the gradients tends to zero. The variational integral is not coersive even when $n = 1$. If we try to obtain coercivity and reflexivity in $C^2(\Omega)$ by changing norm to $\|u\|_{W^{1,2}(\Omega)}$ then we lose completeness, since the limit functions are not necessarily in $C^2(\Omega)$. The Sobolev space seems to have all desirable properties for existence of solutions to PDEs.

2.4 Direct methods in the calculus of variations for more general elliptic PDEs

The variational integral related to the PDE

$$-\sum_{i,j=1}^{n} D_j(a_{ij}D_iu) + cu = f \tag{2.26}$$

is

$$\begin{aligned} I(v) &= \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}D_ivD_jv + cv^2\right)dx - \int_\Omega fv\,dx \\ &= \frac{1}{2}\int_\Omega \left(ADv\cdot Dv + cv^2\right)dx - \int_\Omega fv\,dx, \end{aligned} \tag{2.27}$$

where $A = A(x) = (a_{ij}(x))$ is an $n\times n$ matrix. The PDE (2.26) is called the Euler-Lagrange equation of the variational integral (2.27).

Remark 2.28. By Hölder's inequality, we have

$$\begin{aligned}
|I(v)| &= \left|\frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a_{ij}D_i v D_j v + cv^2\right) dx - \int_\Omega f v\,dx\right| \\
&\leqslant \frac{1}{2}\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} \int_\Omega |Dv|^2\,dx + \frac{1}{2}\int_\Omega |c||v|^2\,dx + \left|\int_\Omega f v\,dx\right| \\
&\leqslant \frac{1}{2}\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} \|Dv\|^2_{L^2(\Omega)} + \frac{1}{2}\|c\|_{L^\infty(\Omega)}\|v\|^2_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \\
&\leqslant \frac{1}{2}\left(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}\right)\|v\|^2_{W^{1,2}(\Omega)} + \|v\|_{W^{1,2}(\Omega)}\|f\|_{L^2(\Omega)} < \infty.
\end{aligned}$$

This shows that the integrand in (2.27) is an integrable function with finite integral for every $v \in W^{1,2}(\Omega)$.

Example 2.29. The variational integral related to Serrin's example in Section 1.5 is

$$I(v) = \int_{B(0,1)} \left(|Dv(x)|^2 + \sigma\left(\frac{x}{|x|}\cdot Dv(x)\right)^2\right) dx,$$

with $\sigma = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)} > 0$. Observe that the integrand

$$F(x,\xi) = \left(|\xi|^2 + \sigma\left(\frac{x}{|x|}\cdot\xi\right)^2\right)$$

satisfies

$$|\xi|^2 \leqslant F(x,\xi) \leqslant (1+\sigma)|\xi|^2,$$

where $\sigma > 0$ can be made arbitrarily small by choosing $\alpha > 0$ small enough.

Definition 2.30. A function $u \in W_0^{1,2}(\Omega)$ is a minimizer of (2.27) with zero boundary values, if $I(u) \leqslant I(v)$ for every $v \in W_0^{1,2}(\Omega)$.

T H E M O R A L : A minimizer u minimizes the variational integral $I(u)$ in the class of functions with zero boundary values, that is,

$$I(u) = \inf\left\{I(v) : v \in W_0^{1,2}(\Omega)\right\}.$$

If there is a minimizer, then infimum can be replaced by minimum.

Remark 2.31. For nonzero boundary values $g \in W^{1,2}(\Omega)$, we may consider

$$I(u) = \inf\left\{I(v) : v \in W^{1,2}(\Omega),\, v - g \in W_0^{1,2}(\Omega)\right\}.$$

Thus a function $u \in W^{1,2}(\Omega)$ is a minimizer of (2.27) with boundary values $g \in W^{1,2}(\Omega)$, if $I(u) \leqslant I(v)$ for every $v \in W^{1,2}(\Omega)$ with $v - g \in W_0^{1,2}(\Omega)$. We consider zero boundary values case in the argument below, but the methods apply to nonzero boundary values as well (exercise).

Theorem 2.32. If $u \in W_0^{1,2}(\Omega)$ is a minimizer of (2.27), then it is a weak solution to (2.27).

T H E M O R A L : A minimizer of a variational integral with given boundary values in the Sobolev sense is a weak solution to the Dirichlet problem for the corresponding Euler-Langrange equation.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon \in \mathbb{R}$. Then

$$\begin{aligned} I(u) &\leqslant I(u+\varepsilon\varphi) \\ &= \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i(u+\varepsilon\varphi) D_j(u+\varepsilon\varphi) + c(u+\varepsilon\varphi)^2 \right) dx - \int_\Omega f(u+\varepsilon\varphi)\,dx \\ &= i(\varepsilon). \end{aligned}$$

Since u is a minimizer, $i(\varepsilon)$ has a minimum at $\varepsilon = 0$, which implies that $i'(0) = 0$. A direct computation shows that

$$\begin{aligned} i(\varepsilon) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij} \left(D_i u D_j u + \varepsilon D_i u D_j \varphi + \varepsilon D_i \varphi D_j u + \varepsilon^2 D_i \varphi D_j \varphi \right) dx \\ + \frac{1}{2}\int_\Omega c(u^2 + 2\varepsilon u\varphi + \varepsilon^2\varphi^2)\,dx - \int_\Omega (fu + \varepsilon f\varphi)\,dx. \end{aligned}$$

Thus

$$\begin{aligned} i'(\varepsilon) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij} \left(D_i u D_j \varphi + D_i \varphi D_j u + 2\varepsilon D_i \varphi D_j \varphi \right) dx \\ + \int_\Omega c(u\varphi + \varepsilon\varphi^2)\,dx - \int_\Omega f\varphi\,dx \end{aligned}$$

and we obtain

$$i'(0) = \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n a_{ij} \left(D_i u D_j \varphi + D_i \varphi D_j u \right) \right) dx + \int_\Omega cu\varphi\,dx - \int_\Omega f\varphi\,dx.$$

As $a_{ij} = a_{ji}$ and $i'(0) = 0$, we obtain

$$\begin{aligned} i'(0) &= \frac{1}{2}\int_\Omega 2\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi\,dx + \int_\Omega cu\varphi\,dx - \int_\Omega f\varphi\,dx \\ &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi + cu\varphi \right) dx - \int_\Omega f\varphi\,dx = 0 \end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. This shows that u is a weak solution to (2.26). □

Lemma 2.33. Assume that $f \in L^2(\Omega)$. The variational integral (2.27) is bounded from below in $W_0^{1,2}(\Omega)$ provided $c \geqslant c_0$, where c_0 is as in the proof of Lemma 2.4.

THE MORAL: We already know that $|I(v)| < \infty$ for every $W_0^{1,2}(\Omega)$. The lemma asserts that there is a constant m such that $I(v) \geqslant m$ for every $W_0^{1,2}(\Omega)$, that is,

$$\inf\left\{I(v) : v \in W_0^{1,2}(\Omega)\right\} > -\infty.$$

This excludes the case that the infimum is $-\infty$.

Proof. By the ellipticity condition, see Definition 1.7, we have

$$\begin{aligned}
I(v) &= \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij} D_i v D_j v\,dx + \frac{1}{2}\int_\Omega cv^2\,dx - \int_\Omega fv\,dx \\
&\geqslant \frac{\lambda}{2}\int_\Omega |Dv|^2\,dx + \frac{1}{2}\int_\Omega cv^2\,dx - \left|\int_\Omega fv\,dx\right| \\
&\geqslant \frac{\lambda}{2}\int_\Omega |Dv|^2\,dx + \frac{c_0}{2}\int_\Omega v^2\,dx - \int_\Omega |f||v|\,dx \qquad (c \geqslant c_0) \\
&\geqslant \frac{\lambda}{2}\int_\Omega |Dv|^2\,dx + \frac{c_0}{2}\int_\Omega v^2\,dx - \frac{\varepsilon}{2}\int_\Omega v^2\,dx - \frac{1}{2\varepsilon}\int_\Omega f^2\,dx \\
&\qquad\qquad \left(0 \leqslant \left(\sqrt{\varepsilon}v - \tfrac{1}{\sqrt{\varepsilon}}f\right)^2 = \varepsilon v^2 - 2vf + \tfrac{1}{\varepsilon}f^2, \text{ see Corollary 1.52}\right) \\
&\geqslant \frac{1}{2}\left(\frac{\lambda}{\mu} + c_0 - \varepsilon\right)\int_\Omega v^2\,dx - \frac{1}{2\varepsilon}\int_\Omega f^2\,dx \qquad \text{(Poincaré inequality)} \\
&\geqslant -\frac{1}{2\varepsilon}\int_\Omega f^2\,dx \qquad \left(\frac{\lambda}{\mu} + c_0 - \varepsilon > 0\right)
\end{aligned}$$

for every $v \in W_0^{1,2}(\Omega)$, when $\varepsilon > 0$ is chosen so small that $\frac{\lambda}{\mu} + c_0 - \varepsilon > 0$. This is possible, since in the proof of Lemma 2.4 we have $c_0 > -\frac{\lambda}{\mu}$, or equivalently, $\frac{\lambda}{\mu} + c_0 > 0$. □

Remark 2.34. From the proof we see that

$$I(v) \geqslant \frac{\lambda}{2}\int_\Omega |Dv|^2\,dx + \frac{c_0 - \varepsilon}{2}\int_\Omega v^2\,dx - \frac{1}{2\varepsilon}\int_\Omega f^2\,dx$$

which implies that

$$\|v\|^2_{W^{1,2}(\Omega)} = \int_\Omega |v|^2\,dx + \int_\Omega |Dv|^2\,dx \leqslant c_1\|f\|^2_{L^2(\Omega)} + c_2 I(v)$$

for every $v \in W_0^{1,2}(\Omega)$. Here c_1 and c_2 are independent of v. In particular, this shows that

$$\|v\|_{W^{1,2}(\Omega)} \to \infty \Longrightarrow I(v) \to \infty.$$

This property is called coercivity.

Theorem 2.35. There exists a constant c_0 such that the variational integral (2.27) has a minimizer $u \in W_0^{1,2}(\Omega)$ for every $f \in L^2(\Omega)$ when $c \geqslant c_0$.

THE MORAL: By Theorem 2.32, every minimizer is a solution to the Euler-Lagrange equation and Theorem 2.35 gives a variational proof of the existence of a solution to the Dirichlet problem. This approach does not use the Hilbert space structure and, as we shall see, it generalizes to nonlinear PDEs as well.

Proof. $\boxed{(1)}$ By Lemma 2.33, the variational integral $I(v)$ is bounded from below in $W_0^{1,2}(\Omega)$ and hence

$$\inf_{v \in W_0^{1,2}(\Omega)} I(v)$$

is a finite number. The definition of infimum implies that there exists a minimizing sequence $u_k \in W_0^{1,2}(\Omega)$, $k = 1, 2, \dots$, such that

$$\lim_{k \to \infty} I(u_k) = \inf_{v \in W_0^{1,2}(\Omega)} I(v).$$

The existence of the limit $\lim_{k\to\infty} I(u_k)$ implies the sequence $(I(u_k))$ is bounded, that is,

$$|I(u_k)| \leqslant M, \quad k = 1, 2, \dots,$$

for some constant $M < \infty$. By Remark 2.34, we see that

$$\|u_k\|^2_{W^{1,2}(\Omega)} \leqslant c_1 \|f\|^2_{L^2(\Omega)} + c_2 M, \quad k = 1, 2, \dots,$$

which shows that (u_k) is a bounded sequence in $W_0^{1,2}(\Omega)$.

$\boxed{(2)}$ By the sequential weak compactness of $W^{1,2}(\Omega)$ there exists a subsequence (u_{k_l}) and a function u in $W_0^{1,2}(\Omega)$ such that $u_{k_l} \to u$ and $Du_{k_l} \to Du$ weakly in $L^2(\Omega)$ as $l \to \infty$. This implies that

$$\lim_{l \to \infty} \int_\Omega f u_{k_l} \, dx = \int_\Omega f u \, dx.$$

By the ellipticity condition, see Definition 1.7, we have

$$\begin{aligned} &\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i (u_{k_l} - u) D_j (u_{k_l} - u) + c(u_{k_l} - u)^2 \right) dx \\ &\qquad \geqslant \lambda \int_\Omega |D(u_{k_l} - u)|^2 \, dx + \int_\Omega c(u_{k_l} - u)^2 \, dx \geqslant 0 \end{aligned}$$

from which we conclude that

$$\begin{aligned} &\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_{k_l} D_j u_{k_l} + c u_{k_l}^2 \right) dx \\ &\quad \geqslant 2 \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_{k_l} D_j u + c u_{k_l} u \right) dx - \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j u + c u^2 \right) dx. \end{aligned}$$

Since $D_i u_{k_l} \to D_i u$ weakly in $L^2(\Omega)$, $i = 1,\dots,n$, and $a_{ij} D_j u \in L^2(\Omega)$, we obtain

$$\begin{aligned}
&\liminf_{l\to\infty} \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_{k_l} D_j u_{k_l} + c u_{k_l}^2 \right) dx \\
&\geqslant 2 \liminf_{l\to\infty} \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_{k_l} D_j u + c u_{k_l} u \right) dx - \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j u + c u^2 \right) dx \\
&= \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j u + c u^2 \right) dx.
\end{aligned}$$

Thus

$$\begin{aligned}
I(u) &= \frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j u + c u^2 \right) dx - \int_\Omega f u \, dx \\
&\leqslant \frac{1}{2} \liminf_{l\to\infty} \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_{k_l} D_j u_{k_l} + c u_{k_l}^2 - f u_{k_l} \right) dx \\
&= \liminf_{l\to\infty} I(u_{k_l}) \\
&= \lim_{k\to\infty} I(u_k),
\end{aligned}$$

and finally

$$\inf_{v \in W_0^{1,2}(\Omega)} I(v) \leqslant I(u) \leqslant \lim_{k\to\infty} I(u_k) = \inf_{v \in W_0^{1,2}(\Omega)} I(v)$$

from which we conclude that

$$I(u) = \inf_{v \in W_0^{1,2}(\Omega)} I(v).$$

□

Remark 2.36. The proof above is based on the following steps:

(1) Choose a minimizing sequence.

(2) Use coercivity, see Remark 2.34 to show that the minimizing sequence is bounded in the Sobolev space.

(3) Use reflexivity to show that there is a weakly converging subsequence.

(4) Use lower semicontinuity of the variational integral to show that the limit is a minimizer.

(5) Use strict convexity of the variational integral to show uniqueness.

Next we discuss an abstract version of the existence result. Let X be a Banach space. We begin with recalling some definitions.

Definition 2.37.

(1) We say that $x_k \in X$, $k = 1,2,\dots$, converges weakly to $x \in X$ if $x^*(x_k) \to x^*(x)$ as $k \to \infty$ for every $x^* \in X^*$. Here X^* denotes the dual of X.

(2) By the Eberlein-Shmulyan theorem a Banach space is reflexive if and only if every bounded sequence has a weakly converging subsequence.

(3) A function $I : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous, if

$$I(u) \leqslant \liminf_{k\to\infty} I(u_k)$$

whenever $u_k \to u$ weakly in X.

(4) A function $I : X \to \mathbb{R}$ is coercive, if

$$\|u_k\|_X \to \infty \Longrightarrow I(u_k) \to \infty$$

as $k \to \infty$.

(5) A function $I : X \to \mathbb{R}$ is convex, if

$$I((1-t)x + ty) \leqslant (1-t)I(x) + tI(y)$$

for every $x, y \in X$ and $t \in [0,1]$. I is strictly convex if

$$I((1-t)x + ty) < (1-t)I(x) + tI(y)$$

for every $x, y \in X$, $x \neq y$ and $t \in (0,1)$.

Theorem 2.38. Assume that $I : X \to \mathbb{R}$ is a coercive, sequentially weakly lower semicontinuous and strictly convex variational integral on a reflexive Banach space X. Then there exists a unique $u \in X$ such that

$$I(u) = \inf_{v\in X} I(v).$$

Proof. $\boxed{(1)}$ We show that $m = \inf_{v\in X} I(v)$ is finite. Assume, for a contradiction, that it is not, in which case $m = -\infty$. By the definition of infimum, there exists a sequence (u_k) such that $I(u_k) \to -\infty$ as $k \to \infty$. If (u_k) is a bounded sequence in X, by reflexivity, it has a weakly converging subsequence such that $u_{k_l} \to u$ weakly as $l \to \infty$ for some $u \in X$. Since I is sequentially weakly lower semicontinuous, we have

$$I(u) \leqslant \liminf_{l\to\infty} I(u_{k_l}) = -\infty$$

and thus $I(u) = -\infty$, which is a contradiction with the fact that $|I(u)| < \infty$. Thus (u_k) is an unbounded sequence in X and there exists a subsequence (u_{k_l}) such that $\|u_{k_l}\| \to \infty$ as $l \to \infty$. By coercivity, $I(u_{k_l}) \to \infty$ as $l \to \infty$. This is a contradiction with $I(u_k) \to -\infty$ as $k \to \infty$. Thus

$$m = \inf_{v\in X} I(v) > -\infty.$$

$\boxed{(2)}$ Let (u_k) be a minimizing sequence such that $I(u_k) \to m$ as $k \to \infty$. As a converging sequence of real numbers $(I(u_k))$ is bounded. We show that (u_k) is a bounded sequence in X. Assume, for a contradiction, that it is unbounded. Then there exists a subsequence (u_{k_l}) such that $\|u_{k_l}\| \to \infty$ as $l \to \infty$. By coercivity, $I(u_{k_l}) \to \infty$ as $l \to \infty$. This is a contradiction with $I(u_k) \to m < \infty$ as $k \to \infty$.

$\boxed{(3)}$ Since (u_k) is a bounded sequence in X, by reflexivity, it has a weakly converging subsequence (u_{k_l}) such that $\|u_{k_l}\| \to u$ as $l \to \infty$ for some $u \in X$. Since I is sequentially weakly lower semicontinuous, we have

$$I(u) \leqslant \liminf_{k\to\infty} I(u_{k_l}) \leqslant \lim_{k\to\infty} I(u_k) = m$$

and

$$m \leqslant I(u) \leqslant \lim_{k\to\infty} I(u_k) = m.$$

This shows that $I(u) = m$ and that u is a minimizer.

$\boxed{(4)}$ To show that the minimizer is unique assume, for a contradiction, that $u_1 \in X$ and $u_2 \in X$ are minimizers with $u_1 \neq u_2$. We consider

$$u = \frac{1}{2}u_1 + \frac{1}{2}u_2.$$

Since $u_1 \neq u_2$, by strict convexity

$$I(u) = I\left(\frac{1}{2}u_1 + \frac{1}{2}u_2\right) < \frac{1}{2}I(u_1) + \frac{1}{2}I(u_2) = m.$$

Thus $I(u) < m$ and this is a contradiction with the fact that u_1 is a minimizer. A similar argument applies for u_2 as well. Thus $u_1 = u_2$ and the minimizer is unique. □

THE MORAL: The variational approach is based on Banach space techniques. This applies to nonlinear variational integrals as well.

Example 2.39. Let $n = 1$ and $\Omega = (0,1)$ and

$$I(u) = \int_0^1 \left(\frac{1}{2}u(x)^2 + (1-u'(x)^2)^2\right) dx, \quad u \in W_0^{1,4}(\Omega).$$

Claim: This variational problem does not have a solution, that is, there does not exist a function $u \in W_0^{1,4}(\Omega)$ such that

$$I(u) = \inf\left\{I(v) : v \in W_0^{1,4}(\Omega)\right\}.$$

Reason. Let $u \in W_0^{1,4}(\Omega)$. By the Sobolev embedding we may assume that u is continuous. Since the integrand is nonnegative, we have $I(u) \geqslant 0$ for every $u \in W_0^{1,4}(\Omega)$. We show that $I(u) > 0$ for every $u \in W_0^{1,4}(\Omega)$. To see this, we note that if $u(x) = 0$ for every $x \in \Omega$, then $I(u) = 1 > 0$. If u is not identically zero, then there exists $k \in \mathbb{N}$ such that

$$\left|\left\{x \in \Omega : |u(x)| > \tfrac{1}{k}\right\}\right| > 0.$$

Thus

$$\begin{aligned} I(u) &= \int_0^1 \left(\frac{1}{2}u(x)^2 + (1-u'(x)^2)^2\right) dx \geqslant \int_0^1 \frac{1}{2}u(x)^2\, dx \\ &\geqslant \frac{1}{2}\left(\frac{1}{k}\right)^2 \left|\left\{x \in \Omega : |u(x)| > \tfrac{1}{k}\right\}\right| > 0. \end{aligned}$$

Consider a sequence of sawtooth functions $u_j \in W_0^{1,4}(\Omega)$, $j = 1,2,\dots$, such that

$$|u_j(x)| \leqslant \frac{1}{2^j}, \quad j = 1,2,\dots, \quad \text{and} \quad |u_j'(x)| = 1 \quad \text{for almost every} \quad x \in \Omega.$$

Then

$$\begin{aligned} I(u_j) &= \int_0^1 \left(\frac{1}{2} u_j(x)^2 + (1 - u_j'(x)^2)^2 \right) dx = \int_0^1 \frac{1}{2} u_j(x)^2 \, dx \\ &\leqslant \int_0^1 \frac{1}{2} \left(\frac{1}{2^j} \right)^2 dx \to 0 \quad \text{as} \quad j \to \infty. \end{aligned}$$

This implies that

$$m = \inf_{v \in W_0^{1,4}(\Omega)} I(v) = 0.$$

Since $I(u) > 0$ for every $u \in W_0^{1,4}(\Omega)$, there does not exist a function $u \in W_0^{1,4}(\Omega)$ such that $I(u) = 0 = m$. ■

THE MORAL: A minimizer may not exist, if the variational integral is not sequentially weakly lower semicontinuous.

Example 2.40. Let $n = 1$, $\Omega = (-1,1)$, $g : \Omega \to \mathbb{R}$, $g(x) = x$, and

$$I(u) = \int_{-1}^1 u'(x)^2 x^4 \, dx,$$

where $u \in W^{1,2}(\Omega)$ such that $u - g \in W_0^{1,2}(\Omega)$. Again, we may assume that u is continuous and $u(-1) = -1$ and $u(1) = 1$.

Claim: This variational problem does not have a solution, that is, there does not exist a function $u \in W^{1,2}(\Omega)$ with $u - g \in W_0^{1,2}(\Omega)$ such that

$$I(u) = \inf\left\{ I(v) : v \in W^{1,2}(\Omega), v - g \in W_0^{1,2}(\Omega) \right\}.$$

Reason. Let $0 < \varepsilon < 1$ and $u_\varepsilon : \Omega \to \mathbb{R}$,

$$u_\varepsilon(x) = \begin{cases} -1, & x \in [-1, -\varepsilon], \\ \frac{x}{\varepsilon}, & x \in (-\varepsilon, \varepsilon), \\ 1, & x \in (\varepsilon, 1]. \end{cases}$$

Then $u_\varepsilon \in W^{1,2}(\Omega)$ with $u_\varepsilon - g \in W_0^{1,2}(\Omega)$

$$0 \leqslant I(u_\varepsilon) = \int_{-\varepsilon}^{\varepsilon} \left(\frac{1}{\varepsilon} \right)^2 x^4 \, dx = \frac{1}{\varepsilon^2} \frac{2\varepsilon^5}{5} = \frac{2\varepsilon^3}{5} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Since $I(v) \geqslant 0$ for every $v \in W^{1,2}(\Omega)$, we conclude that

$$m = \inf\left\{ I(v) : v \in W^{1,2}(\Omega), v - g \in W_0^{1,2}(\Omega) \right\} = 0.$$

Thus the infimum of the variational integral is zero.

Let $u \in W^{1,2}(\Omega)$ with $u - g \in W_0^{1,2}(\Omega)$. Let $\varphi_j \in C^\infty(\Omega)$, $j = 1,2,\dots$, such that $\varphi_j \to u$ in $W^{1,2}(\Omega)$ and

$$\lim_{j\to\infty} \varphi_j(x) = u(x) \quad \text{for every} \quad x \in \Omega.$$

Since $u(1) - u(-1) = 1 - (-1) = 2$ and u is continuous, there exist $x, y \in \Omega$ such that $u(x) - u(y) \geqslant 1$. Thus

$$\begin{aligned} 1 \leqslant u(x) - u(y) &= \lim_{j\to\infty} (\varphi_j(x) - \varphi_j(y)) \\ &= \lim_{j\to\infty} \int_y^x (\varphi_j)'(t)\,dt \leqslant \lim_{j\to\infty} \int_y^x |(\varphi_j)'(t)|\,dt \leqslant \int_{-1}^1 |u'(t)|\,dt. \end{aligned}$$

This implies that there exists $k \in \mathbb{N}$ such that

$$\left|\left\{x \in \Omega : |u'(x)| > \tfrac{1}{k}\right\}\right| > 0$$

and consequently

$$\begin{aligned} I(u) &= \int_{-1}^1 u'(x)^2 x^4\,dx \geqslant \int_{\{x\in\Omega:|u'(x)|>\frac{1}{k}\}} u'(x)^2 x^4\,dx \\ &= \sum_{j=1}^\infty \int_{\{x\in\Omega:|u'(x)|>\frac{1}{k}\}\cap\{x\in\Omega:2^{-j}\leqslant|x|<2^{-j+1}\}} u'(x)^2 x^4\,dx \\ &\geqslant \sum_{j=1}^\infty \frac{1}{k^2}(2^{-j})^4 \left|\left\{x \in \Omega : |u'(x)| > \tfrac{1}{k}\right\} \cap \left\{x \in \Omega : 2^{-j} \leqslant |x| < 2^{-j+1}\right\}\right| > 0, \end{aligned}$$

since at least one of the terms in the sum is positive. Since $I(u) > 0$ for every $u \in W^{1,2}(\Omega)$, there does not exist a function $u \in W^{1,2}(\Omega)$ such that $I(u) = 0 = m$. ■

We observe that I is not coercive, since

$$\begin{aligned} \|u_\varepsilon\|_{W^{1,2}(\Omega)} \geqslant \|(u_\varepsilon)'\|_{L^2(\Omega)} &= \left(\int_{-\varepsilon}^{\varepsilon} \left(\frac{1}{\varepsilon}\right)^2 dx\right)^{\frac{1}{2}} \\ &= \left(\frac{2\varepsilon}{\varepsilon^2}\right)^{\frac{1}{2}} = \left(\frac{2}{\varepsilon}\right)^{\frac{1}{2}} \to \infty \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

but $I(u_\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that the integrand $F(\xi) = \xi^2 x^4$ is convex.

THE MORAL: A minimizer may not exist, if the variational integral is not coercive.

Remark 2.41. We consider the Dirichlet problem for the Laplace equation in the unit disc in the two-dimensional case. Let $\Omega = B(0,1)$ be the unit disc in $\mathbb{R}^2$ and assume that $g \in C(\partial\Omega)$ is a continuous function on the boundary. The problem is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

This problem can be solved by separation of variables with Fourier series in polar coordinates. Recall that any point in the plane can be uniquely determined by its distance from the origin r and the angle θ that the line segment from the origin to the point forms with the x_1-axis, that is,

$$(x_1, x_2) = (r\cos\theta, r\sin\theta), \quad (x_1, x_2) \in \mathbb{R}^2, \quad 0 < r < \infty, \quad -\pi \leqslant \theta < \pi,$$

where $r^2 = x_1^2 + x_2^2$ and $\tan\theta = \frac{x_2}{x_1}$. In polar coordinates, we have

$$\Omega = \{(r,\theta) : 0 < r < 1, -\pi \leqslant \theta < \pi\} \quad \text{and} \quad \partial\Omega = \{(1,\theta) : -\pi \leqslant \theta < \pi\}.$$

The two-dimensional Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 < r < \infty, \quad -\pi \leqslant \theta < \pi.$$

By separation of variables, we obtain

$$u(r,\theta) = \frac{a_0}{2} + \sum_{j=1}^{\infty} r^j \left(a_j \cos(j\theta) + b_j \sin(j\theta)\right),$$

where a_j and b_j are the Fourier cosine and sine coefficients of g, respectively. If $\sum_{j=1}^{\infty}(|a_j| + |b_j|) < \infty$, the series converges uniformly in $\overline{\Omega}$ and its derivatives converge uniformly on compact subsets of Ω. Thus $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. This shows that u is a classical solution to the Dirichlet problem in the unit disc.

$$\begin{aligned}\int_{B(0,\rho)} |Du|^2\,dx &= \int_0^{2\pi}\int_0^{\rho}\left(|u_r|^2 + \frac{1}{r^2}|u_\theta|^2\right) r\,dr\,d\theta \\ &= \pi\sum_{j=1}^{\infty} j\rho^{2j}(a_j^2 + b_j^2).\end{aligned}$$

If we choose

$$u(r,\theta) = \sum_{j=1}^{\infty} \frac{r^{2j!}}{j^2}\sin(j!\theta),$$

then the boundary function is

$$g(\theta) = u(1,\theta) = \sum_{j=1}^{\infty}\frac{1}{j^2}\sin(j!\theta).$$

In this case

$$\int_{B(0,1)} |Du|^2\,dx = \pi\sum_{j=1}^{\infty}\frac{j!}{j^4} = \infty$$

and thus $u \notin W^{1,2}(\Omega)$.

T H E M O R A L : The classical solution of the Dirichlet problem with continuous boundary values may fail to belong to $W^{1,2}(\Omega)$.

Remark 2.42. Let $n=2$ and $\Omega = B(0,1)\setminus\{0\}$. Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

where $g(x) = 1-|x|$. Note that $g \in W^{1,2}(\Omega)\cap C(\overline{\Omega})$. Then $u : \Omega \to \mathbb{R}$, $u(x) = 0$ is the weak solution with boundary values g, that is, $u - g \in W_0^{1,2}(\Omega)$. Observe that

$$0 = \lim_{x\to 0} u(x) \neq \lim_{x\to 0} g(x) = 1.$$

THE MORAL: The boundary values of a weak solution to a Dirichlet problem are not necessarily attained in the classical sense.

2.5 Uniqueness

Let us briefly the discuss uniqueness question. To this end, we need a useful lemma.

Lemma 2.43. If $u \in W_0^{1,2}(\Omega)$ is a weak solution of (2.1), then

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j v + cuv \right) dx = \int_\Omega f v \, dx$$

for every $v \in W_0^{1,2}(\Omega)$.

THE MORAL: The advantage of this result is that we may use $W_0^{1,2}(\Omega)$ functions as test functions in the definition of a weak solution instead of $C_0^\infty(\Omega)$ functions, see Definition 1.18. Especially, we can use a weak solution itself as a test function. The result holds also under the assumption $u \in W^{1,2}(\Omega)$.

Proof. Let $\varphi_k \in C_0^\infty(\Omega)$, $k = 1,2,\dots$, such that $\varphi_k \to v$ in $W^{1,2}(\Omega)$ as $k\to\infty$. Then

$$\begin{aligned} &\left| \int_\Omega a_{ij} D_i u D_j \varphi_k \, dx - \int_\Omega a_{ij} D_i u D_j v \, dx \right| \\ &\qquad = \left| \int_\Omega a_{ij} D_i u (D_j \varphi_k - D_j v) \, dx \right| \\ &\qquad \leqslant \|a_{ij}\|_\infty \|D_i u\|_{L^2(\Omega)} \|D_j \varphi_k - D_j v\|_{L^2(\Omega)} \to 0, \end{aligned}$$

as $k\to\infty$, $i,j = 1,\dots,n$. Thus

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j v \, dx = \lim_{k\to\infty} \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi_k \, dx.$$

Similar arguments show that

$$\int_\Omega cuv \, dx = \lim_{k\to\infty} \int_\Omega cu\varphi_k \, dx$$

and

$$\int_\Omega f v\,dx = \lim_{k\to\infty}\int_\Omega f\varphi_k\,dx.$$

By the definition of a weak solution, see Definition 1.18, we have

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi_k + cu\varphi_k\right) dx = \int_\Omega f\varphi_k\,dx$$

for every $k = 1,2,\dots$, since $\varphi_k \in C_0^\infty(\Omega)$. This implies that

$$\begin{aligned}\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j v + cuv\right) dx &= \lim_{k\to\infty}\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi_k + cu\varphi_k\right) dx\\ &= \lim_{k\to\infty}\int_\Omega f\varphi_k\,dx = \int_\Omega f v\,dx.\end{aligned}$$

□

Theorem 2.44. The solution of (2.1) is unique, provided $c \geqslant c_0$, where c_0 is as in the proof of Lemma 2.4.

Proof. Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be weak solutions. By Lemma 2.43

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_1 D_j v + cu_1 v\right) dx = \int_\Omega f v\,dx$$

and

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u_2 D_j v + cu_2 v\right) dx = \int_\Omega f v\,dx$$

for every $v \in W_0^{1,2}(\Omega)$. By subtracting the equations from each other and choosing $v = u_1 - u_2 \in W_0^{1,2}(\Omega)$, we have

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(D_i u_1 - D_i u_2)(D_j u_1 - D_j u_2) + c(u_1-u_2)(u_1-u_2)\right) dx = 0$$

With the ellipticity property, see Definition 1.7, this implies that

$$\lambda\int_\Omega |Du_1 - Du_2|^2\,dx + \int_\Omega c(u_1-u_2)^2\,dx \leqslant 0.$$

By using the fact that $c \geqslant -\frac{\lambda}{2\mu}$ and the Poincaré inequality, as in the proof of Lemma 2.4, we have

$$\int_\Omega c(u_1-u_2)^2\,dx \geqslant -\frac{\lambda}{2\mu}\int_\Omega (u_1-u_2)^2\,dx \geqslant -\frac{\lambda}{2}\int_\Omega |Du_1 - Du_2|^2\,dx.$$

By combining these estimates, we conclude that

$$-\frac{\lambda}{2}\int_\Omega |Du_1 - Du_2|^2\,dx \leqslant \int_\Omega c(u_1-u_2)^2\,dx \leqslant -\lambda\int_\Omega |Du_1 - Du_2|^2\,dx.$$

Thus

$$\int_\Omega |Du_1 - Du_2|^2\,dx = 0 \quad\text{and}\quad \int_\Omega c(u_1-u_2)^2\,dx = 0.$$

This implies $u_1 = u_2$ almost everywhere in Ω. □

2.6 Comparison and maximum principles

In this section we show that the same technique as in the proof of uniqueness gives certain versions of comparison and maximum principles.

Theorem 2.45 (Comparison principle). Assume that $u, w \in W^{1,2}(\Omega)$ are weak solutions of (2.1) and $c \geqslant c_0$. If $(u-w)_+ \in W_0^{1,2}(\Omega)$, then $u \leqslant w$ in Ω.

THE MORAL: The assumption $(u-w)_+ \in W_0^{1,2}(\Omega)$ means that $u \leqslant w$ on $\partial\Omega$ in Sobolev space sense. Thus the comparison principle asserts that if a solution is above another on the boundary, then it is above also inside the domain.

Proof. The idea is the same as in the proof of the uniqueness. By Lemma 2.43

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv \right) dx = \int_\Omega f v\, dx$$

and

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_j w D_i v + cwv \right) dx = \int_\Omega f v\, dx$$

for every $v \in W_0^{1,2}(\Omega)$. By subtracting the equations from each other we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j(u-w) D_i v + c(u-w)v\, dx = 0.$$

We choose $v = (u-w)_+ \in W_0^{1,2}(\Omega)$ and obtain

$$\begin{aligned} 0 &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_j(u-w) D_i(u-w)_+ + c(u-w)_+^2 \right) dx \\ &\geqslant \int_\Omega \lambda |D(u-w)_+|^2 + c(u-w)_+^2\, dx. \end{aligned}$$

Since $c \geqslant -\frac{\lambda}{2\mu}$ and by the Poincaré inequality, see Remark 1.38, we have

$$\begin{aligned} 0 &\geqslant \int_\Omega \lambda |D(u-w)_+|^2 + c(u-w)_+^2\, dx \\ &\geqslant \int_\Omega \lambda |D(u-w)_+|^2 - \frac{\lambda}{2\mu}(u-w)_+^2\, dx \\ &\geqslant \int_\Omega \lambda |D(u-w)_+|^2 d dx - \frac{\lambda}{2} \int_\Omega |D(u-w)_+|^2\, dx \\ &= \frac{\lambda}{2} \int_\Omega |D(u-w)_+|^2\, dx. \end{aligned}$$

By the Poincaré inequality, we have

$$0 \leqslant \int_\Omega |(u-w)_+|^2\, dx \leqslant \mu \int_\Omega |D(u-w)_+|^2\, dx \leqslant 0.$$

This implies that $(u-w)_+ = 0$ almost everywhere in Ω, that is, $u \leqslant w$ almost everywhere in Ω. $\square$

Remark 2.46. The proof above shows that if $u, w \in W^{1,2}(\Omega)$ are sub- and supersolutions respectively, that is,

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv\, dx \leqslant \int_\Omega f v\, dx$$

and

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j w D_i v + cwv\, dx \geqslant \int_\Omega f v\, dx$$

for every $v \in W_0^{1,2}(\Omega)$ with $v \geqslant 0$, and $(u-w)_+ \in W_0^{1,2}(\Omega)$, then $u \leqslant w$ in Ω.

Theorem 2.47 (Weak maximum principle). Let $u \in W^{1,2}(\Omega)$ be a weak solution of (2.1) with $f = 0$ and $c \geqslant 0$. Then

$$\operatorname*{ess\,sup}_{\Omega} u \leqslant \sup_{\partial\Omega} u_+.$$

THE MORAL: The maximum principle asserts, roughly speaking, that a solution attains its maximum on the boundary of the domain. More precisely, a solution cannot attain a strict maximum inside the domain.

Proof. Set $M = \sup_{\partial\Omega} u_+ \geqslant 0$. Then $(u-M)_+ \in W_0^{1,2}(\Omega)$. To see this, choose a decreasing sequence $l_i \to M$ so that $(u - l_i)_+ = (u_+ - l_i)_+ \in W_0^{1,2}(\Omega)$. Since Ω is bounded, it follows that $u - l_i \to u - M \quad \text{in } W^{1,2}(\Omega)$. This implies

$$(u - l_i)_+ \to (u-M)_+ \quad \text{in } W^{1,2}(\Omega)$$

and thus $(u-M)_+ \in W_0^{1,2}(\Omega)$.

We use $v = (u-M)_+$ as a test function and obtain

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv \right) dx = 0$$

and the constant function M is a weak supersolution, that is,

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_j M D_i v + cMv \right) dx = \int_\Omega cMv\, dx \geqslant 0.$$

Here we used $M, c, v \geqslant 0$. We subtract these from each other and conclude that

$$\lambda \int_\Omega |D(u-M)_+|^2 + c(u-M)_+^2\, dx \leqslant 0.$$

From this it follows that $u \leqslant M$ almost everywhere in Ω. □

3 Higher order regularity

In the previous chapter, we proved existence of a solution by weakening the definition of a solution. In this chapter we study the regularity of weak solutions: are weak solutions of the PDE

$$Lu = f \quad \text{in } \Omega,$$

where L is as in (1.4), smoother than $W^{1,2}_{\text{loc}}(\Omega)$ under suitable assumptions on the coefficients and on the source term f? Are they classical solutions to the problem? Example 1.5 shows that this is not true under the L^∞-assumption on the coefficients, so that additional assumptions have to be imposed. Our main result shows that if the coefficients are smooth and the source term f is smooth, then the solution is smooth.

3.1 Poisson equation

Consider the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \mathbb{R}^n.$$

A formal computation using the integration by parts shows that

$$\begin{aligned}
\int_{\mathbb{R}^n} f^2\,dx &= \int_{\mathbb{R}^n} (\Delta u)^2\,dx = \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}\right)^2 dx \\
&= \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}\right)\left(\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}\right) dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_j^2}\,dx \\
&= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^3 u}{\partial x_i^2 \partial x_j}\frac{\partial u}{\partial x_j}\,dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\,dx \\
&= \int_{\mathbb{R}^n} \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 dx = \int_{\mathbb{R}^n} |D^2 u|^2\,dx,
\end{aligned}$$

where

$$D^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{bmatrix}$$

is the matrix of the second derivatives and

$$|D^2 u|^2 = \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2.$$

T H E M O R A L : This formal argument suggests that the second derivative of a solution to the Poisson equation $-\Delta u = f$ belongs to $L^2(\mathbb{R}^n)$ if $f \in L^2(\mathbb{R}^n)$.

The argument above can be localized. Let $B(x,2r)$ be a ball in $\mathbb{R}^n$ and let $\eta \in C_0^\infty(B(x,2r)$ be a cutoff function with $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $B(x,r)$. Let $v = \eta u$. Then

$$\begin{aligned}
\int_{B(x,r)} |D^2 u|^2\,dy &= \int_{B(x,r)} \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 dy \\
&\leqslant \int_{B(x,2r)} \sum_{i,j=1}^n \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 dy \\
&= -\int_{B(x,2r)} (\Delta v)^2\,dy.
\end{aligned}$$

Since

$$\Delta v = \Delta(\eta u) = \eta \Delta u + 2D\eta \cdot Du + u\Delta\eta,$$

by applying the inequality $(a+b+c)^p \leqslant 3^p(a^p+b^p+c^p)$, $a,b,c \geqslant 0$, $p \geqslant 1$, we have

$$\begin{aligned}
\int_{B(x,2r)} (\Delta v)^2\, dy &= \int_{B(x,2r)} (\eta\Delta u + 2D\eta\cdot Du + u\Delta\eta)^2\, dy \\
&\leqslant \int_{B(x,2r)} (|\eta\Delta u| + 2|D\eta||Du| + |u||\Delta\eta|)^2\, dy \\
&\leqslant 3^2\left(\int_{B(x,2r)} |\eta\Delta u|^2\, dy + \int_{B(x,2r)} (2|D\eta||Du|)^2\, dy + \int_{B(x,2r)} |u\Delta\eta|^2\, dy\right) \\
&\leqslant c\int_{B(x,2r)} \left(|u|^2 + |\Delta u|^2 + |D\eta|^2|Du|^2\right) dy,
\end{aligned}$$

with $c = 36\sup_{B(x,2r)}(\eta^2 + |\Delta\eta|^2)$. Integrating by parts twice and applying the inequality $2ab \leqslant a^2+b^2$ we have

$$\begin{aligned}
\int_{B(x,2r)} |D\eta|^2|Du|^2\, dy &= -\int_{B(x,2r)} |D\eta|^2 u\Delta u\, dy + \frac{1}{2}\int_{B(x,2r)} u^2\Delta(|D\eta|^2)\, dy \\
&\leqslant c\int_{B(x,2r)} \left(u^2 + (\Delta u)^2\right) dy,
\end{aligned}$$

with $c = \sup_{B(x,2r)}(|D\eta|^2 + \Delta(|D\eta|^2))$. It follows that

$$\begin{aligned}
\int_{B(x,r)} |D^2u|^2\, dy &\leqslant c\int_{B(x,2r)} \left(u^2 + (\Delta u)^2\right) dy \\
&= c\int_{B(x,2r)} \left(u^2 + f^2\right) dy.
\end{aligned}$$

For the gradient we have

$$\int_{B(x,r)} |Du|^2\, dy \leqslant \int_{B(x,2r)} |D(\eta u)|^2\, dy = \int_{B(x,2r)} D(\eta u)\cdot D(\eta u)\, dy.$$

We note that

$$D(\eta u)\cdot D(\eta u) = u^2 D\eta\cdot D\eta + D(\eta^2 u)\cdot Du$$

and obtain

$$\int_{B(x,r)} |Du|^2\, dy \leqslant \int_{B(x,2r)} u^2 D\eta\cdot D\eta\, dy + \int_{B(x,2r)} Du\cdot D(\eta^2 u)\, dy.$$

Since $-\Delta u = f$, we have

$$\int_{B(x,2r)} Du\cdot D(\eta^2 u)\, dy = \int_{B(x,2r)} \eta^2 u f\, dy,$$

which implies that

$$\begin{aligned}
\int_{B(x,r)} |Du|^2\, dy &\leqslant \int_{B(x,2r)} |u|^2|D\eta|^2\, dy + \int_{B(x,2r)} \eta^2|u||f|\, dy \\
&\leqslant \int_{B(x,2r)} |u|^2|D\eta|^2\, dy + \frac{1}{2}\int_{B(x,2r)} |u|^2\, dy + \frac{1}{2}\int_{B(x,2r)} |f|^2\, dy \\
&\leqslant c\int_{B(x,2r)} \left(|u|^2 + |f|^2\right) dy,
\end{aligned}$$

with $c = \sup_{B(x,2r)} |D\eta|^2 + \frac{1}{2}$.

By combining the estimates above, we have

$$\begin{aligned}\|u\|_{W^{2,2}(B(x,r))} &= \|u\|_{L^2(B(x,r))} + \|Du\|_{L^2(B(x,r))} + \|D^2 u\|_{L^2(B(x,r))}\\ &\leqslant c\left(\|u\|_{L^2(B(x,r))} + \|f\|_{L^2(B(x,r))}\right),\end{aligned}$$

where c only depends on the radius r.

T H E M O R A L : This formal argument suggests that a solution to the Poisson equation $-\Delta u = f$ belongs to $W^{2,2}_{\text{loc}}(\mathbb{R}^n)$ if $f \in L^2_{\text{loc}}(\mathbb{R}^n)$.

Next we apply these estimates recursively. This is called a bootstrap argument.

$\boxed{\text{Step 1}}$ By the previous computation, the L^2-norm of the second derivatives of u can be estimated by the L^2-norm of f.

$\boxed{\text{Step 2}}$ By differentiating the PDE, we have

$$-\Delta\left(\frac{\partial u}{\partial x_k}\right) = -\frac{\partial}{\partial x_k}(\Delta u) = \frac{\partial f}{\partial x_k}, \quad k = 1, \ldots, n,$$

that is,

$$-\Delta \overline{u} = \overline{f},$$

where

$$\overline{u} = \frac{\partial u}{\partial x_k} \quad \text{and} \quad \overline{f} = \frac{\partial f}{\partial x_k}, \quad k = 1, \ldots, n.$$

Thus the partial derivatives satisfy a similar PDE. By the same method as in Step 1, we can estimate the L^2-norm of the third derivatives of u by the first derivatives of f.

$\boxed{\text{Step 3}}$ Continuing this way, we see that the L^2-norm of the $(m+2)^{nd}$ derivatives of u can be controlled by the L^2-norm of the m^{th} derivatives of f for $m = 0, 1, 2, \ldots$. In particular, if $f \in C_0^\infty(\mathbb{R}^n)$, then $u \in W^{m,2}(\mathbb{R}^n)$ for every $m = 1, 2, \ldots$, and thus $u \in C^\infty(\mathbb{R}^n)$.

T H E M O R A L : This formal argument suggests that u has two more derivatives than f.

Observe, however, that we assumed that u is smooth in the iterative process above, and thus it is not really a proof for smoothness. Next we want to make this heuristic idea more precise. There are two standard approaches to the higher regularity theory:

(1) Schauder estimates $f \in C^{0,\alpha}(\Omega) \Longrightarrow u \in C^{2,\alpha}(\Omega)$, $0 < \alpha < 1$, and

(2) Calderón-Zygmund estimates $f \in L^2(\Omega) \Longrightarrow u \in W^{2,2}(\Omega)$.

We shall focus on the Calderón-Zygmund estimates. For the Schauder estimates, we refer to [1, Chapter 2], [4, Chapter 2], [6, Chapter 3],[7, Chapter 5], [8, Chapter 6], [11, Chapter 3] and [16, Chapter 6]. First we give three examples which show that the claim “u has two derivatives more than f” is not always true.

Examples 3.1:

(1) Let $u : \mathbb{R}^2 \to \mathbb{R}$,

$$u(x,y) = (x^2 - y^2)\log(x^2 + y^2), \quad (x,y) \neq (0,0).$$

Then

$$\frac{\partial^2 u}{\partial x^2}(x,y) = 2\log(x^2+y^2) + \frac{8x^2}{x^2+y^2} - 2\left(\frac{x^2-y^2}{x^2+y^2}\right), \quad (x,y) \neq (0,0),$$

and

$$\frac{\partial^2 u}{\partial y^2}(x,y) = -2\log(x^2+y^2) - \frac{8y^2}{x^2+y^2} + 2\left(\frac{x^2-y^2}{x^2+y^2}\right), \quad (x,y) \neq (0,0).$$

Thus

$$\frac{\partial^2 u}{\partial x^2} \notin L^\infty(\mathbb{R}^2) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \notin L^\infty(\mathbb{R}^2)$$

so that $u \notin W^{2,\infty}(\mathbb{R}^2)$. However, we have

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 8\left(\frac{x^2-y^2}{x^2+y^2}\right), \quad (x,y) \neq (0,0)$$

and $f = \Delta u \in L^\infty(\mathbb{R}^2)$. Thus $f \in L^\infty(\mathbb{R}^2)$ does not necessarily imply that $u \in W^{2,\infty}(\mathbb{R}^2)$.

(2) Let $u : \mathbb{R}^2 \to \mathbb{R}$,

$$u(x,y) = (x^2 - y^2)\log|\log(x^2+y^2)|, \quad (x,y) \neq (0,0).$$

Then $u \notin C^{1,1}(\mathbb{R}^2)$ and $f = \Delta u \in C(\mathbb{R}^2)$. Thus $f \in C(\mathbb{R}^2)$ does not necessarily imply that $u \in C^2(\mathbb{R}^2)$.

(3) Let $u : \mathbb{R}^2 \to \mathbb{R}$,

$$u(x,y) = \log\log\frac{1}{x^2+y^2}, \quad (x,y) \neq (0,0).$$

In polar coordinates with $r^2 = x^2 + y^2$, we have

$$u(r) = \log\log\frac{1}{r}, \quad u_r(r) = \frac{1}{r\log r}, \quad r \neq 0,$$

and

$$\Delta u(r) = u_{rr}(r)\frac{1}{r}u_r = -\frac{\log r + 1}{r^2(\log r)^2} + \frac{1}{r^2\log r} = -\frac{1}{r^2(\log r)^2}, \quad r \neq 0.$$

Then

$$\int_{B(0,\frac{1}{2})} \Delta u\, dx = -2\pi \int_0^{\frac{1}{2}} \frac{1}{r\log r}\, dr < \infty$$

and thus $\Delta u \in L^1(B(0,\frac{1}{2}))$. However, we have

$$\frac{\partial^2 u}{\partial x^2} \notin L^1(B(0,\tfrac{1}{2})) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \notin L^1(B(0,\tfrac{1}{2}))$$

so that $u \notin W^{2,1}(B(0,\frac{1}{2}))$. The correspoding example in $\mathbb{R}^n$, $n \geqslant 2$, can be constructed by considering a function $u(r)$ with $u_r(r) = \frac{r^{1-n}}{\log r}$.

3.2 Difference quotients

The proof of the main result of this section uses difference quotients and thus this approach is called the difference quotient method. We recall the definition and basic properties of difference quotients.

Definition 3.2. Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\Omega' \Subset \Omega$. The k^{th} difference quotient is

$$D^h_k f(x) = \frac{f(x+he_k)-f(x)}{h}, \quad k = 1,\dots,n,$$

for $x \in \Omega'$ and $h \in \mathbb{R}$ such that $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. We denote

$$D^h f = (D^h_1 f, \dots, D^h_n f).$$

THE MORAL: Note that the definition of the difference quotient makes sense at every $x \in \Omega$ whenever $0 < |h| < \operatorname{dist}(x, \partial\Omega)$. If $\Omega = \mathbb{R}^n$, then the definition makes sense for every $h \neq 0$.

The following properties of the difference quotients follow directly from the definition.

Lemma 3.3.

(1) If $f, g \in L^2(\mathbb{R}^n)$ are compactly supported functions, then

$$\int_{\mathbb{R}^n} f(x) D^h_k g(x)\,dx = -\int_{\mathbb{R}^n} g(x) D^{-h}_k f(x)\,dx, \quad k = 1,\dots,n.$$

(2) If f has weak partial derivatives $D_i f$, $i = 1,\dots,n$, then

$$D_i D^h_k f = D^h_k D_i f, \quad i,k = 1,2,\dots,n.$$

(3) If $f, g \in L^2(\mathbb{R}^n)$, then

$$D^h_k(fg) = g^h(x) D^h_k f(x) + f(x) D^h_k g(x),$$

where $g^h(x) = g(x+he_k)$.

Proof. $\boxed{(1)}$

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x) D^h_k g(x)\,dx &= \int_{\mathbb{R}^n} f(x) \frac{g(x+he_k)-g(x)}{h}\,dx \\
&= \int_{\mathbb{R}^n} \frac{g(x+he_k) f(x)}{h}\,dx - \int_{\mathbb{R}^n} \frac{g(x) f(x)}{h}\,dx \\
&= \int_{\mathbb{R}^n} \frac{g(x) f(x-he_k)}{h}\,dx - \int_{\mathbb{R}^n} \frac{g(x) f(x)}{h}\,dx \\
&= -\int_{\mathbb{R}^n} g(x) \frac{f(x-he_k)-f(x)}{(-h)}\,dx.
\end{aligned}$$

$\boxed{(2)}$ Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} D_k^h f \frac{\partial\varphi}{\partial x_i}\,dx &= \int_{\mathbb{R}^n} \frac{f(x+he_k)-f(x)}{h}\frac{\partial\varphi}{\partial x_i}(x)\,dx \\
&= \frac{1}{h}\left(\int_{\mathbb{R}^n} f(x+he_k)\frac{\partial\varphi}{\partial x_i}(x)\,dx - \int_{\mathbb{R}^n} f(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx\right) \\
&= -\frac{1}{h}\left(\int_{\mathbb{R}^n} D_i f(x+he_k)\varphi(x)\,dx - \int_{\mathbb{R}^n} D_i f(x)\varphi(x)\,dx\right) \\
&= -\int_{\mathbb{R}^n} \frac{D_i f(x+he_k)-D_i f(x)}{h}\varphi(x)\,dx \\
&= -\int_{\mathbb{R}^n} D_k^h D_i f\varphi(x)\,dx.
\end{aligned}$$

$\boxed{(3)}$

$$\begin{aligned}
D_k^h(fg) &= \frac{f(x+he_k)g(x+he_k)-f(x)g(x)}{h} \\
&= \frac{1}{h}\big((f(x+he_k)g(x+he_k)-f(x)g(x+he_k)) \\
&\qquad +(f(x)g(x+he_k)-f(x)g(x))\big) \\
&= g(x+he_k)\frac{f(x+he_k)-f(x)}{h}+f(x)\frac{g(x+he_k)-g(x)}{h} \\
&= g^h(x)D_k^h f(x)+f(x)D_k^h g(x).
\end{aligned}$$

□

We recall a characterization of the Sobolev spaces by integrated difference quotients.

Theorem 3.4.

(1) Assume $u \in W^{1,p}(\Omega)$, $1 \leqslant p < \infty$. Then for every $\Omega' \Subset \Omega$, we have

$$\|D^h u\|_{L^p(\Omega')} \leqslant c\|Du\|_{L^p(\Omega)}$$

for some constant $c = c(n,p)$ and all $0 < |h| < \operatorname{dist}(\Omega',\partial\Omega)$.

(2) If $u \in L^p(\Omega')$, $1 < p < \infty$, and there is a constant c such that

$$\|D^h u\|_{L^p(\Omega')} \leqslant c$$

for all $0 < |h| < \operatorname{dist}(\Omega',\partial\Omega)$, then $u \in W^{1,p}(\Omega')$ and

$$\|Du\|_{L^p(\Omega')} \leqslant c.$$

Proof. See Sobolev spaces. □

3.3 Difference quotient method

We assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and we consider a PDE of the type

$$Lu = -\sum_{i,j=1}^n D_j(a_{ij}D_i u) + \sum_{i=1}^n b_i D_i u + cu = f,$$

see (1.4). We continue to require the uniform ellipticity condition, see Definition 1.7 and we will make additional assumptions about the smoothness of the coefficients a_{ij}, b_i and c.

Theorem 3.5 (Second order interior estimate). Assume that

$$a_{ij} \in C^1(\Omega), b_i, c \in L^\infty(\Omega), \quad i,j = 1,\dots,n, \quad \text{and} \quad f \in L^2(\Omega).$$

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, where L is as in (1.4). Then $u \in W^{2,2}_{\text{loc}}(\Omega)$ and for every $\Omega' \Subset \Omega$, we have

$$\|u\|_{W^{2,2}(\Omega')} \leqslant c\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where the constant c depends only on Ω', Ω and the coefficients of L.

T H E M O R A L : This regularity result asserts that the weak solution that assumed to belong to $W^{1,2}(\Omega)$ is more regular and belongs to $W^{2,2}_{\text{loc}}(\Omega)$, if the coefficients $a_{ij} \in C^1(\Omega)$. In addition, this result comes with an estimate. Example 1.5 shows that this cannot hold under the assumption $a_{ij} \in L^\infty(\Omega)$. Note that no boundary conditions are assumed, so that this regularity result applies to PDEs with Dirichlet, Neumann or other boundary conditions.

Remarks 3.6:

(1) Note that we do not require $u \in W^{1,2}_0(\Omega)$, that is, we are not assuming that $u = 0$ on $\partial\Omega$ in the Sobolev sense.

(2) The claim $u \in W^{2,2}_{\text{loc}}(\Omega)$ implies that u actually solves the PDE almost everywhere in Ω, that is,

$$Lu(x) = f(x) \quad \text{for almost every} \quad x \in \Omega.$$

Reason. By the definition of the second order weak derivative gives

$$\begin{aligned}\int_\Omega f\varphi\,dx &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi\right) dx \\ &= \int_\Omega \left(-\sum_{i,j=1}^n D_j(a_{ij} D_i u)\varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi\right) dx\end{aligned}$$

and consequently

$$\int_\Omega \left(-\sum_{i,j=1}^n D_j(a_{ij} D_i u) + \sum_{i=1}^n b_i D_i u + cu - f\right)\varphi\,dx = \int_\Omega (Lu - f)\varphi\,dx = 0$$

for every $\varphi \in C^\infty_0(\Omega)$. This implies that $Lu - f = 0$ almost everywhere in Ω. ■

Proof. $\boxed{(1)}$ Choose Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$. Let $\eta \in C_0^\infty(\Omega'')$ be a cutoff function such that $\eta = 1$ in Ω' and $0 \leqslant \eta \leqslant 1$.

$\boxed{(2)}$ Let u be a weak solution of $Lu = f$ in Ω. Then by Lemma 2.43,

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u D_j v \, dx = \int_\Omega \overline{f} v \, dx \tag{3.7}$$

for every $v \in W_0^{1,2}(\Omega)$, where

$$\overline{f} = f - \sum_{i=1}^{n} b_i D_i u - cu.$$

We point out that Lemma 2.43 holds also without assumption that $b_i = 0$, $i = 1, \dots, n$ (exercise).

$\boxed{(3)}$ Use

$$v = -D_k^{-h}(\eta^2 D_k^h u), \quad k = 1, \dots, n,$$

as a test function in (3.7), where

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}$$

is a difference quotient with $|h| > 0$ small enough. Observe that $v \in W_0^{1,2}(\Omega)$ for small enough $|h| > 0$. We write the resulting expression as $A = B$ for

$$A = \sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u D_j v \, dx \quad \text{and} \quad B = \int_\Omega \overline{f} v \, dx.$$

$\boxed{(4)}$ For A we have

$$\begin{aligned}
A &= -\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u D_j \underbrace{\left(D_k^{-h}(\eta^2 D_k^h u)\right)}_{=D_k^{-h}(D_j(\eta^2 D_k^h u))} dx && \text{(Lemma 3.3 (2))} \\
&= \sum_{i,j=1}^{n} \int_\Omega D_k^h(a_{ij} D_i u) D_j(\eta^2 D_k^h u) \, dx && \text{(Lemma 3.3 (1))} \\
&= \sum_{i,j=1}^{n} \int_\Omega \Big(a_{ij}^h D_k^h(D_i u) D_j(\eta^2 D_k^h u) \\
&\qquad + (D_k^h a_{ij}) D_i u D_j(\eta^2 D_k^h u) \Big) \, dx && \text{(Lemma 3.3 (3))} \\
&= \sum_{i,j=1}^{n} \int_\Omega a_{ij}^h D_k^h(D_i u) D_k^h(D_j u) \eta^2 \, dx \\
&\quad + \sum_{i,j=1}^{n} \int_\Omega \Big(a_{ij}^h D_k^h(D_i u) D_k^h u 2\eta D_j \eta && \text{(Leibniz)} \\
&\qquad + (D_k^h a_{ij}) D_i u D_k^h(D_j u) \eta^2 + (D_k^h a_{ij}) D_i u D_k^h u 2\eta D_j \eta \Big) \, dx \\
&= A_1 + A_2.
\end{aligned}$$

Recall that $a_{ij}^h(x) = a_{ij}(x + he_k)$. The uniform ellipticity (see Definition 1.7) implies that

$$A_1 = \sum_{i,j=1}^{n} \int_\Omega a_{ij}^h D_k^h(D_i u) D_k^h(D_j u)\eta^2\,dx \geqslant \lambda \int_\Omega \eta^2 |D_k^h Du|^2\,dx.$$

On the other hand, by using the properties $a_{ij} \in L^\infty(\Omega)$, $D_k^h a_{ij} \in L^\infty(\Omega)$, $\eta^2 \leqslant \eta$ in Ω and Young's inequality with ε, see Corollary 1.52, with $p = 2$, we have

$$\begin{aligned}
|A_2| &\leqslant c \int_\Omega \left(\eta |D_k^h Du||D_k^h u| + \eta |D_k^h Du||Du| + \eta |D_k^h u||Du|\right) dx \\
&\leqslant c\varepsilon \int_\Omega \eta^2 |D_k^h Du|^2\,dx + c(\varepsilon) \int_{\Omega''} \left(|D_k^h u|^2 + |Du|^2\right) dx \\
&\leqslant c\varepsilon \int_\Omega \eta^2 |D_k^h Du|^2\,dx + c(\varepsilon) \int_\Omega |Du|^2\,dx.
\end{aligned}$$

In the last inequality we used the fact that

$$\|D_k^h u\|_{L^2(\Omega'')} \leqslant c\|Du\|_{L^2(\Omega)}, \quad k = 1, \dots, n,$$

for some constant $c = c(n, p)$ and all $0 < |h| < \operatorname{dist}(\Omega'', \partial\Omega)$, see Theorem 3.4 (1). By choosing $\varepsilon > 0$ so that $c\varepsilon = \frac{\lambda}{2}$, we have

$$|A_2| \leqslant \frac{\lambda}{2} \int_\Omega \eta^2 |D_k^h Du|^2\,dx + c \int_\Omega |Du|^2\,dx.$$

This gives the lower bound

$$\begin{aligned}
A &\geqslant A_1 - |A_2| \\
&\geqslant \lambda \int_\Omega \eta^2 |D_k^h Du|^2\,dx - \frac{\lambda}{2} \int_\Omega \eta^2 |D_k^h Du|^2\,dx - c \int_\Omega |Du|^2\,dx \\
&= \frac{\lambda}{2} \int_\Omega \eta^2 |D_k^h Du|^2\,dx - c \int_\Omega |Du|^2\,dx.
\end{aligned}$$

$\boxed{(5)}$ We estimate B by using Young's inequality with ε, see Corollary 1.52, and obtain

$$\begin{aligned}
|B| &\leqslant \int_\Omega |\overline{f}||v|\,dx = \int_\Omega \left| f - \sum_{i=1}^{n} b_i D_i u - cu \right| |v|\,dx \\
&\leqslant c \int_\Omega (|f| + |Du| + |u|)|v|\,dx \\
&\leqslant c\varepsilon \int_\Omega |v|^2\,dx + c(\varepsilon) \int_\Omega (|f| + |Du| + |u|)^2\,dx \\
&\leqslant c\varepsilon \int_\Omega |v|^2\,dx + c(\varepsilon) \int_\Omega (|f|^2 + |u|^2 + |Du|^2)\,dx,
\end{aligned}$$

where

$$\begin{aligned}
\int_{\Omega} |v|^2\,dx &= \int_{\Omega''} |v|^2\,dx = \int_{\Omega''} |D_k^{-h}(\eta^2 D_k^h u)|^2\,dx \\
&\leqslant c\int_{\Omega} |D(\eta^2 D_k^h u)|^2\,dx = c\int_{\Omega''} |D(\eta^2 D_k^h u)|^2\,dx \\
&\leqslant c\int_{\Omega''} \left(2\eta|D\eta||D_k^h u| + \eta^2|D(D_k^h u)|\right)^2\,dx \qquad \text{(Leibniz)} \\
&\leqslant c\int_{\Omega''} \eta^2|D\eta|^2|D_k^h u|^2\,dx + c\int_{\Omega''} \eta^4|D_k^h Du|^2\,dx \\
&\leqslant c\int_{\Omega} |Du|^2\,dx + c\int_{\Omega''} \eta^2|D_k^h Du|^2\,dx. \qquad (\eta^4 \leqslant \eta^2)
\end{aligned}$$

Thus

$$|B| \leqslant c\varepsilon\int_{\Omega} \eta^2|D_k^h Du|^2\,dx + c(\varepsilon)\int_{\Omega} (|f|^2 + |u|^2 + |Du|^2)\,dx.$$

By choosing $\varepsilon > 0$ so that $c\varepsilon = \frac{\lambda}{4}$, we obtain

$$|B| \leqslant \frac{\lambda}{4}\int_{\Omega} \eta^2|D_k^h Du|^2\,dx + c\int_{\Omega} (|f|^2 + |u|^2 + |Du|^2)\,dx.$$

$\boxed{(6)}$ A combination of estimates from (4) and (5) gives

$$\begin{aligned}
\frac{\lambda}{2}\int_{\Omega} \eta^2|D_k^h Du|^2\,dx - c\int_{\Omega} |Du|^2\,dx &\leqslant A \\
= B &\leqslant \frac{\lambda}{4}\int_{\Omega} \eta^2|D_k^h Du|^2\,dx + c\int_{\Omega} (|f|^2 + |u|^2 + |Du|^2)\,dx.
\end{aligned}$$

Thus

$$\int_{\Omega'} |D_k^h Du|^2\,dx \leqslant \int_{\Omega} \eta^2|D_k^h Du|^2\,dx \leqslant c\int_{\Omega} (|f|^2 + |u|^2 + |Du|^2)\,dx$$

for $k = 1, \dots, n$ and all sufficiently small $|h| \neq 0$. The characterization of Sobolev spaces by integrated difference quotients, see Theorem 3.4 (2), implies $Du \in W^{1,2}(\Omega')$ and thus $u \in W^{2,2}(\Omega')$ with the estimate

$$\|u\|_{W^{2,2}(\Omega')} \leqslant c\left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).$$

This is almost what we want except that there is the Sobolev norm $\|u\|_{W^{1,2}(\Omega)}$ instead of $\|u\|_{L^2(\Omega)}$ on the right-hand side.

$\boxed{(7)}$ To complete the proof, choose a cutoff function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ on Ω'' and $0 \leqslant \eta \leqslant 1$. By Lemma 2.43 we may apply

$$v = \eta^2 u \in W_0^{1,2}(\Omega)$$

as a test function in (3.7), that is,

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u D_j(\eta^2 u)\,dx = \int_{\Omega} \overline{f}\eta^2 u\,dx.$$

Now

$$\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iuD_j(\eta^2u)\,dx=\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iu\left(\eta^2D_ju+2u\eta D_j\eta\right)dx$$
$$\geqslant\lambda\int_\Omega\eta^2|Du|^2\,dx-\left|\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iu2u\eta D_j\eta\,dx\right|,$$

where

$$\left|\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iu2u\eta D_j\eta\,dx\right|\leqslant c\sum_{i,j=1}^{n}\int_\Omega\eta|D_iu||u|\,dx\leqslant c\int_\Omega\eta|Du||u|\,dx.$$

Thus

$$\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iuD_j(\eta^2u)\,dx\geqslant\lambda\int_\Omega\eta^2|Du|^2\,dx-c\int_\Omega\eta|Du||u|\,dx.$$

On the other hand, we can use Young's inequality with ε to obtain

$$\int_\Omega\overline{f}\eta^2u\,dx\leqslant c\int_\Omega(|f|+|Du|+|u|)\eta^2u\,dx$$
$$\leqslant c\varepsilon\int_\Omega\eta^2|Du|^2\,dx+c(\varepsilon)\int_\Omega|u|^2\,dx+c\int_\Omega|f|^2\,dx.$$

Choosing $\varepsilon>0$ such that $c\varepsilon=\frac{\lambda}{2}$ and combining the previous estimates, we have

$$\int_\Omega\eta^2|Du|^2\,dx\leqslant c\int_\Omega\left(|f|^2+|u|^2\right)dx+c\int_\Omega\eta|u||Du|\,dx,$$

where the last term can again be estimated by Young's inequality as

$$c\int_\Omega\eta|u||Du|\,dx\leqslant c\varepsilon\int_\Omega\eta^2|Du|^2\,dx+c(\varepsilon)\int_\Omega|u|^2\,dx.$$

By choosing $\varepsilon>0$ such that $c\varepsilon=\frac{1}{2}$, we finally have

$$\int_\Omega\eta^2|Du|^2\,dx\leqslant c\int_\Omega(|f|^2+|u|^2)dx+\frac{1}{2}\int_\Omega\eta^2|Du|^2\,dx+c\int_\Omega|u|^2\,dx,$$

which implies

$$\int_{\Omega''}|Du|^2\,dx\leqslant\int_\Omega\eta^2|Du|^2\,dx\leqslant c\int_\Omega(|f|^2+|u|^2)dx. \tag{3.8}$$

$\boxed{8}$ The argument in (6), with Ω replaced by Ω'', combined with (7) gives

$$\|u\|_{W^{2,2}(\Omega')}\leqslant c\left(\|f\|_{L^2(\Omega'')}+\|u\|_{W^{1,2}(\Omega'')}\right).$$
$$\leqslant c\left(\|f\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)}\right).$$

This completes the proof. □

THE MORAL: The proof is based on choosing appropriate test functions. In step (2) we use $v=-D_k^{-h}(\eta^2D_k^hu)$, $k=1,\dots,n$, and in step (7) we use $v=\eta^2u$ as a test function in (3.7). These are the only points in the proof where we use the PDE.

Remark 3.9. The proof of the previous theorem gives an extremely useful energy estimate (Caccioppoli estimate). Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set, $a_{ij}, b_i, c \in L^\infty(\Omega)$, $i, j = 1, \dots, n$ and $f \in L^2(\Omega)$. Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, where L is as in (1.4). Then by (3.8), there exists a constant $c = c(\lambda, \Omega')$ such that

$$\int_{\Omega'} |Du|^2 \, dx \leqslant c \int_\Omega (|f|^2 + |u|^2) \, dx,$$

whenever $\Omega' \Subset \Omega$. Observe, that Poincaré inequality states that

$$\int_\Omega |u|^2 \, dx \leqslant c(\operatorname{diam} \Omega)^2 \int_\Omega |Du|^2 \, dx$$

for every $u \in W_0^{1,2}(\Omega)$. Thus the energy estimate above is a reverse Poincaré inequality.

3.4 A bootstrap argument

Motivation: Our goal is to use Theorem 3.5 recursively provided the coefficients and the right-hand side of the PDE are smooth enough. To this end, we would like to show that weak derivatives of a weak solution are solutions to certain PDE as well. Assume that a_{ij} are constants, $b_i = 0$, $i, j = 1, \dots, n$, $c = 0$ and $f = 0$. Then we have

$$Lu = -\sum_{i,j=1}^n D_j(a_{ij} D_i u) = -\sum_{i,j=1}^n a_{ij} D_j D_i u = 0.$$

Let $\psi \in C_0^\infty(\Omega)$ and choose

$$\varphi = D_k \psi \in C_0^\infty \Omega), \quad k = 1, \dots, n,$$

as a test function in the definition of a weak solution. This gives

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx = 0.$$

Recall that by Theorem 3.5 we have $u \in W^{2,2}_{\text{loc}}(\Omega)$ and thus

$$D_k u \in W^{1,2}_{\text{loc}}(\Omega), \quad k = 1, \dots, n.$$

By the definition of the weak derivative, that is, integration by parts, this gives

$$\begin{aligned}
\sum_{i,j=1}^n \int_\Omega (a_{ij} D_i u)(D_j D_k \psi) \, dx &= \sum_{i,j=1}^n \int_\Omega (a_{ij} D_i u)(D_k D_j \psi) \, dx \\
&= -\sum_{i,j=1}^n \int_\Omega D_k (a_{ij} D_i u) D_j \psi \, dx \\
&= -\sum_{i,j=1}^n \int_\Omega (a_{ij} D_k D_i u) D_j \psi \, dx \\
&= -\int_\Omega \sum_{i,j=1}^n a_{ij} D_i (D_k u) D_j \psi \, dx = 0.
\end{aligned}$$

THE MORAL: This means that $D_k u$, $k = 1,\dots,n$, is a weak solution to the same PDE as u. This procedure can be iterated and used to show smoothness of weak solutions.

Next we extend this argument to more general PDEs.

Theorem 3.10 (Higher order interior estimate). Let m be a nonnegative integer. Assume that

$$a_{ij}, b_i, c \in C^{m+1}(\Omega), \quad i,j = 1,\dots,n, \quad \text{and} \quad f \in W^{m,2}(\Omega).$$

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, where L is as in (1.4). Then $u \in W^{m+2,2}_{\text{loc}}(\Omega)$ and for every $\Omega' \Subset \Omega$, we have

$$\|u\|_{W^{m+2,2}(\Omega')} \leqslant c\left(\|f\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where the constant c depends only on Ω', Ω and the coefficients of L.

THE MORAL: This regularity result asserts that a weak solution belongs locally to a higher order Sobolev space, if the coefficients and the right-hand side of the PDE are smooth enough. In addition, this result comes with an estimate. In this sense u has two more derivatives than f. Thus the degree of regularity can be increased stepwise provided the data is smooth.

Proof. $\boxed{(1)}$ We prove the claim by induction on m. The case $m = 0$ follows from Theorem 3.5.

$\boxed{(2)}$ Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω. Assume that for some nonnegative integer m, we have $u \in W^{m+2,2}_{\text{loc}}(\Omega)$ and

$$\|u\|_{W^{m+2,2}(\Omega')} \leqslant c\left(\|f\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)}\right) \tag{3.11}$$

for every $\Omega' \Subset \Omega$, where the constant c depends only on Ω', Ω and the coefficients of L. We shall show that the claim holds for $m+1$. To this end, assume that

$$a_{ij}, b_i, c \in C^{m+2}(\Omega), \quad i,j = 1,\dots,n, \quad \text{and} \quad f \in W^{m+1,2}(\Omega). \tag{3.12}$$

Recall that by the induction hypothesis we have $u \in W^{m+2,2}_{\text{loc}}(\Omega)$.

$\boxed{(3)}$ Assume that $\Omega' \Subset \Omega'' \Subset \Omega$. Let α be any multi-index with $|\alpha| = m+1$. Let $\phi \in C_0^\infty(\Omega'')$ and use

$$\varphi = (-1)^{|\alpha|} D^\alpha \phi$$

as a test function in

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + c u \varphi\right) dx = \int_\Omega f \varphi \, dx.$$

This gives

$$\sum_{i,j=1}^{n}(-1)^{|\alpha|}\int_{\Omega}a_{ij}D_iuD_j(D^\alpha\phi)\,dx+\sum_{i=1}^{n}(-1)^{|\alpha|}\int_{\Omega}b_iD_iuD^\alpha\phi\,dx$$
$$+(-1)^{|\alpha|}\int_{\Omega}cuD^\alpha\phi\,dx=(-1)^{|\alpha|}\int_{\Omega}fD^\alpha\phi\,dx.$$

After a number of integrations by parts, we obtain

$$\int_{\Omega}\left(\sum_{i,j=1}^{n}a_{ij}D_i\widetilde{u}D_j\phi+\sum_{i=1}^{n}b_iD_i\widetilde{u}\phi+c\widetilde{u}\phi\right)dx=\int_{\Omega}\widetilde{f}\phi\,dx,$$

where $\widetilde{u}=D^\alpha u\in W^{1,2}(\Omega'')$ and

$$\widetilde{f}=D^\alpha f-\sum_{\beta\leqslant\alpha,\beta\neq\alpha}\binom{\alpha}{\beta}\left[-\sum_{i,j=1}^{n}D_j\left(D^{\alpha-\beta}a_{ij}D^\beta D_iu\right)\right.$$
$$\left.+\sum_{i=1}^{n}D^{\alpha-\beta}b_iD^\beta D_iu+D^{\alpha-\beta}cD^\beta u\right],\tag{3.13}$$

where $\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}$. This shows that $\widetilde{u}$ is a weak solution to

$$L\widetilde{u}=\widetilde{f}\quad\text{in}\quad\Omega''.$$

By (3.13), (3.11) and (3.12) we conclude that $\widetilde{f}\in L^2(\Omega'')$ with

$$\|\widetilde{f}\|_{L^2(\Omega'')}\leqslant c\left(\|f\|_{W^{m+1,2}(\Omega)}+\|u\|_{L^2(\Omega)}\right).$$

$\boxed{(4)}$ Theorem 3.5 implies $\widetilde{u}\in W^{2,2}(\Omega')$ with the estimate

$$\begin{aligned}\|\widetilde{u}\|_{W^{2,2}(\Omega')}&\leqslant c\left(\|\widetilde{f}\|_{L^2(\Omega'')}+\|\widetilde{u}\|_{L^2(\Omega'')}\right)\\&\leqslant c\left(\|f\|_{W^{m+1,2}(\Omega)}+\|u\|_{L^2(\Omega)}\right).\end{aligned}$$

This holds true for every multi-index α with $|\alpha|=m+1$ and $\widetilde{u}=D^\alpha u$. This implies $u\in W^{m+3,2}(\Omega')$ and

$$\begin{aligned}\|u\|_{W^{m+3,2}(\Omega')}&\leqslant\|u\|_{W^{m+2,2}(\Omega')}+\|D^\alpha u\|_{W^{2,2}(\Omega')}\\&\leqslant c\left(\|f\|_{W^{m,2}(\Omega)}+\|u\|_{L^2(\Omega)}\right)+c\left(\|f\|_{W^{m+1,2}(\Omega)}+\|u\|_{L^2(\Omega)}\right)\\&\leqslant c\left(\|f\|_{W^{m+1,2}(\Omega)}+\|u\|_{L^2(\Omega)}\right).\end{aligned}$$

This completes the proof. □

Remark 3.14. By the higher order Sobolev embedding, we obtain that $u\in C^1(\Omega)$ when $2(m+2)>n$ and $u\in C^2(\Omega)$ when $2m>n$. Thus

$$\bigcap_{m=0}^{\infty}W^{m+2,2}_{\text{loc}}(\Omega)=C^\infty(\Omega).$$

Theorem 3.10 can be applied recursively with $m=0,1,2,\dots$ to conclude smoothness of a weak solution if the data is smooth.

Theorem 3.15 (Smoothness). Assume that

$$a_{ij}, b_i, c \in C^\infty(\Omega), \quad i,j = 1,\dots,n, \quad \text{and} \quad f \in C^\infty(\Omega).$$

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, where L is as in (1.4). Then $u \in C^\infty(\Omega)$.

THE MORAL: A weak solution is smooth if the data is smooth. Note that no boundary conditions are assumed, so that this regularity result applies to PDEs with Dirichlet, Neumann or other boundary conditions. Moreover, it shows that possible singularities on the boundary do not propagate inside the domain. Observe that these regularity results are based on estimates that are proved from structural ellipticity properties of the PDE. Thus the result holds for a whole class of PDEs instead of a particular PDE.

Remark 3.16. For the corresponding estimates up to the boundary, we refer to [2], p. 336–345.

Remark 3.17. We discuss very formally Hilbert's XIXth problem (1900) on the calculus of variations. For a detailed presentation, we refer to [4, Chapter 3]. Consider the variational integral

$$I(v) = \int_\Omega F(Dv)\,dx,$$

where F is smooth and uniformly convex and $\Omega \subset \mathbb{R}^n$ is an open set. Roughly speaking Hilbert's XIXth problem is the following: Is it true that all local minimizers of the variational integral above are smooth? Let $u \in W^{1,2}_{\text{loc}}(\Omega)$ be a weak solution to the associated Euler-Lagrange equation

$$-\operatorname{div} A(Du) = 0$$

with $A = (A_1,\dots,A_n)$, $A_i(\xi) = \frac{\partial}{\partial \xi_i} F(\xi)$, $i = 1,2,\dots,n$. Let us assume that u is smooth enough so that it satisfies

$$-\sum_{i,j=1}^n \left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} F(Du(x)) \right) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0 \quad \text{in} \quad \Omega.$$

Let us consider this as a linear PDE with the coefficients

$$a_{ij}(x) = \frac{\partial^2}{\partial \xi_i \partial \xi_j} F(Du(x)), \quad i,j = 1,\dots,n.$$

By the uniform convexity, the coefficient matrix $A = (a_{ij}(x))$ is satisfies the ellipticity condition. Moreover, if $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$, then $a_{ij} \in C^{0,\alpha}_{\text{loc}}(\Omega)$. By Schauder estimates, we have

$$u \in C^{0,\alpha}_{\text{loc}}(\Omega) \Longrightarrow a_{ij} \in C^{0,\alpha}_{\text{loc}}(\Omega) \Longrightarrow u \in C^{2,\alpha}_{\text{loc}}(\Omega).$$

We can then apply a bootstrap argument and obtain

$$\begin{aligned} u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega) &\Longrightarrow Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega) \Longrightarrow a_{ij} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \Longrightarrow u \in C^{3,\alpha}_{\mathrm{loc}}(\Omega) \\ &\Longrightarrow \cdots \Longrightarrow u \in C^{\infty}(\Omega) \end{aligned}$$

Later we shall show that a weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is locally Hölder continuous, which is required in the initial step in the boostrap argument.

4
Local Hölder continuity

In the previous chapter we discussed regularity of weak solutions under smoothness assumptions on the coefficients, but this chapter focuses regularity of weak solutions under the assumption that the coefficients are only bounded and measurable functions. We give a treatment of a remarkable De Giorgi-Nash-Moser result that weak solutions of the equation

$$-\operatorname{div}(ADu) = -\sum_{i,j=1}^{n} D_j(a_{ij}D_i u) = 0 \quad \text{in} \quad \Omega \tag{4.1}$$

are locally Hölder continuous under the ellipticity assumption

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad 0 < \lambda \leqslant \Lambda,$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. See Definitions 1.7 and 1.18 for precise definitions. This result was proved by De Giorgi and Nash independently in the 1950's and it is one of the major results in PDEs. We shall consider Moser's proof of this result. Throughout we assume that $a_{ij} \in L^\infty(\Omega)$, $i, j = 1, \dots, n$, that is, the coefficients are only bounded and measurable functions. Instead of the general equation (1.4), we only consider the case $b_i = 0$, $i = 1, \dots, n$, $c = 0$ and $f = 0$ in this chapter. Essential features and challenges of the theory are already visible in this case.

4.1 Super- and subsolutions

Motivation: Assume that $u \in C^2(\Omega)$, $a_{ij} \in C^1(\Omega)$ satisfies

$$-\sum_{i,j=1}^{n} D_j(a_{ij}D_i u) \geqslant 0 \quad \text{in} \quad \Omega.$$

Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$. Then we can integrate by parts and obtain

$$0 \leqslant \int_\Omega -\sum_{i,j=1}^n D_j(a_{ij}D_iu)\varphi\,dx = \int_\Omega \sum_{i,j=1}^n a_{ij}D_iuD_j\varphi\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

On the other hand, if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}D_iuD_j\varphi\,dx \geqslant 0$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$, then

$$\int_\Omega -\sum_{i,j=1}^n D_j(a_{ij}D_iu)\varphi\,dx \geqslant 0$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$ and consequently

$$-\sum_{i,j=1}^n D_j(a_{ij}D_iu) \geqslant 0 \quad \text{in} \quad \Omega.$$

THE MORAL: A function $u \in C^2(\Omega)$ is a classical supersolution of (4.1) if and only if it is a weak supersolution of (4.1) in the sense of the definition below. Observe that the negative sign in front of the second order terms disappears after integration by parts.

Definition 4.2. $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak supersolution of (4.1), if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)D_iuD_j\varphi\,dx \geqslant 0$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$. For a subsolution, we require

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)D_iuD_j\varphi\,dx \leqslant 0$$

for all such test functions.

THE MORAL: Every weak solution is a weak super- and subsolution. The advantage is that the properties of super- and subsolutions can be considered separately.

Remarks 4.3:

(1) By Lemma 2.43, a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak supersolution (subsolution and solution, respectively) in Ω if and only if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}D_iuD_jv\,dx \geqslant 0$$

for every $v \in W_0^{1,2}(\Omega)$ with $v \geqslant 0$ almost everywhere in Ω (exercise).

(2) $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution if and only if it is both super- and subsolution in Ω (exercise).

(3) u is a weak supersolution if and only if $-u$ is a weak subsolution (exercise).

Lemma 4.4. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak subsolution of (4.1), then $u^+ = \max\{u,0\}$ is a weak subsolution in Ω.

THE MORAL: The class of weak subsolutions is closed with respect to truncation from below. The class of weak solutions does not have the corresponding property.

Proof. By properties of Sobolev spaces, we have $u^+ \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$. Denote

$$v_k = \min\{ku^+, 1\}, \quad k = 1,2,\dots.$$

Then (v_k) is an increasing sequence, $0 \leqslant v_k \leqslant 1$, $k = 1,2,\dots$,

$$\lim_{k\to\infty} v_k(x) = \chi_{\{x\in\Omega: u(x)>0\}}(x), \quad x \in \Omega,$$

and we choose $v_k\varphi \in W_0^{1,2}(\Omega)$ as a test function. Notice that $v_k\varphi \geqslant 0$ and that

$$D_j v_k = \begin{cases} kD_j u & \text{almost everywhere in } \{x\in\Omega : 0 < ku(x) < 1\}, \\ 0 & \text{almost everywhere in } \{x\in\Omega : ku(x) \geqslant 1\} \cup \{x \in \Omega : u(x) \leqslant 0\}. \end{cases}$$

The Leibniz rule gives

$$\begin{aligned} 0 &\geqslant \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j(v_k\varphi)\,dx \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u(\varphi D_j v_k + v_k D_j\varphi)\,dx \\ &= k\int_{\{x\in\Omega: 0<u(x)<\frac{1}{k}\}} \sum_{i,j=1}^n a_{ij}\varphi D_i u D_j u\,dx + \int_\Omega \sum_{i,j=1}^n a_{ij} v_k D_i u D_j\varphi\,dx. \end{aligned}$$

The previous estimate together with the ellipticity implies that

$$\begin{aligned} \int_\Omega \sum_{i,j=1}^n a_{ij} v_k D_i u D_j \varphi\,dx &\leqslant -k \int_{\{x\in\Omega: 0<u(x)<\frac{1}{k}\}} \varphi \sum_{i,j=1}^n a_{ij} D_i u D_j u\,dx \\ &\leqslant -k\lambda \int_{\{x\in\Omega: 0<u(x)<\frac{1}{k}\}} \varphi |Du|^2\,dx \leqslant 0. \end{aligned}$$

Since

$$\begin{aligned} \left| \sum_{i,j=1}^n a_{ij} v_k D_i u D_j \varphi \right| &\leqslant \sum_{i,j=1}^n |a_{ij}||v_k||D_i u||D_j\varphi| \\ &\leqslant \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} \|D_j\varphi\|_{L^\infty(\Omega)} |D_i u| \\ &\leqslant c\sum_{i,j=1}^n |D_i u| \in L^1(\Omega), \end{aligned}$$

we may use the Lebesgue dominated convergence theorem to conclude that

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u^+ D_j \varphi \, dx &= \int_\Omega \lim_{k\to\infty} \sum_{i,j=1}^n a_{ij} v_k D_i u D_j \varphi \, dx \\ &= \lim_{k\to\infty} \int_\Omega \sum_{i,j=1}^n a_{ij} v_k D_i u D_j \varphi \, dx \leqslant 0\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$. □

THE MORAL: The proof is based on a clever choice of a test function.

Remark 4.5. The following versions of the previous result are left as exercises.

(1) If u is a weak subsolution, then $\max\{u,k\}$, $k \in \mathbb{Z}$, is a weak subsolution.
(2) If u,v are weak subsolutions, then $\max\{u,v\}$ is a weak subsolution.
(3) If u is a weak supersolution, then $\min\{u,k\}$, $k \in \mathbb{Z}$, is a weak supersolution.
(4) If u,v are weak supersolutions, then $\min\{u,v\}$ is a weak supersolution.
(5) If u is a weak subsolution and $f \in C^2(\mathbb{R})$ with $f(0) = 0$, $f'' \geqslant 0$ (f is convex) and $f' \geqslant 0$, then $f \circ u$ is a weak subsolution.
(6) If u is a weak supersolution and $f \in C^2(\mathbb{R})$ with $f(0) = 0$, $f'' \leqslant 0$ (f is concave) and $f' \geqslant 0$, then $f \circ u$ is a weak supersolution.
(7) If u is a weak solution and $f \in C^2(\mathbb{R})$ is convex, then $f \circ u$ is a weak subsolution.

In properties (5)–(7) we assume $f \in C^2(\mathbb{R})$ is such that the chain rule holds for $f \circ u$.

THE MORAL: The classes of super- and subsolutions are more flexible than solutions. In particular, super- and subsolutions can be modified as above. The corresponding modifications are not possible in the class of weak solutions.

4.2 Caccioppoli estimates

Next we prove a Caccioppoli type energy estimate. The purpose of Caccioppoli type estimates is to provide estimates for the gradient of the solutions with respect to the function itself. A combination of a Caccioppoli type estimate and Sobolev embedding provides us reverse Hölder inequalities. In many cases the PDE is used only to prove Caccioppoli estimates and the rest of the argument applies to all functions that satisfy the corresponding estimate. This is a powerful method, since it applies to a whole class of PDEs simultaneously.

Theorem 4.6. Assume that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak subsolution of (4.1) in Ω. There exists a constant $c = c(\lambda, \Lambda)$ such that

$$\int_\Omega \varphi^2 |Du|^2 \, dx \leqslant c \int_\Omega u^2 |D\varphi|^2 \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

THE MORAL: The energy estimate above asserts that if u is small in average, the gradient of u is small in average. This contains nontrivial information about a weak solution, since by considering highly oscillating functions with small amplitude we note that this is not true for arbitrary functions $u \in W^{1,2}_{\text{loc}}(\Omega)$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and define $v = \varphi^2 u \in W_0^{1,2}(\Omega)$. Then v is compactly supported in Ω and

$$D_j v = \varphi^2 D_j u + 2\varphi u D_j \varphi, \quad j = 1, \dots, n,$$

almost everywhere in Ω. Since u is a weak solution and $v \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} 0 &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j v \, dx \\ &= \int_\Omega \varphi^2 \sum_{i,j=1}^n a_{ij} D_i u D_j u \, dx + 2 \int_\Omega \varphi u \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx. \end{aligned}$$

This implies that

$$\begin{aligned} \int_\Omega \varphi^2 \sum_{i,j=1}^n a_{ij} D_i u D_j u \, dx &\leqslant 2 \left| \int_\Omega \varphi u \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx \right| \\ &\leqslant 2 \int_\Omega |\varphi| |u| \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} |D_i u| |D_j \varphi| \, dx \\ &\leqslant c \int_\Omega |\varphi| |u| |Du| |D\varphi| \, dx. \end{aligned}$$

Next we first apply the uniform ellipticity condition to the previous estimate, and then we use Young's inequality with epsilon to have

$$\begin{aligned} \lambda \int_\Omega \varphi^2 |Du|^2 \, dx &\leqslant c \int_\Omega |\varphi| |u| |Du| |D\varphi| \, dx \\ &\leqslant \frac{\lambda}{2} \int_\Omega \varphi^2 |Du|^2 \, dx + c \int_\Omega u^2 |D\varphi|^2 \, dx. \end{aligned}$$

Both terms on the right-hand side are finite, since $u \in W^{1,2}_{\text{loc}}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. The claim follows by absorbing the first term on the right-hand side. □

By the Poincaré inequality in Remark 1.43, we have

$$\int_{B(x,2r)} |u - u_{B(x,2r)}|^2 \, dy \leqslant c r^2 \int_{B(x,2r)} |Du|^2 \, dy$$

for every $u \in W^{1,2}_{\text{loc}}(\Omega)$. As a consequence of the energy estimate, we obtain the following reverse Poincaré inequality for weak solutions. Compare to Remark 3.9.

Lemma 4.7. Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak subsolution of (4.1) in Ω. There exists a constant $c = c(\lambda, \Lambda)$ such that

$$\int_{B(z,r)} |Du|^2\,dx \leqslant \frac{c}{r^2}\int_{B(z,2r)} |u - u_{B(z,2r)}|^2\,dx$$

for every ball $B(z,r)$ with $B(z,2r) \Subset \Omega$.

Proof. Let $\varphi \in C_0^\infty(B(z,2r))$ be a cutoff function with $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $B(z,r)$ and $|D\varphi| \leqslant \frac{c}{r}$. Then $u - u_{B(z,2r)} \in W^{1,2}(B(z,2r))$ is a weak solution to (4.1) in $B(z,2r)$. Theorem 4.6 implies

$$\begin{aligned}\int_{B(z,r)} |Du|^2\,dx &\leqslant \int_{B(z,2r)} \varphi^2 |D(u - u_{B(z,2r)})|^2\,dx\\ &\leqslant c\int_{B(z,2r)} |u - u_{B(z,2r)}|^2 |D\varphi|^2\,dx\\ &\leqslant \frac{c}{r^2}\int_{B(z,2r)} |u - u_{B(z,2r)}|^2\,dx.\end{aligned}$$

□

Next we discuss a Caccioppoli estimate for weak subsolutions. Observe, that the result also holds for weak solutions.

Theorem 4.8 (Caccioppoli estimate for subsolutions). Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak subsolution of (4.1) in Ω and let $\alpha > 0$. Then there exists $c = c(\lambda, \Lambda)$ such that

$$\int_{\{x\in\Omega: u(x)>0\}} u^{\alpha-1}|Du|^2\varphi^2\,dx \leqslant \frac{c}{\alpha^2}\int_{\{x\in\Omega: u(x)>0\}} u^{\alpha+1}|D\varphi|^2\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$.

Proof. By Lemma 4.4 we may assume that $u = u^+$. We would like to apply $u^\alpha\varphi^2$ as a test function, but it is not clear that this function belongs to $W_0^{1,2}(\Omega)$. Thus we modify the test function in the following manner. Let

$$\psi_k = \varphi^2 \min\{u^\alpha, ku\} \quad k = 1, 2, \dots,$$

Observe that $\psi_k \in W_0^{1,2}(\Omega)$ and $\psi_k \geqslant 0$, $k = 1, 2, \dots$. Moreover, (ψ_k) is an increasing sequence,

$$\lim_{k\to\infty} \psi_k(x) = u(x)^\alpha \varphi(x)^2, \quad x \in \Omega,$$

and

$$D_j\psi_k = 2\varphi(D_j\varphi)\min\{u^\alpha, ku\} + (D_j \min\{u^\alpha, ku\})\varphi^2, \quad j = 1, \dots, n.$$

Since u is a weak subsolution, we have

$$\begin{aligned}0 &\geqslant \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j \psi_k\,dx\\ &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u (D_j \min\{u^\alpha, ku\})\varphi^2\,dx + 2\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u (D_j\varphi)\varphi \min\{u^\alpha, ku\}\,dx.\end{aligned}$$

Denote

$$\Omega_k = \{x \in \Omega : 0 < u^\alpha(x) \leqslant ku(x)\}, \quad k = 1, 2, \dots.$$

Notice that $D_j u = 0$ almost everywhere in the set where $u = 0$. Therefore we have

$$D_j \min\{u^\alpha, ku\} = \begin{cases} \alpha u^{\alpha-1} D_j u & \text{almost everywhere in } \Omega_k, \\ k D_j u & \text{almost everywhere in } \Omega \setminus \Omega_k. \end{cases}$$

The previous inequality implies that

$$\begin{aligned}
\alpha \int_{\Omega_k} \sum_{i,j=1}^n a_{ij} D_i u D_j u\, u^{\alpha-1} \varphi^2\, dx &+ k \int_{\Omega\setminus\Omega_k} \sum_{i,j=1}^n a_{ij} D_i u D_j u \varphi^2\, dx \\
&\leqslant 2 \left| \int_\Omega \varphi \min\{u^\alpha, ku\} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi\, dx \right| \\
&\leqslant 2c \int_{\Omega_k} \varphi u^\alpha \sum_{i,j=1}^n |D_i u||D_j \varphi|\, dx + 2kc \int_{\Omega\setminus\Omega_k} \varphi u \sum_{i,j=1}^n |D_i u||D_j \varphi|\, dx \\
&\leqslant c \int_{\Omega_k} \varphi u^\alpha |Du||D\varphi|\, dx + kc \int_{\Omega\setminus\Omega_k} \varphi u |Du||D\varphi|\, dx.
\end{aligned}$$

Next we first apply the uniform ellipticity condition to the previous estimate, and then we use Young's inequality with epsilon (exercise) to have

$$\begin{aligned}
\alpha\lambda \int_{\Omega_k} u^{\alpha-1} |Du|^2 \varphi^2\, dx &+ k\lambda \int_{\Omega\setminus\Omega_k} |Du|^2 \varphi^2\, dx \\
&\leqslant c \int_{\Omega_k} \varphi u^\alpha |Du||D\varphi|\, dx + kc \int_{\Omega\setminus\Omega_k} \varphi u |Du||D\varphi|\, dx \\
&\leqslant \frac{\alpha\lambda}{2} \int_{\Omega_k} u^{\alpha-1} |Du|^2 \varphi^2\, dx + \frac{\lambda k}{2} \int_{\Omega\setminus\Omega_k} |Du|^2 \varphi^2\, dx \\
&\quad + \frac{c}{\alpha} \int_{\Omega_k} u^{\alpha+1} |D\varphi|^2\, dx + ck \int_{\Omega\setminus\Omega_k} u^2 |D\varphi|^2\, dx.
\end{aligned}$$

Since $u^\alpha \leqslant ku$ in Ω_k and $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we have

$$\int_{\Omega_k} u^{\alpha-1} |Du|^2 \varphi^2\, dx \leqslant k \int_{\Omega_k} |Du|^2 \varphi^2\, dx < \infty$$

and

$$\int_{\Omega\setminus\Omega_k} |Du|^2 \varphi^2\, dx \leqslant \int_\Omega |Du|^2 \varphi^2\, dx < \infty,$$

so that these terms can be absorbed into the left-hand side. This gives

$$\begin{aligned}
\alpha \int_{\Omega_k} u^{\alpha-1} |Du|^2 \varphi^2\, dx &+ k \int_{\Omega\setminus\Omega_k} |Du|^2 \varphi^2\, dx \\
&\leqslant \frac{c}{\alpha} \int_{\Omega_k} u^{\alpha+1} |D\varphi|^2\, dx + ck \int_{\Omega\setminus\Omega_k} u^2 |D\varphi|^2\, dx,
\end{aligned}$$

where

$$k \int_{\Omega\setminus\Omega_k} u^2 |D\varphi|^2\, dx \leqslant \int_{\Omega\setminus\Omega_k} u^{\alpha+1} |D\varphi|^2\, dx \to 0 \quad \text{as} \quad k \to \infty,$$

since $ku \leqslant u^\alpha$ in $\Omega \setminus \Omega_k$. Here we may assume that $u^{\alpha+1}|D\varphi|^2 \in L^1(\Omega)$, since otherwise the claim is clear. Consequently

$$\begin{aligned}\int_\Omega u^{\alpha-1}|Du|^2\varphi^2\,dx &= \lim_{k\to\infty}\int_{\Omega_k} u^{\alpha-1}|Du|^2\varphi^2\,dx\\ &\leqslant \lim_{k\to\infty}\left(\frac{c}{\alpha^2}\int_{\Omega_k} u^{\alpha+1}|D\varphi|^2\,dx + \frac{ck}{\alpha}\int_{\Omega\setminus\Omega_k} u^2|D\varphi|^2\,dx\right)\\ &= \frac{c}{\alpha^2}\int_\Omega u^{\alpha+1}|D\varphi|^2\,dx.\end{aligned}$$

The last equality follows from the Lebesgue dominated convergence theorem. □

Theorem 4.9 (Caccioppoli estimate for supersolutions). Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $u \geqslant 0$, is a weak supersolution of (4.1) in Ω and let $\alpha < 0$. Then there exists $c = c(\lambda, \Lambda)$ such that

$$\int_{\{x\in\Omega:u(x)>0\}} u^{\alpha-1}|Du|^2\varphi^2\,dx \leqslant \frac{c}{|\alpha|^2}\int_{\{x\in\Omega:u(x)>0\}} u^{\alpha+1}|D\varphi|^2\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$.

THE MORAL: This is the same estimate as in Theorem 4.8 for negative values of α.

Proof. Let $u_k = u + \frac{1}{k}$, $k = 1,2,\dots$, and apply $u_k^\alpha\varphi^2 \in W_0^{1,2}(\Omega)$ as a test function. Then

$$D_j(u_k^\alpha\varphi^2) = 2\varphi(D_j\varphi)u_k^\alpha + \alpha u_k^{\alpha-1}(D_ju_k)\varphi^2, \quad j = 1,\dots,n.$$

Since u is a weak supersolution, we have

$$\begin{aligned}0 &\leqslant \int_\Omega \sum_{i,j=1}^n a_{ij}D_iuD_j(u_k^\alpha\varphi^2)\,dx\\ &= 2\int_\Omega \varphi u_k^\alpha \sum_{i,j=1}^n a_{ij}D_iuD_j\varphi\,dx + \alpha\int_\Omega \varphi^2u_k^{\alpha-1}\sum_{i,j=1}^n a_{ij}D_iuD_ju\,dx.\end{aligned}$$

By using the previous equation and ellipticity, we obtain the estimate

$$\begin{aligned}\int_\Omega \varphi^2u_k^{\alpha-1}|Du|^2\,dx &\leqslant \frac{1}{\lambda}\int_\Omega \varphi^2u_k^{\alpha-1}\sum_{i,j}^n a_{ij}D_iuD_ju\,dx\\ &\leqslant -\frac{2}{\alpha\lambda}\int_\Omega \varphi u_k^\alpha\sum_{i,j=1}^n a_{ij}D_iuD_j\varphi\,dx\\ &\leqslant \frac{c}{|\alpha|}\int_\Omega \varphi u_k^\alpha|Du||D\varphi|\,dx \qquad (a_{ij}\in L^\infty(\Omega))\\ &= \frac{c}{|\alpha|}\int_\Omega \varphi u_k^{\frac{\alpha-1}{2}}u_k^{\frac{\alpha+1}{2}}|Du||D\varphi|\,dx\\ &\leqslant \frac{1}{2}\int_\Omega \varphi^2u_k^{\alpha-1}|Du|^2\,dx + \frac{c}{|\alpha|^2}\int_\Omega u_k^{\alpha+1}|D\varphi|^2\,dx. \qquad \text{(Young with } \varepsilon)\end{aligned}$$

Thus

$$\int_\Omega \varphi^2 u_k^{\alpha-1}|Du|^2\,dx \leqslant \frac{c}{|\alpha|^2}\int_\Omega u_k^{\alpha+1}|D\varphi|^2\,dx.$$

By the monotone and dominated convergence theorem, we conclude that

$$\begin{aligned}\int_\Omega \varphi^2 u^{\alpha-1}|Du|^2\,dx &= \int_\Omega \lim_{k\to\infty}\varphi^2 u_k^{\alpha-1}|Du|^2\,dx\\ &= \lim_{k\to\infty}\int_\Omega \varphi^2 u_k^{\alpha-1}|Du|^2\,dx\\ &\leqslant \lim_{k\to\infty}\frac{c}{|\alpha|^2}\int_\Omega u_k^{\alpha+1}|D\varphi|^2\,dx\\ &= \frac{c}{|\alpha|^2}\int_\Omega \lim_{k\to\infty} u_k^{\alpha+1}|D\varphi|^2\,dx\\ &\leqslant \frac{c}{|\alpha|^2}\int_\Omega u^{\alpha+1}|D\varphi|^2\,dx.\end{aligned}$$

Observe that if $\alpha+1<0$, we may use the monotone convergence theorem in taking the limit inside the integral. If $-1\leqslant\alpha<0$, then $u_k^{\alpha+1}\leqslant(u+1)^{\alpha+1}$ and we may apply the dominated convergence theorem. □

Theorem 4.10 (Logarithmic Caccioppoli inequality). If $u>0$ is a weak supersolution of (4.1) in Ω, then there exists $c=c(\lambda,\Lambda)$ such that

$$\int_\Omega \varphi^2|D\log u|^2\,dx \leqslant c\int_\Omega |D\varphi|^2\,dx$$

for every $\varphi\in C_0^\infty(\Omega)$, $\varphi\geqslant 0$.

THE MORAL: This is a uniform bound for the logarithm of the gradient, since the right hand side is independent of u.

Proof. Theorem 4.9 with $\alpha=-1$ gives

$$\int_\Omega \varphi^2|D\log u|^2\,dx = \int_\Omega \varphi^2\frac{|Du|^2}{u^2}\,dx \leqslant c\int_\Omega |D\varphi|^2\,dx.$$

□

4.3 Integral averages

Our goal is to obtain estimates for the maximum and the minimum of a solution to a PDE. Since functions in Sobolev spaces are defined only up to a set of measure zero, we recall the definition of essential supremum and infimum.

Definition 4.11. Let $A\subset\mathbb{R}^n$ be a Lebesgue measurable set and $f:A\to[-\infty,\infty]$ a Lebesgue measurable function. The essential supremum of f is

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in A} f(x) &= \inf\{M : f(x)\leqslant M \text{ for almost every } x\in A\}\\ &= \inf\{M : |\{x\in A : f(x)>M\}| = 0\}\end{aligned}$$

and the essential infimum of f is

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in A} f(x) &= \sup\{m : f(x) \geqslant m \text{ for almost every } x \in A\} \\ &= \sup\{m : |\{x \in A : |f(x)| < m\}| = 0\}.\end{aligned}$$

T H E M O R A L : Essential supremum is supremum outside sets of measure zero.

Remark 4.12. Observe that for the standard supremum we have

$$\sup_{x\in A} f(x) = \inf\{M : \{x \in A : f(x) > M\} = \emptyset\}.$$

Analogously, essential infimum is infimum outside sets of measure zero.

$$\inf_{x\in A} f(x) = \sup\{m : \{x \in A : f(x) < m\} = \emptyset\}.$$

Moreover,

$$f(x) \leqslant \operatorname*{ess\,sup}_{x\in A} f(x) \quad \text{for almost every} \quad x \in A$$

and

$$f(x) \geqslant \operatorname*{ess\,inf}_{x\in A} f(x) \quad \text{for almost every} \quad x \in A.$$

The integral average of f in A, $0 < |A| < \infty$, is denoted by

$$\fint_A f\,dx = \frac{1}{|A|}\int_A f\,dx.$$

Let $-\infty < p < q < \infty$, $p \neq 0$, $q \neq 0$ and assume that $0 < |A| < \infty$. By Hölder's, or Jensen's, inequality

$$\operatorname*{ess\,inf}_A |f| \leqslant \left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} \leqslant \left(\fint_A |f|^q\,dx\right)^{\frac{1}{q}} \leqslant \operatorname*{ess\,sup}_A |f|.$$

Thus the integral average is an increasing function of the power.

Theorem 4.13. Let $f : A \to [-\infty,\infty]$ be a Lebesgue measurable function and $0 < |A| < \infty$. Then

(1) $\displaystyle\lim_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} = \operatorname*{ess\,sup}_A |f|$ and

(2) $\displaystyle\lim_{p\to\infty}\left(\fint_A |f|^{-p}\,dx\right)^{-\frac{1}{p}} = \operatorname*{ess\,inf}_A |f|.$

T H E M O R A L : This gives a method to derive estimates for supremum and infimum by uniform estimates for integral averages with powers. The Moser iteration technique is based on this observation.

Remark 4.14. The integral average can be replaced with the integral.

Proof. $\boxed{(1)}$ Assume that $\operatorname{ess\,sup}_A |f| < \infty$. Then

$$\int_A |f|^p\,dx \leqslant \operatorname*{ess\,sup}_A |f|^p \int_A 1\,dx = |A|(\operatorname*{ess\,sup}_A |f|)^p,$$

which implies that for every p, $1 \leqslant p < \infty$,

$$\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} \leqslant \operatorname*{ess\,sup}_A |f|$$

and, in particular, that

$$\limsup_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} \leqslant \operatorname*{ess\,sup}_A |f|.$$

This clearly holds true also in the case $\operatorname{ess\,sup}_A |f| = \infty$.

Denote $E_\lambda = \{x \in A : |f(x)| > \lambda\}$. For every λ with $0 \leqslant \lambda < \operatorname{ess\,sup}_A |f|$, we have $|E_\lambda| > 0$. Since $|f|^p \geqslant \lambda^p$ in E_λ, we obtain

$$\lambda^p |E_\lambda| \leqslant \int_{E_\lambda} |f|^p\,dx \leqslant \int_A |f|^p\,dx.$$

By taking the pth root we have

$$\lambda |E_\lambda|^{\frac{1}{p}} \leqslant \left(\int_A |f|^p\,dx\right)^{\frac{1}{p}}.$$

Observe that for any E_λ with $0 < |E_\lambda| < \infty$, we have $|E_\lambda|^{\frac{1}{p}} \to 1$ as $p \to \infty$. Thus

$$\lambda \leqslant \liminf_{p\to\infty}\left(\int_A |f|^p\,dx\right)^{\frac{1}{p}}$$

As $0 < |A| < \infty$, we have also that $|A|^{\frac{1}{p}} \to 1$ as $p \to \infty$ and thus

$$\lambda \leqslant \liminf_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}}.$$

By letting $\lambda \to \operatorname{ess\,sup}_A |f|$, we obtain

$$\operatorname*{ess\,sup}_A |f| \leqslant \liminf_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}}.$$

All together we have now proved that

$$\operatorname*{ess\,sup}_A |f| \leqslant \liminf_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} \leqslant \limsup_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} \leqslant \operatorname*{ess\,sup}_A |f|,$$

which implies that the limit exists and

$$\operatorname*{ess\,sup}_A |f| = \lim_{p\to\infty}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}}.$$

$\boxed{(2)}$ Clearly

$$\int_A |f|^{-p}\,dx \leqslant (\operatorname*{ess\,inf}_A |f|)^{-p}|A|,$$

and thus

$$\left(\fint_A |f|^{-p}\,dx\right)^{-\frac{1}{p}} \geqslant \operatorname*{ess\,inf}_A |f|.$$

By letting $p \to \infty$ we see that

$$\liminf_{p\to\infty}\left(\fint_A |f|^{-p}\,dx\right)^{-\frac{1}{p}} \geqslant \operatorname*{ess\,inf}_A |f|.$$

Let $F_\lambda = \{x \in A : |f(x)| < \lambda\}$. For every $\lambda > \operatorname{ess\,inf}_A |f|$, we have $|F_\lambda| > 0$ and by using the fact that $|f|^{-p} \geqslant \lambda^{-p}$ in F_λ, we obtain

$$\lambda^{-p}|F_\lambda| \leqslant \int_{F_\lambda} |f|^{-p}\,dx \leqslant \int_A |f|^{-p}\,dx.$$

This is equivalent to

$$\lambda|F_\lambda|^{-\frac{1}{p}} \geqslant \left(\int_A |f|^{-p}\,dx\right)^{-\frac{1}{p}}.$$

As $|F_\lambda|^{-\frac{1}{p}} \to 1$ as $p \to \infty$, we conclude

$$\lambda \geqslant \limsup_{p\to\infty}\left(\int_A |f|^{-p}\,dx\right)^{-\frac{1}{p}} = \limsup_{p\to\infty}\left(\fint_A |f|^{-p}\,dx\right)^{-\frac{1}{p}}.$$

Since this holds for every $\lambda > \operatorname{ess\,inf}_A |f|$, we obtain

$$\operatorname*{ess\,inf}_A |f| \geqslant \limsup_{p\to\infty}\left(\fint_A |f|^{-p}\,dx\right)^{-\frac{1}{p}}.$$

□

Remark 4.15. Part (2) of the theorem above could be also proved by applying the part (1) to the function $\frac{1}{|f|}$.

The following result is sometimes useful in the Moser iteration technique. We will not apply it later, but we discuss it for the sake of curiosity.

Theorem 4.16. Let $f : A \to [-\infty,\infty]$ be a Lebesgue measurable function with $\int_A |f|^{p_0}\,dx < \infty$ for some $0 < p_0 < \infty$ and $0 < |A| < \infty$. Then

$$\lim_{p\to 0}\left(\fint_A |f|^p\,dx\right)^{\frac{1}{p}} = e^{\fint_A \log|f|\,dx}.$$

Proof. Let $I : [0,p_0] \to [0,\infty)$, $I(p) = \fint_A |f|^p\,dx$, with the interpretation $I(0) = 1$. We observe that $|f|^p \leqslant \max\{1,|f|^{p_0}\} \in L^1(A)$, $0 \leqslant p \leqslant p_0$, and that I is a continuous function by the dominated convergence theorem.

We show that I is differentiable at $p = 0$. For a fixed parameter $t > 0$, consider the function $g_t : [0, p_0] \to \mathbb{R}$, $g_t(p) = \frac{t^p - 1}{p}$. The function $p \mapsto t^p$ is convex, so that the function $p \mapsto g_t(p)$ is increasing and

$$\lim_{p\to 0} g_t(p) = \frac{d}{dp} t^p \bigg|_{p=0} = \log t.$$

By the monotone convergence theorem, we have

$$\begin{aligned} I'(0) &= \lim_{p\to 0} \frac{I(p) - I(0)}{p - 0} = \lim_{p\to 0} \fint_A \frac{|f|^p - 1}{p}\, dx \\ &= \fint_A \lim_{p\to 0} \frac{|f|^p - 1}{p}\, dx = \fint_A \log|f|\, dx. \end{aligned}$$

Here we use the convention

$$\lim_{p\to 0} \frac{|0|^p - 1}{p} = -\lim_{p\to 0} \frac{1}{p} = -\infty = \log|0|.$$

This shows that $I'(0)$ exists and $I'(0) = \fint_A \log|f|\, dx$, with the interpretation that $I'(0)$ may be $-\infty$.

On the other hand, we have

$$\left(\fint_A |f|^p\, dx \right)^{\frac{1}{p}} = e^{\frac{1}{p} \log \fint_A |f|^p\, dx} = e^{\frac{1}{p} \log I(p)} = e^{\frac{1}{p} (\log I(p) - \log I(0))}.$$

By the chain rule

$$\frac{d}{dp} \log I(p) \bigg|_{p=0} = \frac{I'(0)}{I(0)} = I'(0) = \fint_A \log|f|\, dx.$$

Form this we conclude

$$\lim_{p\to 0} \left(\fint_A |f|^p\, dx \right)^{\frac{1}{p}} = e^{I'(0)} = e^{\fint_A \log|f|\, dx}. \qquad \square$$

4.4 Estimates from above

The next result shows that a weak subsolution to an elliptic PDE with measurable coefficients is locally bounded from above. The proof is based on the Moser iteration technique together with a Caccioppoli inequality and a Sobolev inequality. Sometimes this result is called the weak maximum principle, since it gives an estimate of the supremum in terms of positive powers of integral averages. This is a counterpart of the mean value property of subharmonic functions for more general PDEs.

Theorem 4.17 (Local boundedness from above). Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak subsolution of (4.1) in Ω and let $\beta > 1$. There exists constants $c = c(n,\lambda,\Lambda,\beta)$ and $\tau = \tau(n) > 0$ such that

$$\operatorname*{ess\,sup}_{B(x,r)} u^+ \leqslant c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}}$$

whenever $B(x,R) \Subset \Omega$, $0 < r < R$.

T H E M O R A L : By choosing $R = 2r$, we have

$$\operatorname*{ess\,sup}_{B(x,r)} u \leqslant \operatorname*{ess\,sup}_{B(x,r)} u^+ \leqslant c\left(\fint_{B(x,2r)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}} \leqslant c\left(\fint_{B(x,2r)} |u|^{\beta}\,dy\right)^{\frac{1}{\beta}}$$

whenever $B(x,2r) \Subset \Omega$. Weak subsolutions are locally bounded from above. Observe that for $\beta = 2$, and by Hölder's inequality also for $0 < \beta \leqslant 2$, the assumption $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ implies that the integral average on the right-hand side is finite. It follows from the result that the integral average on the right-hand side is finite for every $\beta > 0$.

Proof. Assume that $u \in L^{\beta}_{\mathrm{loc}}(\Omega)$. Observe that for $\beta = 2$, and by Hölder's inequality also for $0 < \beta \leqslant 2$, this follows from the assumption $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. By Lemma 4.4, we may assume that $u = u^+$. Choose a cutoff function $\varphi \in C_0^{\infty}(B(x,R))$ with $\varphi = 1$ in $B(x,r)$, $0 \leqslant \varphi \leqslant 1$ and $|D\varphi| \leqslant \frac{c}{R-r}$. By the Caccioppoli estimate, Theorem 4.8, we have

$$\begin{aligned}
\int_{\Omega} |\varphi D(u^{\frac{\beta}{2}})|^2\,dy &= \int_{\Omega} \left|\frac{\beta}{2} u^{\frac{\beta}{2}-1} Du\right|^2 \varphi^2\,dy \\
&= \left(\frac{\beta}{2}\right)^2 \int_{\Omega} u^{\beta-2} |Du|^2 \varphi^2\,dy \\
&= \left(\frac{\beta}{2}\right)^2 \int_{\{x\in\Omega: u(x)>0\}} u^{\beta-2} |Du|^2 \varphi^2\,dy \\
&\leqslant c(\lambda,\Lambda)\left(\frac{\beta}{\beta-1}\right)^2 \int_{\{x\in\Omega: u(x)>0\}} u^{\beta} |D\varphi|^2\,dy \\
&= c(\lambda,\Lambda)\left(\frac{\beta}{\beta-1}\right)^2 \int_{\Omega} u^{\beta} |D\varphi|^2\,dy.
\end{aligned}$$

By the Leibniz rule,

$$\left|D\left(\varphi u^{\frac{\beta}{2}}\right)\right| \leqslant \left|\varphi D\left(u^{\frac{\beta}{2}}\right)\right| + \left|u^{\frac{\beta}{2}} D\varphi\right|,$$

and thus

$$\begin{aligned}
\int_{\Omega} |D(\varphi u^{\frac{\beta}{2}})|^2\,dy &\leqslant 2\left(\int_{\Omega} |\varphi D(u^{\frac{\beta}{2}})|^2\,dy + \int_{\Omega} |u^{\frac{\beta}{2}} D\varphi|^2\,dy\right) \\
&\leqslant c(\lambda,\Lambda)\left(\left(\frac{\beta}{\beta-1}\right)^2 + 1\right) \int_{\Omega} |u^{\frac{\beta}{2}} D\varphi|^2\,dy.
\end{aligned}$$

Notice that

$$\left(\frac{\beta}{\beta-1}\right)^2+1=\frac{\beta^2+(\beta-1)^2}{(\beta-1)^2}\leqslant\frac{4\beta^2+4\beta+1}{(\beta-1)^2}=\left(\frac{2\beta+1}{\beta-1}\right)^2.$$

By the Sobolev inequality in Theorem 1.44, we obtain

$$\begin{aligned}\left(\fint_{B(x,R)}|\varphi u^{\frac{\beta}{2}}|^{2\kappa}\,dy\right)^{\frac{1}{2\kappa}}&\leqslant c(n)R\left(\fint_{B(x,R)}|D(\varphi u^{\frac{\beta}{2}})|^2\,dy\right)^{\frac12}\\&\leqslant c(n,\lambda,\Lambda)R\left(\left(\frac{2\beta+1}{\beta-1}\right)^2\fint_{B(x,R)}|u^{\frac{\beta}{2}}D\varphi|^2\,dy\right)^{\frac12},\end{aligned}$$

where $\kappa>1$ is defined, for example, by

$$\kappa=\begin{cases}\frac{n}{n-2}, & n\geqslant 3,\\ 2, & n=2.\end{cases}$$

By combining the previous estimates and using the properties of the cutoff function, we obtain the estimate

$$\begin{aligned}\left(\fint_{B(x,r)}u^{\kappa\beta}\,dy\right)^{\frac{1}{\kappa\beta}}&\leqslant\left(\frac{|B(x,R)|}{|B(x,r)|}\fint_{B(x,R)}|\varphi u^{\frac{\beta}{2}}|^{2\kappa}\,dy\right)^{\frac{1}{\kappa\beta}}\\&\leqslant c(n\lambda,\Lambda)^{\frac{2}{\beta}}\left(\frac{R}{r}\right)^{\frac{n}{\kappa\beta}}\left(R^2\left(\frac{2\beta+1}{\beta-1}\right)^2\fint_{B(x,R)}|u^{\frac{\beta}{2}}D\varphi|^2\,dy\right)^{\frac{1}{\beta}}\qquad(4.18)\\&\leqslant c(n,\lambda,\Lambda)^{\frac{2}{\beta}}\left(\frac{R}{r}\right)^{\frac{n}{\beta}}\left(\left(\frac{2\beta+1}{\beta-1}\,\frac{R}{R-r}\right)^2\fint_{B(x,R)}u^{\beta}\,dy\right)^{\frac{1}{\beta}}.\end{aligned}$$

This is a reverse Hölder inequality. Observe that from $u\in L^{\beta}_{\text{loc}}(\Omega)$, we may conclude that $u\in L^{\kappa\beta}_{\text{loc}}(\Omega)$ with $\kappa>1$. This gives us a bootstrap method to increase the level of local integrability stepwise. In particular, starting from $\beta=2$, we may iterate (4.18) and conclude that $u\in L^{\beta}_{\text{loc}}(\Omega)$ for every $1<\beta<\infty$. Thus all integrals in this proof are finite.

We show that the claim of Theorem 4.17 holds for $\beta_0>1$. Note that if $\beta_0>1$ and $\beta\geqslant\beta_0$, then

$$\frac{2\beta+1}{\beta-1}\leqslant\frac{2\beta_0+1}{\beta_0-1}=c(\beta_0)$$

and by (4.18) there exists a constant $c=c(n,\lambda,\Lambda,\beta_0)$ such that

$$\left(\fint_{B(x,r)}u^{\kappa\beta}\,dy\right)^{\frac{1}{\kappa\beta}}\leqslant c^{\frac{2}{\beta}}\left(\frac{R}{r}\right)^{\frac{n}{\beta}}\left(\frac{R}{R-r}\right)^{\frac{2}{\beta}}\left(\fint_{B(x,R)}u^{\beta}\,dy\right)^{\frac{1}{\beta}}\qquad(4.19)$$

for every $\beta\geqslant\beta_0$.

We apply this estimate recursively. Let $r_0=R$ and

$$r_k=r+\frac{R-r}{2^k},\quad k=1,2,\dots.$$

Then

$$r < r_{k+1} < r_k \leqslant R, \quad \frac{r_k}{r_{k+1}} \leqslant 2 \quad \text{and} \quad \frac{r_k}{r_k - r_{k+1}} \leqslant \frac{2^{k+1}R}{R-r}$$

for every $k = 0, 1, 2, \dots$.

$\boxed{\text{Step 1}}$ By (4.19) we have

$$\left(\fint_{B(x,r_1)} u^{\kappa\beta_0}\, dy \right)^{\frac{1}{\kappa\beta_0}} \leqslant c^{\frac{2}{\beta_0}} 2^{\frac{n}{\beta_0}} \left(\frac{R}{R-r} \right)^{\frac{2}{\beta_0}} \left(\fint_{B(x,r_0)} u^{\beta_0}\, dy \right)^{\frac{1}{\beta_0}},$$

where $c = c(n, \lambda, \Lambda, \beta_0)$.

$\boxed{\text{Step 2}}$ By applying (4.19) twice we have

$$\begin{aligned}
\left(\fint_{B(x,r_2)} u^{\kappa^2\beta_0}\, dy \right)^{\frac{1}{\kappa^2\beta_0}} &\leqslant c^{\frac{2}{\kappa\beta_0}} 2^{\frac{n}{\kappa\beta_0}} \left(\frac{2^2 R}{R-r} \right)^{\frac{2}{\kappa\beta_0}} \left(\fint_{B(x,r_1)} u^{\kappa\beta_0}\, dy \right)^{\frac{1}{\kappa\beta_0}} \\
&\leqslant c^{\frac{2}{\beta_0} + \frac{2}{\kappa\beta_0}} \cdot 2^{\frac{n}{\beta_0} + \frac{n}{\kappa\beta_0}} \cdot 2^{\frac{2}{\beta_0} + \frac{2\cdot 2}{\kappa\beta_0}} \left(\frac{R}{R-r} \right)^{\frac{2}{\beta_0} + \frac{2}{\kappa\beta_0}} \left(\fint_{B(x,r_0)} u^{\beta_0}\, dy \right)^{\frac{1}{\beta_0}}.
\end{aligned}$$

$\boxed{\text{Step } k}$ By applying (4.19) recursively we have

$$\begin{aligned}
\left(\fint_{B(x,r_k)} u^{\kappa^k\beta_0}\, dy \right)^{\frac{1}{\kappa^k\beta_0}} &\leqslant c^{\frac{2}{\beta_0}\sum_{i=1}^k \frac{1}{\kappa^{i-1}}} \cdot 2^{\frac{n}{\beta_0}\sum_{i=1}^k \frac{i}{\kappa^{i-1}}} \cdot 2^{\frac{2}{\beta_0}\sum_{i=1}^k \frac{i}{\kappa^{i-1}}} \\
&\quad \cdot \left(\frac{R}{R-r} \right)^{\frac{2}{\beta_0}\sum_{i=1}^k \frac{i}{\kappa^{i-1}}} \left(\fint_{B(x,r_0)} u^{\beta_0}\, dy \right)^{\frac{1}{\beta_0}}
\end{aligned} \tag{4.20}$$

for every $k = 1, 2, \dots$. Let us compute the sums that appear in (4.20). The sum of a geometric series gives

$$\sum_{i=1}^k \frac{1}{\kappa^{i-1}} \xrightarrow{k\to\infty} \sum_{i=1}^\infty \frac{1}{\kappa^{i-1}} = \frac{1}{1-\frac{1}{\kappa}} = \frac{\kappa}{\kappa-1},$$

and by recognizing the derivative of a geometric series we obtain

$$\sum_{i=1}^k \frac{i}{\kappa^{i-1}} \xrightarrow{k\to\infty} \sum_{i=1}^\infty \frac{i}{\kappa^{i-1}} = \frac{1}{(1-\frac{1}{\kappa})^2}.$$

Hence we conclude from (4.20) that

$$\begin{aligned}
\lim_{k\to\infty} \left(\fint_{B(x,r)} u^{\kappa^k\beta_0}\, dy \right)^{\frac{1}{\kappa^k\beta_0}} &\leqslant \lim_{k\to\infty} \left(\left(\frac{r_k}{r} \right)^n \fint_{B(x,r_k)} u^{\kappa^k\beta_0}\, dy \right)^{\frac{1}{\kappa^k\beta_0}} \\
&\leqslant c \left(\frac{R}{R-r} \right)^{\frac{2}{\beta_0}\frac{\kappa}{\kappa-1}} \left(\fint_{B(x,r_0)} u^{\beta_0}\, dy \right)^{\frac{1}{\beta_0}},
\end{aligned}$$

where $c = c(n, \lambda, \Lambda, \beta_0)$. By Theorem 4.13, we conclude that u is essentially bounded in the ball $B(x,r)$ and

$$\begin{aligned}
\operatorname*{ess\,sup}_{x\in B(x,r)} u(x) &= \lim_{k\to\infty} \left(\fint_{B(x,r)} u^{\kappa^k\beta_0}\, dy \right)^{\frac{1}{\kappa^k\beta_0}} \\
&\leqslant c \left(\left(\frac{R}{R-r} \right)^{\frac{2\kappa}{\kappa-1}} \fint_{B(x,R)} u^{\beta_0}\, dy \right)^{\frac{1}{\beta_0}},
\end{aligned}$$

where $c = c(n,\lambda,\Lambda,\beta_0)$. This implies the claim with $\tau = \frac{2\kappa}{\kappa-1}$. The claim follows from this, since we denoted $u = u^+$. □

Remark 4.21. For $n > 2$, by the proof of Theorem 4.17, we may choose $\kappa = \frac{n}{n-2}$ and thus $\tau = \frac{2\kappa}{\kappa-1} = n$. For $n = 2$, the proof gives $\tau = \frac{2\kappa}{\kappa-1} = 4 > 2 = n$.

Corollary 4.22 (Local boundedness). Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution of (4.1) in Ω and let $\beta > 1$. There exists constants $c = c(n,\lambda,\Lambda,\beta)$ and $\tau = \tau(n) > 0$ such that

$$\operatorname*{ess\,sup}_{B(x,r)} |u| \leqslant c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} |u|^{\beta}\,dy\right)^{\frac{1}{\beta}}$$

whenever $B(x,R) \Subset \Omega$, $0 < r < R$.

T H E M O R A L : By choosing $R = 2r$ and $\beta = 2$, we have

$$\operatorname*{ess\,sup}_{B(x,r)} |u| \leqslant c\left(\fint_{B(x,2r)} |u|^{2}\,dy\right)^{\frac{1}{2}}.$$

whenever $B(x,2r) \Subset \Omega$. In particular, every weak solution is locally bounded.

Proof. By Lemma 4.4, $u^+ \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak subsolution and thus by Theorem 4.17 we have $u^+ \in L^{\infty}_{\mathrm{loc}}(\Omega)$ with

$$\operatorname*{ess\,sup}_{B(x,r)} u^+ \leqslant c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}},$$

where $c = c(n,\lambda,\Lambda,\beta)$. On the other hand, since u is a weak solution $-u$ is a weak solution as well. Again by Lemma 4.4, $(-u)^+ = u^- \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak subsolution, and by Theorem 4.17 we have $u^- \in L^{\infty}_{\mathrm{loc}}(\Omega)$ with

$$\operatorname*{ess\,sup}_{B(x,r)} u^- \leqslant c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^-)^{\beta}\,dy\right)^{\frac{1}{\beta}},$$

where $c = c(n,\lambda,\Lambda,\beta)$. This shows that $u = u^+ - u^- \in L^{\infty}_{\mathrm{loc}}(\Omega)$. Moreover,

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(x,r)} |u| &= \operatorname*{ess\,sup}_{B(x,r)} (u^+ + u^-) \leqslant \operatorname*{ess\,sup}_{B(x,r)} u^+ + \operatorname*{ess\,sup}_{B(x,r)} u^- \\
&\leqslant c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}} + c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^-)^{\beta}\,dy\right)^{\frac{1}{\beta}} \\
&\leqslant c\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} |u|^{\beta}\,dy\right)^{\frac{1}{\beta}},
\end{aligned}$$

where $c = c(n,\lambda,\Lambda,\beta)$. □

Remark 4.23. The example at the end of Section 1.5 shows that there may exist unbounded weak solutions, if the assumption $u \in W^{1,2}_{\text{loc}}(\Omega)$ is relaxed. Let us discuss this issue in more detail. We consider the function in (1.33) in the two-dimensional case, that is, $n = 2$ and $0 < \varepsilon < 1$. In this case we have $u : B(0,1) \to \mathbb{R}$,

$$u(x) = u(x_1, x_2) = x_1|x|^{-1-\varepsilon}.$$

We have $u \in W^{1,p}(\Omega)$, for $1 \leqslant p < \frac{2}{1+\varepsilon}$, but $u \notin W^{1,p}(\Omega)$, for $p = \frac{2}{1+\varepsilon}$. In particular $u \notin W^{1,2}(\Omega)$. However, as in Section 1.5, we see that

$$\int_{B(0,1)} \sum_{i,j=1}^{2} a_{ij} D_i u D_j \varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(B(0,1))$, where

$$a_{ij}(x) = \delta_{ij} + (a-1)\frac{x_i x_j}{|x|^2}, \quad i,j = 1,2,$$

and $a = \frac{1}{\varepsilon^2}$. In this sense u is a (very) weak solution to

$$-\sum_{i,j=1}^{2} D_j(a_{ij} D_i u) = 0$$

in $B(0,1)$, but $u \notin W^{1,2}(\Omega)$ for every $0 < \varepsilon < 1$. Clearly the function u is not locally bounded.

The uniform ellipticity condition in Definition 1.7 is satisfied with $\lambda = 1$ and $\Lambda = a$. Observe that $\Lambda > 1$ can be made arbitrarily close to one by choosing $0 < \varepsilon < 1$ close enough to one. Thus for every $\Lambda > 1$, there exists an unbouded (very) weak solution to an elliptic equation.

THE MORAL: The previous examples show that for every $1 \leqslant p < 2$, there exists an unbounded (very) weak solution $u \in W^{1,p}(\Omega)$ to an elliptic equation, in the above sense. This shows that the assumption $u \in W^{1,2}_{\text{loc}}(\Omega)$ in Corollary 4.22 is essentially sharp.

We will next present a technical lemma, which will be used in proving that Theorem 4.17 actually holds for all $\beta > 0$.

Lemma 4.24. Let $\psi : [0,T] \to \mathbb{R}$ be a nonnegative bounded function. If there exists $A > 0$, $\alpha > 0$ and $0 < \varepsilon < 1$ such that

$$\Psi(r) \leqslant A(R-r)^{-\alpha} + \varepsilon \Psi(R)$$

for every $0 \leqslant r < R \leqslant T$, then there exists $c = c(\alpha, \varepsilon)$ such that

$$\Psi(r) \leqslant cA(R-r)^{-\alpha}$$

for every $0 \leqslant r < R \leqslant T$.

Proof. Let $0 < \tau < 1$, $t_0 = r$ and

$$t_{i+1} = t_i + (1-\tau)\tau^i (R-r), \quad i = 0,1,2,\dots.$$

Then $r \leqslant t_i < t_{i+1} \leqslant R$ for every $i = 0,1,2,\dots$ and

$$\begin{aligned}
\Psi(t_0) &\leqslant \varepsilon\Psi(t_1) + A(t_1 - t_0)^{-\alpha} && \text{(assumption)}\\
&= \varepsilon\Psi(t_1) + A(1-\tau)^{-\alpha}(R-r)^{-\alpha} && \text{(definition of } t_i)\\
&\leqslant \varepsilon(\varepsilon\Psi(t_2) + A(t_2-t_1)^{-\alpha}) + A(1-\tau)^{-\alpha}(R-r)^{-\alpha} && \text{(assumption)}\\
&= \varepsilon^2\Psi(t_2) + \varepsilon A\tau^{-\alpha}(1-\tau)^{-\alpha}(R-r)^{-\alpha} + A(1-\tau)^{-\alpha}(R-r)^{-\alpha} && \text{(definition of } t_i)\\
&= \varepsilon^2\Psi(t_2) + A(1-\tau)^{-\alpha}(R-r)^{-\alpha}(\varepsilon\tau^{-\alpha}+1).
\end{aligned}$$

Recursively, we obtain

$$\Psi(r) = \Psi(t_0) \leqslant \varepsilon^k\Psi(t_k) + A(R-r)^{-\alpha}(1-\tau)^{-\alpha}\sum_{i=0}^{k-1}\varepsilon^i\tau^{-i\alpha}$$

for every $k = 1,2,\dots$. Since Ψ is bounded, here $\varepsilon^k\Psi(t_k) \to 0$ as $k \to \infty$. By choosing $\tau = \tau(\varepsilon,\alpha)$ with $\frac{\varepsilon}{\tau^\alpha} < 1$, we conclude that

$$\begin{aligned}
\Psi(r) &\leqslant \lim_{k\to\infty}\left(\varepsilon^k\Psi(t_k) + A(R-r)^{-\alpha}(1-\tau)^{-\alpha}\sum_{i=0}^{k-1}\varepsilon^i\tau^{-i\alpha}\right)\\
&= c(\alpha,\varepsilon)A(R-r)^{-\alpha}.
\end{aligned}$$

Here the first term on the right-hand side converges to zero because Ψ is bounded. □

Lemma 4.25. Theorem 4.17 and Corollary 4.22 hold for every $\beta > 0$.

THE MORAL: We can choose the power $\beta > 0$ as close to zero as we want in Theorem 4.17 and Corollary 4.22. This will be useful in Harnack estimates below.

Proof. We may assume that $0 < \beta \leqslant 1$, since for $\beta > 1$ the results are covered by Theorem 4.17 and Corollary 4.22 respectively. By Remark 4.21 we may also assume that $\tau = \tau(n) \geqslant n$ in Theorem 4.17. Let $B(x,R) \Subset \Omega$, $0 < r < R \leqslant T$. Since $B(x,R) \Subset \Omega$, Theorem 4.17 implies

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(x,r)} u^+ &\leqslant c\left(\left(\frac{R}{R-r}\right)^\tau \fint_{B(x,R)} (u^+)^2\,dy\right)^{\frac12}\\
&\leqslant c\left(\left(\frac{R}{R-r}\right)^\tau \fint_{B(x,R)} (u^+)^\beta(\operatorname*{ess\,sup}_{B(x,R)} u^+)^{2-\beta}\,dy\right)^{\frac12}\\
&= c\left(\left(\frac{R}{R-r}\right)^\tau \fint_{B(x,R)} (u^+)^\beta\,dy\right)^{\frac12}(\operatorname*{ess\,sup}_{B(x,R)} u^+)^{1-\frac{\beta}{2}},
\end{aligned}$$

where $c = c(n, \lambda, \Lambda, \beta)$. Let $0 < \varepsilon < 1$. By Young's inequality

$$\begin{aligned} c(n,\lambda,\Lambda,\beta)&\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^+)^{\beta}\,dy\right)^{\frac{1}{2}} (\operatorname*{ess\,sup}_{B(x,R)} u^+)^{1-\frac{\beta}{2}} \\ &\leqslant \varepsilon \operatorname*{ess\,sup}_{B(x,R)} u^+ + c(n,\lambda,\Lambda,\beta,\varepsilon)\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}} \\ &\leqslant \varepsilon \operatorname*{ess\,sup}_{B(x,R)} u^+ + A(R-r)^{-\frac{\tau}{\beta}}, \end{aligned}$$

where

$$A = c(n,\lambda,\Lambda,\beta,\varepsilon)T^{\frac{\tau-n}{\beta}}\left(\int_{B(x,T)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}} < \infty.$$

Here we used the facts that $\tau \geqslant n$ and that by Theorem 4.17 we have $u^+ \in L^{\infty}_{\mathrm{loc}}(\Omega)$. It is important to use T instead of R above, since x is not allowed to depend on R. Without loss of generality, we may assume that $T > 0$. Let $\Psi(0) = 0$ and

$$\Psi(r) = \operatorname*{ess\,sup}_{B(x,r)} u^+ \leqslant \operatorname*{ess\,sup}_{B(x,T)} u^+ < \infty$$

for every $0 < r \leqslant T$. By Lemma 4.24 we obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x,r)} u^+ &\leqslant c(n,\lambda,\Lambda,\beta,\varepsilon)(R-r)^{-\frac{\tau}{\beta}}T^{\frac{\tau-n}{\beta}}\left(\int_{B(x,T)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}} \\ &\leqslant c(n,\lambda,\Lambda,\beta,\varepsilon)\left(\left(\frac{T}{R-r}\right)^{\tau} \fint_{B(x,T)} (u^+)^{\beta}\,dy\right)^{\frac{1}{\beta}} \end{aligned}$$

whenever $0 < r < R \leqslant T$. By choosing $R = T$ we conclude that Theorem 4.17 holds for every $\beta > 0$. Finally, the proof of Corollary 4.22 together with the knowledge that Theorem 4.17 holds for every $\beta > 0$ shows that also Corollary 4.22 holds for every $\beta > 0$. □

4.5 Estimates from below

The following property of super- and subsolutions gives us a tool to apply Theorem 4.17 to obtain a lower bound for the infimum of supersolutions in terms of negative powers of integral averages.

Lemma 4.26. If $u \geqslant \varepsilon > 0$ is a weak supersolution of (4.1) in Ω, then $v = \frac{1}{u}$ is a weak subsolution in Ω.

Proof. Since $v = \frac{1}{u}$ and $u \geqslant \varepsilon > 0$, we have $0 < \varepsilon^2 v \leqslant u$ and by the chain rule

$$D_i v = -u^{-2} D_i u, \quad i = 1, \dots, n,$$

almost everywhere in Ω. Thus $\varepsilon^2 |Dv| \leqslant |Du|$ almost everywhere in Ω and so $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. If $\psi = u^{-2}\varphi$, then $\psi \in W_0^{1,2}(\Omega)$ and

$$D_j\psi = -2u^{-3}D_ju\varphi + u^{-2}D_j\varphi, \quad j = 1,\dots,n,$$

and

$$\begin{aligned}
0 &\leqslant \int_\Omega \sum_{i,j=1}^n a_{ij}D_iuD_j\psi\,dx \\
&= -2\int_\Omega u^{-3}\sum_{i,j=1}^n a_{ij}D_iuD_ju\varphi\,dx + \int_\Omega \sum_{i,j=1}^n a_{ij}u^{-2}D_iuD_j\varphi\,dx \\
&\leqslant -\int_\Omega \sum_{i,j=1}^n a_{ij}D_ivD_j\varphi\,dx
\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. Here we used the facts that

$$\int_\Omega u^{-3}\sum_{i,j=1}^n a_{ij}D_iuD_ju\varphi\,dx \geqslant \lambda\int_\Omega u^{-3}|Du|^2\varphi\,dx \geqslant 0$$

and $D_iv = -u^{-2}D_iu$, $i = 1,\dots,n$. □

Next we discuss a version of Theorem 4.17 for supersolutions.

Lemma 4.27. Let $u \geqslant 0$ be a weak supersolution of (4.1) in Ω. There exists constants $c = c(n,\lambda,\Lambda,\beta)$ and $\tau = \tau(n) > 0$ such that

$$\left(\left(\frac{R}{R-r}\right)^\tau \fint_{B(x,R)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}} \leqslant c\operatorname*{ess\,inf}_{B(x,r)} u$$

whenever $B(x,R) \Subset \Omega$, $0 < r < R$.

T H E M O R A L : By choosing $R = 2r$ we have

$$\left(\fint_{B(x,2r)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}} \leqslant c\operatorname*{ess\,inf}_{B(x,r)} u$$

whenever $B(x,2r) \Subset \Omega$.

Proof. Without loss of generality, we may assume that

$$\left(\fint_{B(x,R)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}} > 0.$$

Since we can add constants to weak supersolutions, the function $u_k = u + \frac{1}{k}$, $k = 1,2,\dots$, is a weak supersolution. By Lemma 4.26, $\frac{1}{u_k}$, $k = 1,2,\dots$, is a weak subsolution. By Theorem 4.17 and Lemma 4.25, we have

$$\operatorname*{ess\,sup}_{B(x,r)} \frac{1}{u_k} \leqslant c\left(\left(\frac{R}{R-r}\right)^\tau \fint_{B(x,R)} \left(\tfrac{1}{u_k}\right)^\beta\,dy\right)^{\frac{1}{\beta}},$$

where $c = c(n, \lambda, \Lambda, \beta)$, or equivalently

$$\begin{aligned}\left(\left(\frac{R}{R-r}\right)^{\tau} \fint_{B(x,R)} u_k^{-\beta}\,dy\right)^{-\frac{1}{\beta}} &\leqslant c\left(\operatorname*{ess\,sup}_{B(x,r)} \frac{1}{u_k}\right)^{-1}\\ &= c \operatorname*{ess\,inf}_{B(x,r)} u_k\\ &= c\left(\operatorname*{ess\,inf}_{B(x,r)} u + \frac{1}{k}\right).\end{aligned}$$

The claim follows from the monotone convergence theorem by letting $k \to \infty$, since

$$\begin{aligned}0 < \left(\fint_{B(x,R)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}} &= \left(\lim_{k\to\infty} \fint_{B(x,R)} u_k^{-\beta}\,dy\right)^{-\frac{1}{\beta}}\\ &= \lim_{k\to\infty}\left(\fint_{B(x,R)} u_k^{-\beta}\,dy\right)^{-\frac{1}{\beta}}.\end{aligned}$$

□

Remark 4.28. Another way to prove Lemma 4.27 is to run the Moser iteration technique as in the proof of Theorem 4.17 using Theorem 4.9 for weak supersolutions. This approach completely avoids Lemma 4.26 (exercise).

4.6 Harnack's inequality

Recall that Harnack's inequality for nonnegative solutions of the Laplace equation can be proved by the mean value property. If $u \geqslant 0$ is a weak solution to (4.1) in Ω, then by Theorem 4.17 there exist a constant $c = c(n, \lambda, \Lambda, \beta)$ such that

$$\operatorname*{ess\,sup}_{B(x,r)} u \leqslant c\left(\fint_{B(x,2r)} u^{\beta}\,dy\right)^{\frac{1}{\beta}}$$

and by Lemma 4.27 we have

$$\left(\fint_{B(x,2r)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}} \leqslant c \operatorname*{ess\,inf}_{B(x,r)} u,$$

whenever $B(x,2r) \Subset \Omega$. Next we prove the missing inequality

$$\left(\fint_{B(x,2r)} u^{\beta}\,dy\right)^{\frac{1}{\beta}} \leqslant c\left(\fint_{B(x,2r)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}}.$$

THE MORAL: This is a reverse Hölder inequality, since by Hölder's, or Jensen's, inequality we always have

$$\left(\fint_{B(x,r)} u^{-\gamma}\,dy\right)^{-\frac{1}{\gamma}} \leqslant \left(\fint_{B(x,r)} u^{\gamma}\,dy\right)^{\frac{1}{\gamma}}.$$

Reverse Hölder inequalities are very powerful tools in harmonic analysis and PDEs.

With the reverse Hölder inequality above, we obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{B(x,r)} u &\leqslant c\left(\fint_{B(x,2r)} u^{\beta}\,dy\right)^{\frac{1}{\beta}}\\ &\leqslant c\left(\fint_{B(x,2r)} u^{-\beta}\,dy\right)^{-\frac{1}{\beta}} \leqslant c\operatorname*{ess\,inf}_{B(x,r)} u\end{aligned}$$

whenever $B(x,2r) \Subset \Omega$. This is Harnack's inequality for nonnegative weak solutions. Harnack's inequality states that locally the supremum of a positive solution is bounded by a constant times the infimum of the solution. However, since a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is defined only up to a set of measure zero, we consider the essential supremum and infimum. For a continuous function, these can be replaced by the standard supremum and infimum.

The only missing piece is the passage over zero. We shall use the theory of BMO functions, in particular, the John-Nirenberg lemma, to overcome this problem. For the theory of BMO functions we refer to the Harmonic Analysis course.

In the theory of BMO it is more convenient to use cubes instead of balls. This is just a technical point and we could work either with cubes or balls throughout.

Definition 4.29. A closed cube is a bounded interval in $\mathbb{R}^n$, whose sides are parallel to the coordinate axes and equally long, that is,

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

with $b_1 - a_1 = \ldots = b_n - a_n$. The side length of a cube Q is denoted by $l(Q)$. In case we want to specify the center, we write

$$Q(x,l) = \left\{ y \in \mathbb{R}^n : |y_i - x_i| \leqslant \frac{l}{2},\, i = 1, \ldots, n \right\}$$

for a cube with center at $x \in \mathbb{R}^n$ and side length $l > 0$. Clearly

$$|Q(x,l)| = l^n \quad \text{and} \quad \operatorname{diam}(Q(x,l)) = \sqrt{n}\,l.$$

Assume then that $u \geqslant 0$ is a weak solution to (4.1) in Ω. Denote

$$v_k = \log\left(u + \frac{1}{k}\right), \quad k = 1, 2, \ldots.$$

Then $u + \frac{1}{k} \geqslant \frac{1}{k} > 0$, $k = 1, 2, \ldots$, is a solution to (4.1). Let $Q(x,2l) \Subset \Omega$ and take a cutoff function $\varphi \in C_0^\infty(Q(x,2l))$, $0 \leqslant \varphi \leqslant 1$ such that $\varphi = 1$ on $Q(x,l)$ and $|D\varphi| \leqslant \frac{c(n)}{l}$.

By the logarithmic Caccioppoli estimate, Theorem 4.10, we have

$$\begin{aligned}\int_{Q(x,l)} |Dv_k|^2\,dy &= \int_{Q(x,l)} \left|D\log\left(u+\frac{1}{k}\right)\right|^2 dy\\ &\leqslant \int_\Omega \varphi^2 \left|D\log\left(u+\frac{1}{k}\right)\right|^2 dy\\ &\leqslant c(\lambda,\Lambda)\int_\Omega |D\varphi|^2\,dy\\ &\leqslant \frac{c(n,\lambda,\Lambda)}{l^2}\int_{Q(x,2l)} 1\,dy\\ &= c(n,\lambda,\Lambda)l^{n-2}, \quad k=1,2,\dots.\end{aligned}$$

In particular, this implies that $|Dv_k|\in L^2_{\text{loc}}(\Omega)$. On the other hand, since $u+\frac{1}{k}\geqslant\frac{1}{k}$ and $u\in L^2_{\text{loc}}(\Omega)$, we conclude that $v_k\in L^2_{\text{loc}}(\Omega)$. This implies that $v_k\in W^{1,2}_{\text{loc}}(\Omega)$, $k=1,2,\dots$. By the Poincaré inequality we have

$$\begin{aligned}\fint_{Q(x,l)} \left|v_k-(v_k)_{Q(x,l)}\right|^2 dy &\leqslant c(n)l^2\fint_{Q(x,l)} |Dv_k|^2\,dy\\ &\leqslant c(n,\lambda,\Lambda)l^2 l^{-n} l^{n-2} = c(n,\lambda,\Lambda), \quad k=1,2,\dots,\end{aligned}$$

for every cube $Q(x,l)$ such that $Q(x,2l)\Subset\Omega$. By Hölder's, or Jensen's, inequality

$$\fint_{Q(x,l)} \left|v_k-(v_k)_{Q(x,l)}\right| dy \leqslant \left(\fint_{Q(x,l)} \left|v_k-(v_k)_{Q(x,l)}\right|^2 dy\right)^{\frac{1}{2}} \leqslant c<\infty, \quad k=1,2,\dots,$$

for every cube $Q(x,l)$ such that $Q(x,2l)\Subset\Omega$ with $c=c(n,\lambda,\Lambda)$. Observe, that the constant c is independent of u and k. This shows that v_k is of bounded mean oscillation (BMO) over such cubes. By the exponential integrability result for BMO-functions, there exist $\gamma=\gamma(n,\lambda,\Lambda)>0$ and $c=c(n)<\infty$, such that

$$\fint_{Q(x,l)} e^{\gamma|v_k-(v_k)_{Q(x,l)}|}\,dy \leqslant c, \quad k=1,2,\dots,$$

for every cube $Q(x,l)$ such that $Q(x,2l)\Subset\Omega$. This implies that

$$\begin{aligned}\fint_{B(x,r)} e^{\gamma v_k}\,dy \fint_{B(x,r)} e^{-\gamma v_k}\,dy &\leqslant c(n)\fint_{Q(x,2r)} e^{\gamma v_k}\,dy \fint_{Q(x,2r)} e^{-\gamma v_k}\,dy\\ &= c(n)\fint_{Q(x,2r)} e^{\gamma(v_k-(v_k)_{Q(x,2r)})}\,dy \fint_{Q(x,2r)} e^{-\gamma(v_k-(v_k)_{Q(x,2r)})}\,dy\\ &\leqslant c(n)\left(\fint_{Q(x,2r)} e^{\gamma|v_k-(v_k)_{Q(x,2r)}|}\,dy\right)^2 \leqslant c(n), \quad k=1,2,\dots,\end{aligned}$$

whenever $Q(x,4r)\Subset\Omega$. We note that $Q(x,4r)\subset B(x,2\sqrt{n}r)$, so that $Q(x,4r)\Subset\Omega$ if $B(x,2\sqrt{n}r)\Subset\Omega$. Thus the estimate above holds whenever $B(x,2\sqrt{n}r)\Subset\Omega$. Observe that the constants in the estimate above are independent of $k\in\mathbb{N}$. Since $u+1\in L^\beta_{\text{loc}}(\Omega)$, we can apply both dominated and monotone convergence theorems to conclude that

$$\begin{aligned}\left(\fint_{B(x,r)} u^\gamma\,dy\right)^{\frac{1}{\gamma}} &= \lim_{k\to\infty}\left(\fint_{B(x,r)}\left(u+\frac{1}{k}\right)^\gamma dy\right)^{\frac{1}{\gamma}}\\ &= \lim_{k\to\infty}\left(\fint_{B(x,r)} e^{\gamma v_k}\,dy\right)^{\frac{1}{\gamma}}\\ &\leqslant c(n)\lim_{k\to\infty}\left(\fint_{B(x,r)} e^{-\gamma v_k}\,dy\right)^{-\frac{1}{\gamma}}\\ &= c(n)\lim_{k\to\infty}\left(\fint_{B(x,r)}\left(u+\frac{1}{k}\right)^{-\gamma} dy\right)^{-\frac{1}{\gamma}}\\ &= c(n)\left(\fint_{B(x,r)} u^{-\gamma}\,dy\right)^{-\frac{1}{\gamma}}\end{aligned} \tag{4.30}$$

whenever $B(x,2\sqrt{n}r)\Subset\Omega$.

Theorem 4.31 (Harnack's inequality). Assume that $u\geqslant 0$ is a weak solution to (4.1) in Ω. Then there exists a constant $c=c(n,\lambda,\Lambda)$ such that

$$\operatorname*{ess\,sup}_{B(x,r)} u \leqslant c \operatorname*{ess\,inf}_{B(x,r)} u$$

for every ball $B(x,r)$ such that $B(x,2r)\Subset\Omega$.

THE MORAL: Harnack's inequality is a quantitative version of the strong maximum principle. It asserts that if $u\geqslant 0$ is a nontrivial weak solution in $B(x,2r)$, then it does not only hold that $u>0$ in $B(x,r)$ but we also have $u\geqslant c^{-1}\sup_{B(x,r)} u$ in $B(x,r)$.

Proof. By Theorem 4.17, Lemma 4.25, Lemma 4.27 and (4.30), there exist $\gamma=\gamma(n,\lambda,\Lambda)>0$ and $c=c(n,\lambda,\Lambda)$ such that

$$\operatorname*{ess\,sup}_{B(x,r)} u \leqslant c\left(\fint_{B(x,2r)} u^\gamma\,dy\right)^{\frac{1}{\gamma}} \leqslant c\left(\fint_{B(x,2r)} u^{-\gamma}\,dy\right)^{-\frac{1}{\gamma}} \leqslant c \operatorname*{ess\,inf}_{B(x,r)} u \tag{4.32}$$

whenever $B(x,2\sqrt{n}r)\Subset\Omega$.

Assume then that $B(x,r)$ is a ball with $B(x,2r)\Subset\Omega$. We apply a chaining argument. Let $\rho=\frac{r}{2\sqrt{n}}$ and $B_i=B(x_i,\rho)$, $i=1,\dots,N$, be finitely many balls such that $x_i\in B(x,r)$, $i=1,\dots,N$, $B_i\cap B_{i+1}\neq\emptyset, i=1,\dots,N-1$, and $B(x,r)\subset\bigcup_{i=1}^N B_i$.

Then $B(x_i, 2\sqrt{n}\rho) \subset B(x, 2r) \Subset \Omega$, $i = 1, \dots, N$. By (4.32), we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{B_i \cup B_{i+1}} u &= \max\Big\{\operatorname*{ess\,sup}_{B_i} u, \operatorname*{ess\,sup}_{B_{i+1}} u\Big\} \\
&\leqslant c \max\Big\{\operatorname*{ess\,inf}_{B_i} u, \operatorname*{ess\,inf}_{B_{i+1}} u\Big\} \\
&\leqslant c \operatorname*{ess\,inf}_{B_i \cap B_{i+1}} u \leqslant c \operatorname*{ess\,sup}_{B_i \cap B_{i+1}} u \\
&\leqslant c \min\Big\{\operatorname*{ess\,sup}_{B_i} u, \operatorname*{ess\,sup}_{B_{i+1}} u\Big\} \\
&\leqslant c^2 \min\Big\{\operatorname*{ess\,inf}_{B_i} u, \operatorname*{ess\,inf}_{B_{i+1}} u\Big\} \\
&= c^2 \operatorname*{ess\,inf}_{B_i \cup B_{i+1}} u, \quad i = 1, \dots, N-1.
\end{aligned}$$

By applying this estimate recursively, we obtain

$$\operatorname*{ess\,sup}_{B(x,r)} u \leqslant \operatorname*{ess\,sup}_{\bigcup_{i=1}^N B_i} u \leqslant c^N \operatorname*{ess\,inf}_{\bigcup_{i=1}^N B_i} u \leqslant c^N \operatorname*{ess\,inf}_{B(x,r)} u. \qquad \square$$

Remarks 4.33:

(1) Harnacks' inequality is a uniform estimate in the sense that the constant in Harnack's inequality does not depend on the radius of the ball. The requirement $B(x, 2r) \Subset \Omega$ is chosen for convenience, but the proof shows that we could assume $B(x, \sigma r) \Subset \Omega$ for any $\sigma > 1$. In this case the constant in Harnack's inequality also depends on σ.

(2) By a chaining argument Harnack's inequality gives the pointwise estimate

$$\operatorname*{ess\,sup}_{\Omega'} u \leqslant c \operatorname*{ess\,inf}_{\Omega'} u.$$

for almost every points $x, y \in \Omega'$ where $\Omega' \Subset \Omega$ is a connected set. This means that the values of nonnegative weak solution are comparable in Ω'. Thus if u is small (or large) somewhere in Ω' it is small (or large) everywhere in Ω'. In particular, if $u(y) = 0$ for some $y \in \Omega$, then $u(x) = 0$ for every $x \in \Omega$. The assumption $u \geqslant 0$ is essential in the result.

4.7 Local Hölder continuity

Next we shall prove that Harnack's inequality implies that weak solutions of (4.1) are locally Hölder continuous after a possible redefinition on a set of measure zero. Observe that a weak solution belongs to $W^{1,2}_{\text{loc}}(\Omega)$ and is defined only up to a set of measure zero and a function in $W^{1,2}_{\text{loc}}(\Omega)$ is not necessarily continuous.

Assume $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to (4.1) and $B(x, 2R) \Subset \Omega$. We denote

$$m(r) = \operatorname*{ess\,inf}_{B(x,r)} u,$$

$$M(r) = \operatorname*{ess\,sup}_{B(x,r)} u,$$

and the oscillation of u by

$$\operatorname*{osc}_{B(x,r)} u = M(r) - m(r).$$

By Corollary 4.22, we have $-\infty < m(r) \leqslant M(r) < \infty$ for $0 < r \leqslant R$. Since constants can be added to weak solutions, we see that the functions $u - m(2r)$ and $M(2r) - u$ are weak solutions of (4.1) as well. Notice that $u - m(2r) \geqslant 0$ and $M(2r) - u \geqslant 0$ in $B(x,2r)$. By Harnack's inequality, Theorem 4.31, there exists a constant $c = c(n,\lambda,\Lambda) > 1$ such that

$$\begin{aligned} M(r) - m(2r) &= \operatorname*{ess\,sup}_{B(x,r)} (u - m(2r)) \\ &\leqslant c \operatorname*{ess\,inf}_{B(x,r)} (u - m(2r)) \\ &= c(m(r) - m(2r)) \end{aligned}$$

for $0 < r \leqslant R$. A similar argument gives

$$\begin{aligned} M(2r) - m(r) &= \operatorname*{ess\,sup}_{B(x,r)} (M(2r) - u) \\ &\leqslant c \operatorname*{ess\,inf}_{B(x,r)} (M(2r) - u) \\ &= c(M(2r) - M(r)) \end{aligned}$$

for $0 < r \leqslant R$. By combining these estimates we have

$$M(r) - m(2r) + M(2r) - m(r) \leqslant c\,(m(r) - m(2r) + M(2r) - M(r)),$$

which is implies

$$M(r) - m(r) \leqslant \frac{c-1}{c+1}(M(2r) - m(2r)),$$

that is,

$$\operatorname*{osc}_{B(x,r)} u \leqslant \gamma \operatorname*{osc}_{B(x,2r)} u \tag{4.34}$$

for $\gamma = \frac{c-1}{c+1}$ with $0 < \gamma < 1$. This is an oscillation decay estimate.

THE MORAL: Harnacks' inequality implies oscillation decay.

For $0 < r \leqslant R$, we may choose i such that

$$\frac{R}{2^{i+1}} < r \leqslant \frac{R}{2^i}.$$

Then by iterating (4.34), we obtain

$$\operatorname*{osc}_{B(x,r)} u \leqslant \operatorname*{osc}_{B(x,\frac{R}{2^i})} u \leqslant \gamma^i \operatorname*{osc}_{B(x,R)} u \leqslant \frac{1}{\gamma}\left(\frac{r}{R}\right)^{\alpha} \operatorname*{osc}_{B(x,R)} u, \tag{4.35}$$

where $\gamma = \gamma(n,\lambda,\Lambda)$, $\alpha = -\frac{\log\gamma}{\log 2}$, $0 < r \leqslant R$ and $B(x,2R) \Subset \Omega$. The last inequality follows, because

$$\frac{r}{R} \geqslant \frac{1}{2^{i+1}} = \left(\frac{1}{2}\right)^{i+1}$$

implies that

$$i \geqslant \frac{\log\frac{r}{R}}{\log\frac{1}{2}} - 1,$$

and as $\gamma < 1$, we have

$$\gamma^i = e^{i\log\gamma} \leqslant e^{\left(\frac{\log(r/R)}{\log(1/2)}-1\right)\log\gamma} = \frac{1}{\gamma}\left(\frac{r}{R}\right)^{\alpha}.$$

Let $B(z,10r) \Subset \Omega$ and $x,y \in B(z,r)$, $x \neq y$. Denote $R = 4r$. Then $0 < 2|x-y| < 2\cdot 2r = 4r = R$ and $B(x,2R) = B(x,2\cdot 4r) = B(x,8r) \subset B(z,9r) \Subset \Omega$. By (4.35), there exists a constant $c = c(n,\lambda,\Lambda)$, such that

$$\begin{aligned}
|u(x)-u(y)| &\leqslant \operatorname*{osc}_{B(x,2|x-y|)} u \leqslant c\left(\frac{2|x-y|}{4r}\right)^{\alpha} \operatorname*{osc}_{B(x,4r)} u \\
&\leqslant c\left(\frac{|x-y|}{r}\right)^{\alpha} \operatorname*{osc}_{B(z,5r)} u \leqslant c\left(\frac{|x-y|}{r}\right)^{\alpha} \operatorname*{ess\,sup}_{B(z,5r)} |u|,
\end{aligned}$$

for almost every $x,y \in B(z,r)$. By Corollary 4.22, there exists $c = c(n,\lambda,\Lambda)$ such that

$$\operatorname*{ess\,sup}_{B(x_0,5r)} |u| \leqslant c\left(\fint_{B(x_0,10r)} |u|^2\,dy\right)^{\frac{1}{2}} < \infty.$$

Thus for every $z \in \Omega$ there exists $r = r(z) > 0$ such that $B(z,10r) \Subset \Omega$ and a constant $c = c(n,\lambda,\Lambda,z)$ such that

$$|u(x)-u(y)| \leqslant c|x-y|^{\alpha},$$

for almost every $x,y \in B(z,r)$. Observe that r and c may depend on z, but $\alpha = \alpha(n,\lambda,\Lambda)$ is independent of z. This implies that u is locally Hölder continuous by redefining it on a set of measure zero. The argument to show that there exists a locally Hölder continuous representative is similar as in the proof of Morrey's inequality.

Theorem 4.36 (Local Hölder continuity). Every weak solution of (4.1) is locally Hölder continuous.

THE MORAL: Oscillation decay implies local Hölder continuity.

Remark 4.37. The example in Section 1.5 shows that for every $0 < \alpha < 1$ there exists a weak solution to an elliptic equation such that the weak gradient is unbounded. This shows that local Hölder continuity is essentially the best regularity result we can hope for a general elliptic equation with bounded and measurable coefficients.

Remark 4.38. Let $\Omega = B(0,r)$ and $y \in \partial B(0,r)$. The Poisson kernel for the ball $B(0,r)$ gives the function

$$u(x) = \frac{1}{n\alpha(n)r}\frac{r^2-|x|^2}{|x-y|^n}, \quad x \in B(0,r).$$

Then $\Delta u(x) = 0$ for every $x \in \Omega$, but is not Hölder continuous in Ω for any $0 < \alpha \leqslant 1$.

Reason. If u is Hölder continuous in Ω for some $0 < \alpha \leqslant 1$, then it is Hölder continuous in $\overline{\Omega}$ with the same α. This implies that $u \in L^\infty(\Omega)$. This is not possible, since $u \notin L^\infty(\Omega)$. However, u is locally Hölder continuous in Ω. ■

THE MORAL: Weak solutions are locally Hölder continuous, but not in general Hölder continuous in the whole domain.

Finally we show that Harnack's inequality implies that weak solutions of (4.1) satisfy the strong maximum principle.

Theorem 4.39 (Strong maximum principle). If a weak solution of (4.1) attains its maximum in a connected open set Ω, then it is a constant function.

Proof. If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x\in\Omega} u(x),$$

then $u(x_0) - u(x)$ is a nonnegative weak solution in Ω. By Harnack's inequality, Theorem 4.31, we have

$$\sup_{x\in B(x_0,r)} (u(x_0)-u(x)) \leqslant c \min_{x\in B(x_0,r)} (u(x_0)-u(x)) = 0$$

whenever $B(x_0,2r) \Subset \Omega$. Thus $u(x_0) - u(x) = 0$ for every $x \in B(x_0,r)$.

Let $x \in \Omega$. Since Ω is connected, a point x can be connected to the point x_0 with a finite chain of balls $B(x_i,r_i)$, $i = 0,1,\dots,N$, such that $x_N = x$ and

$$B(x_i,r_i) \cap B(x_{i+1},r_{i+1}) \neq \emptyset, \quad i = 1,\dots,N-1$$

and $B(x_i,2r) \subset \Omega$ for every $i = 0,1,\dots,N$. By using Harnack's inequality in every ball, we have $u(x) = u(x_0)$. □

Remarks 4.40:

(1) An analogous argument gives a strong minimum principle as well.

(2) The strong maximum principle implies the standard maximum principle: if $u \in C(\overline{\Omega})$ is a weak solution in a bounded open set Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Theorem 4.41 (Comparison principle). Let u and v be weak solutions in Ω. By Theorem 4.36 we may assume that they both are continuous functions in Ω. If $\Omega' \Subset \Omega$ and $u \leqslant v$ on $\partial\Omega'$ then $u \leqslant v$ in Ω'.

Proof. As $u \leqslant v$ on $\partial\Omega'$, $u - v \leqslant 0$ on $\partial\Omega'$. The partial differential equation in (4.1) is linear and therefore $u - v$ is a weak solution in Ω. The maximum principle, Theorem 4.39, implies that

$$\max_{\overline{\Omega}'}(u-v) = \max_{\partial\Omega'}(u-v) \leqslant 0.$$

Therefore $u - v \leqslant 0$ in Ω' and thus $u \leqslant v$ in Ω'. □

Remark 4.42. This argument uses the linearity, but the result holds true also for certain nonlinear partial differential equations.

5 Gradient estimates

5.1 Maximal functions

In this section it is more convenient to use cubes instead of balls. A closed cube is a bounded interval in $\mathbb{R}^n$, whose sides are parallel to the coordinate axes and equally long, see Definition 4.29. First we introduce an appropriate maximal function on cubes.

Definition 5.1. Let $1 \leqslant p < \infty$, let $Q_0 \subset \mathbb{R}^n$ be a cube, and assume that $f \in L^p(Q_0)$ is a nonnegative function. The noncentered maximal function $M^p_{Q_0} f$ on Q_0 is defined as

$$M^p_{Q_0} f(x) = \sup_{Q \ni x} \left(\fint_Q f(y)^p \, dy \right)^{\frac{1}{p}},$$

where the supremum is taken over all cubes $Q \subset Q_0$ with $x \in Q$.

For $p = 1$, we have the standard Hardy-Littlewood maximal function on Q_0 and we denote $M^1_{Q_0} f = M_{Q_0} f$. Observe that

$$M^p_{Q_0} f(x) = \big(M_{Q_0}(f^p)(x)\big)^{\frac{1}{p}}$$

for every $x \in Q_0$, so that in principle it would be enough to consider the standard Hardy-Littlewood maximal function. However, the new notation turns out to be useful below. For a sign changing function f, we consider $|f|$ in the definition above. By the Lebesgue differentiation theorem and Hölder's inequality

$$f(x) \leqslant M^p_{Q_0} f(x) \leqslant M^q_{Q_0} f(x), \quad 1 \leqslant p \leqslant q < \infty,$$

for almost every $x \in Q_0$. Let $f, g \in L^1(Q_0)$ and $x \in Q_0$. It follows immediately from the definition that $M_{Q_0} f(x) \geqslant 0$,

$$M^p_{Q_0}(f+g)(x) \leqslant M^p_{Q_0} f(x) + M^p_{Q_0} g(x),$$

and

$$M^p_{Q_0}(af)(x) = aM^p_{Q_0}f(x)$$

for every $a \geqslant 0$. It is enough to assume that $f : Q_0 \to [0,\infty]$ is a measurable function in the definition above, but the assumption $f \in L^p(Q_0)$ guarantees that the integral averages are finite. We prove a weak type estimate for the maximal function.

Lemma 5.2. Let $1 \leqslant p < \infty$ and let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $f \in L^p(Q_0)$ is a nonnegative function. There exist a constant $c = c(n)$ (we may take $c(n) = 2\cdot 5^n$) such that

$$|\{x \in Q_0 : M^p_{Q_0}f(x) > t\}| \leqslant \frac{c}{t^p}\int_{\{x\in Q_0 : f(x) > \frac{t}{2}\}} f(x)^p\,dx \tag{5.3}$$

for every $t > 0$.

Proof. Let $E_t = \{x \in Q_0 : M^p_{Q_0}f(x) > t\}$. By the definition of the maximal function, for every $x \in E_t$, there exits a cube Q_x such that $x \in Q_x \subset Q_0$ and

$$\left(\fint_{Q_x} f(y)^p\,dy\right)^{\frac{1}{p}} > t.$$

Thus $\mathscr{F} = \{Q_x : x \in E_t\}$ is a collection of subcubes of Q_0 and

$$E_t \subset \bigcup_{Q\in\mathscr{F}} Q.$$

By a Vitali type covering theorem, there exists a countable subcollection of pairwise disjoint cubes $Q(x_i,l_i) \in \mathscr{F}$, $i = 1,2,\dots$, such that

$$\bigcup_{Q\in\mathscr{F}} Q \subset \bigcup_{i=1}^{\infty} Q(x_i,5l_i).$$

Thus

$$\begin{aligned}
|\{x \in Q_0 : M^p_{Q_0}f(x) > t\}| &\leqslant \left|\bigcup_{i=1}^{\infty} Q(x_i,5l_i)\right| = 5^n\sum_{i=1}^{\infty}|Q(x_i,l_i)| \\
&\leqslant \frac{5^n}{t^p}\sum_{i=1}^{\infty}\int_{Q(x_i,l_i)} f(y)^p\,dy \\
&\leqslant \frac{5^n}{t^p}\int_{Q_0} f(y)^p\,dy.
\end{aligned} \tag{5.4}$$

Let

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } f(x) \geqslant \frac{t}{2}, \\ 0, & \text{otherwise.}\end{cases}$$

Then $f \leqslant \tilde{f} + \frac{t}{2}$, which implies that

$$M^p_{Q_0}f(x) \leqslant M^p_{Q_0}\left(\tilde{f} + \frac{t}{2}\right)(x) \leqslant M^p_{Q_0}\tilde{f}(x) + \frac{t}{2}$$

for every $x \in \mathbb{R}^n$. Consequently,

$$\{x \in Q_0 : M^p_{Q_0} f(x) > t\} \subset \left\{x \in Q_0 : M^p_{Q_0} \widetilde{f}(x) > \frac{t}{2}\right\}.$$

Observe that

$$\widetilde{f}(x) = \chi_{\{y\in Q_0:\widetilde{f}(y)\neq 0\}}(x)\widetilde{f}(x)^{1-p}\widetilde{f}(x)^p \leqslant \left(\frac{t}{2}\right)^{1-p} f(x)^p,$$

which implies $\widetilde{f} \in L^1(Q_0)$. By (5.4) we have

$$\begin{aligned} |\{x \in Q_0 : M^p_{Q_0} f(x) > t\}| &\leqslant \left|\left\{x \in Q_0 : M^p_{Q_0} \widetilde{f}(x) > \frac{t}{2}\right\}\right| \\ &\leqslant \frac{2\cdot 5^n}{t} \int_{Q_0} \widetilde{f}(x)^p\,dx \\ &= \frac{2\cdot 5^n}{t} \int_{\{x\in Q_0:f(x)\geqslant \frac{t}{2}\}} f(x)^p\,dx. \end{aligned} \tag{5.5}$$

□

The following Calderón-Zygmund decomposition will be extremely useful in harmonic analysis.

Theorem 5.6. Let Q_0 be a cube in $\mathbb{R}^n$ and assume that $f \in L^1(Q_0)$ is a nonnegative function. Then for every $t \geqslant \fint_{Q_0} f(y)\,dy$ there are countably many subcubes Q_i, $i = 1,2,\dots$, of Q_0 such that

(1) the interiors of Q_i, $i = 1,2,\dots$, are pairwise disjoint,

(2) $t < \fint_{Q_i} f(y)\,dy \leqslant 2^n t$ for every $i = 1,2,\dots$ and

(3) $f(x) \leqslant t$ for almost every $x \in Q_0 \setminus \bigcup_{i=1}^\infty Q_i$.

The collection of cubes Q_i, $i = 1,2,\dots$, is called the Calderón-Zygmund cubes in Q_0 at level t.

T H E   M O R A L : A cube can be divided into good and bad parts so that in the good part (complement of the Calderón-Zygmund cubes) the function is small and in the bad part (union of the Calderón-Zygmund cubes) the integral average of a function is in control. Note that the Calderón-Zygmund cubes cover the set $\{x \in Q : |f(x)| > t\}$, up to a set of measure zero, and thus the bad part contains the set where the function is unbounded. Next we discuss a reverse weak type inequality for this maximal operator.

Lemma 5.7. Let $1 \leqslant p < \infty$ and let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $f \in L^p(Q_0)$ is a nonnegative function. There exists a constant $c = c(n)$ (we may take $c(n) = 2^n$) such that

$$\int_{\{x\in Q_0:f(x)>t\}} f(x)^p\,dx \leqslant ct^p|\{x \in Q_0 : M^p_{Q_0} f(x) > t\}| \tag{5.8}$$

whenever $t^p \geqslant \fint_{Q_0} f(y)^p\,dy$.

Proof. By the Calderón-Zygmund lemma for f^p at the level t^p and obtain a collection subcubes Q_i, $i = 1,2,\dots$, of Q_0 such that the interiors of Q_i, $i = 1,2,\dots$, are pairwise disjoint,

$$t^p < \fint_{Q_i} f(y)^p \, dy \leqslant 2^n t^p \quad \text{for every} \quad i = 1,2,\dots$$

and

$$f(x) \leqslant t \quad \text{for almost every} \quad x \in Q_0 \setminus \bigcup_{i=1}^{\infty} Q_i.$$

This implies

$$\begin{aligned}\int_{\{x \in Q_0 : f(x) > t\}} f(x)^p \, dx &\leqslant \sum_{i=1}^{\infty} \int_{Q_i} f(x)^p \, dx = \sum_{i=1}^{\infty} |Q_i| \fint_{Q_i} f(x)^p \, dx \\ &\leqslant 2^n t^p \sum_{i=1}^{\infty} |Q_i| = 2^n t^p \left| \bigcup_{i=1}^{\infty} Q_i \right| \\ &\leqslant 2^n t^p |\{x \in Q_0 : M_{Q_0} f^p(x) > t^p\}|.\end{aligned}$$

In the last inequality we applied the fact that $M_{Q_0} f^p(x) > t^p$ if $x \in Q \in \mathscr{D}_{t^p}$. □

5.2 A general self-improvement result

Let $E \subset \mathbb{R}^n$ be a μ-measurable set with $\mu(E) < \infty$ and let f be a nonnegative μ-measurable function on E. For short, we denote the distribution set as

$$\{f > t\} = \{x \in E : f(x) > t\}.$$

By Cavalieri's principle, we have

$$\int_E f(x)^q \, d\mu(x) = q \int_0^{\infty} t^{q-1} \mu(\{f > t\}) \, dt, \quad 0 < q < \infty.$$

Next we discuss a truncated version of Cavalieri's principle.

Lemma 5.9. Let μ be a measure in $\mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a μ-measurable set with $\mu(E) < \infty$ and let f be a nonnegative μ-measurable function on E. For $0 < q < \infty$ and $0 \leqslant t_0 \leqslant t_1 < \infty$, we have

$$\begin{aligned}&\int_{\{t_0 < f \leqslant t_1\}} f(x)^q \, d\mu(x) \\ &\quad = q \int_{t_0}^{t_1} t^{q-1} \mu(\{f > t\}) \, dt + t_0^q \mu(\{f > t_0\}) - t_1^q \mu(\{f > t_1\}).\end{aligned} \tag{5.10}$$

Proof. Cavalieri's principle implies

$$\int_{\{t_0 < f \leqslant t_1\}} f(x)^q \, d\mu(x) = q \int_0^{\infty} t^{q-1} \mu(\{t_0 < f \leqslant t_1\} \cap \{f > t\}) \, dt,$$

where

$$\begin{aligned}
&\int_0^\infty t^{q-1}\mu(\{t_0 < f \leqslant t_1\} \cap \{f > t\})\,dt \\
&\quad = \int_0^{t_0} t^{q-1}\mu(\{t_0 < f \leqslant t_1\})\,dt + \int_{t_0}^{t_1} t^{q-1}\mu(\{t < f \leqslant t_1\})\,dt \\
&\quad = \frac{t_0^q}{q}\mu(\{t_0 < f \leqslant t_1\}) + \int_{t_0}^{t_1} t^{q-1}\mu(\{t < f \leqslant t_1\})\,dt.
\end{aligned}$$

Since $\{t < f \leqslant t_1\} = \{f > t\} \setminus \{f > t_1\}$ and the measures of the sets are finite by the assumption $\mu(E) < \infty$, we obtain

$$\mu(\{t < f \leqslant t_1\}) = \mu(\{f > t\}) - \mu(\{f > t_1\})$$

for every $t_0 \leqslant t \leqslant t_1$. Consequently

$$\begin{aligned}
&\int_{t_0}^{t_1} t^{q-1}\mu(\{t < f \leqslant t_1\})\,dt \\
&\quad = \int_{t_0}^{t_1} t^{q-1}\mu(\{f > t\})\,dt - \mu(\{f > t_1\})\int_{t_0}^{t_1} t^{q-1}\,dt \\
&\quad = \int_{t_0}^{t_1} t^{q-1}\mu(\{f > t\})\,dt - \frac{t_1^q - t_0^q}{q}\mu(\{f > t_1\}).
\end{aligned}$$

The claim follows by combining the equations above. □

The next lemma is a core of the self-improving result for reverse Hölder inequalities.

Lemma 5.11. Let $1 < p < \infty$ and let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $f, g \in L^p(Q_0)$ are nonnegative functions and that there exist $t_0 \geqslant 0$ and $c_1 > 1$ such that

$$\int_{\{f>t\}} f(x)^p\,dx \leqslant c_1\left(t^{p-1}\int_{\{f>t\}} f(x)\,dx + \int_{\{g>t\}} g(x)^p\,dx\right) \tag{5.12}$$

for every $t_0 \leqslant t < \infty$. Let $q > p$ with $c_1 \frac{q-p}{q-1} < 1$. Then there exists a constant $c = c(p,q,c_1)$ such that

$$\int_{Q_0} f(x)^q\,dx \leqslant c\left(t_0^{q-p}\int_{Q_0} f(x)^p\,dx + \int_{Q_0} g(x)^q\,dx\right).$$

T H E M O R A L : We assume that $f, g \in L^p(Q_0)$ satisfy a uniform estimate over the distribution sets in (5.12) for every $t \geqslant t_0$. This implies that $f \in L^q(Q_0)$ for some $q > p$ and, consequently f is integrable to a higher power than assumed in the beginning. This is an example of a phenomenon called higher integrability or self-improvement.

Proof. Clearly

$$\begin{aligned}
\int_{Q_0} f(x)^q\,dx &= \int_{\{f\leqslant t_0\}} f(x)^q\,dx + \int_{\{f>t_0\}} f(x)^q\,dx \\
&\leqslant t_0^{q-p}\int_{\{f\leqslant t_0\}} f(x)^p\,dx + \int_{\{f>t_0\}} f(x)^q\,dx.
\end{aligned} \tag{5.13}$$

It suffices to estimate the second integral on the right-hand side. Let $t_1 > t_0$. Using equation (5.10) with the exponent $q-p>0$ and the measure $\mu(E)=\int_E f^p\,dx$ for a measurable $E\subset\mathbb{R}^n$, we obtain

$$\begin{aligned}\int_{\{t_0<f\leqslant t_1\}} f(x)^q\,dx &= \int_{\{t_0<f\leqslant t_1\}} f(x)^{q-p}\,d\mu(x)\\ &= (q-p)\int_{t_0}^{t_1} t^{q-p-1}\int_{\{f>t\}} f(x)^p\,dx\,dt\\ &\quad + t_0^{q-p}\int_{\{f>t_0\}} f(x)^p\,dx - t_1^{q-p}\int_{\{f>t_1\}} f(x)^p\,dx.\end{aligned}$$

Assumption (5.12) implies

$$\begin{aligned}&\int_{t_0}^{t_1} t^{q-p-1}\int_{\{f>t\}} f(x)^p\,dx\,dt\\ &\quad\leqslant c_1\int_{t_0}^{t_1} t^{q-2}\int_{\{f>t\}} f(x)\,dx\,dt + c_1\int_{t_0}^{t_1} t^{q-p-1}\int_{\{g>t\}} g(x)^p\,dx\,dt.\end{aligned}$$

By (5.10), with the exponent $q-1>0$ and the measure $\mu(E)=\int_E f\,dx$ for a measurable $E\subset\mathbb{R}^n$, we obtain

$$\begin{aligned}&\int_{t_0}^{t_1} t^{q-2}\int_{\{f>t\}} f(x)\,dx\,dt\\ &\quad\leqslant \frac{1}{q-1}\left(\int_{\{t_0<f\leqslant t_1\}} f(x)^q\,dx + t_1^{q-1}\int_{\{f>t_1\}} f(x)\,dx\right).\end{aligned}$$

On the other hand, with the measure $\mu(E)=\int_E g^p\,dx$ for a measurable $E\subset\mathbb{R}^n$, we have

$$\begin{aligned}\int_{t_0}^{t_1} t^{q-p-1}\int_{\{g>t\}} g(x)^p\,dx\,dt &\leqslant \int_0^\infty t^{q-p-1}\mu(\{g>t\})\,dt\\ &= \frac{1}{q-p}\int_{Q_0} g(x)^{q-p}\,d\mu(x)\\ &= \frac{1}{q-p}\int_{Q_0} g(x)^q\,dx.\end{aligned}$$

Consequently

$$\begin{aligned}\int_{\{t_0<f\leqslant t_1\}} f(x)^q\,dx &\leqslant c_1\frac{q-p}{q-1}\int_{\{t_0<f\leqslant t_1\}} f(x)^q\,dx + t_0^{q-p}\int_{\{f>t_0\}} f(x)^p\,dx\\ &\quad + \left(c_1\frac{q-p}{q-1}-1\right)t_1^{q-p}\int_{\{f>t_1\}} f(x)^p\,dx + c_1\int_{Q_0} g(x)^q\,dx,\end{aligned}\tag{5.14}$$

where we also applied the estimate

$$\int_{\{f>t_1\}} f(x)\,dx \leqslant \int_{\{f>t_1\}} f(x)\left(\frac{f(x)}{t_1}\right)^{p-1}dx = t_1^{1-p}\int_{\{f>t_1\}} f(x)^p\,dx.$$

Since

$$\int_{\{t_0<f\leqslant t_1\}} f(x)^q\,dx \leqslant t_1^q\big|\{t_0<f\leqslant t_1\}\big| \leqslant t_1^q|Q_0|<\infty,$$

we obtain from (5.14) that

$$\left(1-c_1\frac{q-p}{q-1}\right)\int_{\{t_0<f\leqslant t_1\}} f(x)^q\,dx \leqslant t_0^{q-p}\int_{\{f>t_0\}} f(x)^p\,dx + \left(c_1\frac{q-p}{q-1}-1\right)t_1^{q-p}\int_{\{f>t_1\}} f(x)^p\,dx + c_1\int_{Q_0} g(x)^q\,dx.$$

Here $0<1-c_1\frac{q-p}{q-1}<1$, and thus

$$\begin{aligned}\int_{\{t_0<f\leqslant t_1\}} & f(x)^q\,dx \\ &\leqslant ct_0^{q-p}\int_{\{f>t_0\}} f(x)^p\,dx - t_1^{q-p}\int_{\{f>t_1\}} f(x)^p\,dx + c\int_{Q_0} g(x)^q\,dx \\ &\leqslant ct_0^{q-p}\int_{\{f>t_0\}} f(x)^p\,dx + c\int_{Q_0} g(x)^q\,dx,\end{aligned}$$

with $c=c(p,q,c_1)\geqslant 1$. This upper bound does not depend on t_1, and by letting $t_1\to\infty$ and using Fatou's lemma, we obtain

$$\int_{\{f>t_0\}} f(x)^q\,dx \leqslant ct_0^{q-p}\int_{\{f>t_0\}} f(x)^p\,dx + c\int_{Q_0} g(x)^q\,dx.$$

Finally, by (5.13), we arrive at

$$\int_{Q_0} f(x)^q\,dx \leqslant c(p,q,c_1)\left(t_0^{q-p}\int_{Q_0} f(x)^p\,dx + \int_{Q_0} g(x)^q\,dx\right),$$

which is the required estimate. □

5.3 Reverse Hölder inequalities

Next we discuss reverse Hölder inequalities. Let $1<p<\infty$ and let $Q_0\subset\mathbb{R}^n$ be a cube. Assume that $f\in L^p(Q_0)$ is a nonnegative function. By Hölder's or Jensen's inequality

$$\fint_Q f(x)\,dx \leqslant \left(\fint_Q f(x)^p\,dx\right)^{\frac{1}{p}}$$

for every cube $Q\subset Q_0$. We are interested in functions that satisfy an inequality to the reverse direction. Assume that there exists a constant c_1 such that

$$\left(\fint_Q f(x)^p\,dx\right)^{\frac{1}{p}} \leqslant c_1\fint_Q f(x)\,dx$$

for every cube $Q\subset Q_0$. This kind of functions occur in Harnack's inequality for nonnegative weak solutions of a PDE, see Theorem 4.31 and in the theory of Muckenhoupt weights in harmonic analysis. By the following Gehring lemma, a uniform reverse Hölder inequality implies a stronger uniform reverse Hölder inequality.

Theorem 5.15 (The Gehring lemma (1973)). Let $1 < p < \infty$ and let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $f \in L^p(Q_0)$ is a nonnegative function and that there exists a constant c_1 such that

$$\left(\fint_Q f(x)^p \, dx \right)^{\frac{1}{p}} \leqslant c_1 \fint_Q f(x) \, dx \tag{5.16}$$

for every cube $Q \subset Q_0$. Then there exist an exponent $q = q(n, p, c_1) > p$ and a constant $c = c(n, p, c_1)$ such that

$$\left(\fint_Q f(x)^q \, dx \right)^{\frac{1}{q}} \leqslant c \fint_Q f(x) \, dx \tag{5.17}$$

for every cube $Q \subset Q_0$.

THE MORAL: A uniform reverse Hölder inequality is self-improving. Since we assume that $f \in L^1(Q_0)$, by (5.16) we have $f \in L^p(Q_0)$. By (5.17) we have $f \in L^q(Q_0)$ for some $q > p$ and, consequently f is integrable to a higher power than assumed in the beginning. Gehring's lemma applies to $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ that satisfy (5.16) for every cube $Q \subset \mathbb{R}^n$. This phenomenon is called local higher integrability.

Proof. It suffices to prove that (5.17) holds for $Q = Q_0$. We may clearly assume that $f_{Q_0} > 0$. Let $M^p_{Q_0} f$ be the maximal function in Definition 5.1. By the reverse Hölder inequality in (5.16), we have

$$M^p_{Q_0} f(x) \leqslant c_1 M_{Q_0} f(x) \quad \text{for every } x \in Q_0. \tag{5.18}$$

By Lemma 5.7 and (5.18), we have

$$\begin{aligned} \int_{\{x \in Q_0 : f(x) > t\}} f(x)^p \, dx &\leqslant 2^n t^p |\{x \in Q_0 : M^p_{Q_0} f(x) > t\}| \\ &\leqslant 2^n t^p \left| \left\{ x \in Q_0 : M_{Q_0} f(x) > \frac{t}{c_1} \right\} \right| \end{aligned}$$

for every

$$t \geqslant t_0 = \left(\fint_{Q_0} f(x)^p \, dx \right)^{\frac{1}{p}}. \tag{5.19}$$

On the other hand, by Lemma 5.2, we have

$$\left| \left\{ x \in Q_0 : M_{Q_0} f(x) > \frac{t}{c_1} \right\} \right| \leqslant c_1 \frac{2 \cdot 5^n}{t} \int_{\{x \in Q_0 : f(x) > \frac{t}{2}\}} f(x) \, dx$$

for every $t > 0$. Denote $F_t = \{x \in Q_0 : f(x) > t\}$. Then

$$\begin{aligned} \int_{F_t} f(x)^p \, dx &\leqslant 2^n t^p \left| \left\{ x \in Q_0 : M_{Q_0} f(x) > \frac{t}{c_1} \right\} \right| \\ &\leqslant c_1 2^{n+1} 5^n t^{p-1} \int_{F_{\frac{t}{2}}} f(x) \, dx \\ &= c t^{p-1} \int_{F_{\frac{t}{2}}} f(x) \, dx \end{aligned}$$

with $c = c(n,p,c_1)$.

On the other hand, we have

$$\int_{F_{\frac{t}{2}}\setminus F_t} f(x)^p\,dx = \int_{F_{\frac{t}{2}}\setminus F_t} f(x)^{p-1} f(x)\,dx \leqslant t^{p-1}\int_{F_{\frac{t}{2}}} f(x)\,dx,$$

and combination of the estimates above shows that

$$\int_{F_{\frac{t}{2}}} f(x)^p\,dx \leqslant c(n,c_1)\left(\frac{t}{2}\right)^{p-1}\int_{F_{\frac{t}{2}}} f(x)\,dx$$

for $t \geqslant t_0$. We apply Lemma 5.11, with $g = 0$, and obtain $q = q(n,p,c_1) > p$ and a constant $c = c(n,p,c_1)$ such that

$$\int_{Q_0} f(x)^q\,dx \leqslant c t_0^{q-p}\int_{Q_0} f(x)^p\,dx.$$

Finally (5.19) and (5.16) give

$$\begin{aligned}
\int_{Q_0} f(x)^q\,dx &\leqslant c\left(\fint_{Q_0} f(x)^p\,dx\right)^{\frac{q}{p}-1}\int_{Q_0} f(x)^p\,dx \\
&\leqslant c\left(\fint_{Q_0} f(x)^p\,dx\right)^{\frac{q}{p}}|Q_0|\left(\int_{Q_0} f(x)^p\,dx\right)^{-1}\int_{Q_0} f(x)^p\,dx \\
&= c\left(\fint_{Q_0} f(x)^p\,dx\right)^{\frac{q}{p}}|Q_0| \\
&\leqslant c|Q_0|\left(\fint_{Q_0} f(x)\,dx\right)^{q}.
\end{aligned} \tag{5.20}$$

This proves (5.17) for Q_0. □

For the gradient of a weak solution to a PDE, we usually have a weaker reverse Hölder inequality of type

$$\left(\fint_{Q(z,l)} f(x)^p\,dx\right)^{\frac{1}{p}} \leqslant c_1\fint_{Q(z,2l)} f(x)\,dx$$

for every cube $Q(z,l)$ with $Q(z,2l)\subset\Omega$. The difference compared to (5.16) is that there is a larger cube on the right-hand side. Next we discuss a self-improving result for a general class of weak reverse Hölder inequalities.

Theorem 5.21. Let $1 < p < \infty$ and $c_1 > 0$, and let $\Omega \subset \mathbb{R}^n$ be an open set. There exist $\theta = \theta(n,p) > 0$, $q = q(n,p,c_1) > p$ and $c = c(n,p,c_1) \geqslant 1$ such that, if $f, g \in L^p_{\mathrm{loc}}(\Omega)$ are nonnegative functions satisfying

$$\begin{aligned}
\left(\fint_{Q(z,l)} f(x)^p\,dx\right)^{\frac{1}{p}} &\leqslant c_1\left[\fint_{Q(z,2l)} f(x)\,dx + \left(\fint_{Q(z,2l)} g(x)^p\,dx\right)^{\frac{1}{p}}\right] \\
&\quad + \theta\left(\fint_{Q(z,2l)} f(x)^p\,dx\right)^{\frac{1}{p}},
\end{aligned} \tag{5.22}$$

for every cube $Q(z,l)$ with $Q(z,2l) \subset \Omega$, then

$$\left(\fint_{Q(z,l)} f(x)^q\,dx\right)^{\frac{1}{q}} \leqslant c\left(\fint_{Q(z,2l)} f(x)^p\,dx\right)^{\frac{1}{p}} + c\left(\fint_{Q(z,2l)} g(x)^q\,dx\right)^{\frac{1}{q}} \tag{5.23}$$

for every cube $Q(z,l)$ with $\overline{Q(z,2l)} \subset \Omega$.

THE MORAL: We assume that $f \in L^p(Q_0)$ satisfies a uniform weak reverse Hölder inequality in (5.22). This implies that $f \in L^q(Q_0)$ for some $q > p$ and, consequently f is integrable to a higher power than assumed in the beginning. Moreover, there is a uniform weak reverse Hölder type estimate with the exponent $q > p$. Thus a uniform weak reverse Hölder inequality is self-improving.

Remarks 5.24:

(1) If we assume

$$\left(\fint_{Q(z,l)} f(x)^p\,dx\right)^{\frac{1}{p}} \leqslant c_1 \fint_{Q(z,2l)} f(x)\,dx$$

instead of (5.22), that is $g = 0$, we have

$$\left(\fint_{Q(z,l)} f(x)^q\,dx\right)^{\frac{1}{q}} \leqslant c\left(\fint_{Q(z,2l)} f(x)^p\,dx\right)^{\frac{1}{p}}$$

in (5.23).

(2) The inequality

$$\left(\fint_{Q(z,l)} f(x)^p\,dx\right)^{\frac{1}{p}} \leqslant c_1\left[\fint_{Q(z,2l)} f(x)\,dx + \left(\fint_{Q(z,2l)} g(x)^p\,dx\right)^{\frac{1}{p}}\right],$$

correspoding the case $\theta = 0$, clearly implies (5.22) and thus the result also applies in this case.

Proof. Let $Q_0 = Q(x_0, l_0)$ be a cube with $\overline{Q}_0 \subset \Omega$. We begin by constructing a specific Whitney type decomposition $\mathscr{W}$ of Q_0. Let

$$Q_i = Q\big(x_0, (1-2^{-i})l_0\big), \quad i = 1,2,\dots.$$

We divide each Q_i into $(2^{i+1}-2)^n$ dyadic subcubes of Q_0, with common side length $2^{-i}l_0$, which have pairwise disjoint interiors and cover Q_i. Denote this collection by $\mathscr{F}_i$. We define recursively a collection $\mathscr{W}_i$, $i = 1,2,\dots$, of cubes by setting $\mathscr{W}_1 = \mathscr{F}_1$ and

$$\mathscr{W}_{i+1} = \{Q \in \mathscr{F}_{i+1} : Q \cap \widetilde{Q} = \emptyset \text{ for every } \widetilde{Q} \in \mathscr{W}_i\}$$

for every $i = 1,2,\dots$. Let $\mathscr{W} = \bigcup_{i=1}^{\infty} \mathscr{W}_i$. The cubes in $\mathscr{W}$ have pairwise disjoint interiors and they they cover the interior of Q_0 and thus they cover Q_0 up to measure zero. Moreover, if $Q = Q(z,r) \in \mathscr{W}$, then the doubled cube $Q(z,2r)$ is a subset of Q_0.

Let $f,g \in L^p_{\mathrm{loc}}(\Omega)$ be nonnegative functions such that (5.22) holds for some $0<\theta<1$, to be specified later, and let

$$t_0 = \left(\fint_{Q_0} f(x)^p \, dx \right)^{\frac{1}{p}} < \infty. \tag{5.25}$$

Without loss of generality, we may assume that $t_0 > 0$.

Let $t \geqslant t_0$. For $Q \in \mathscr{W}$, we have

$$\fint_Q f(x)^p \, dx \leqslant \frac{1}{|Q|} \int_{Q_0} f(x)^p \, dx = \frac{|Q_0|}{|Q|} \fint_{Q_0} f(x)^p \, dx \leqslant a_Q t^p, \tag{5.26}$$

where $a_Q = \frac{|Q_0|}{|Q|} > 1$. Define functions $\widetilde{f}$ and $\widetilde{g}$ in the interior of Q_0 by setting

$$\widetilde{f}(x) = a_Q^{-\frac{1}{p}} f(x) \quad \text{and} \quad \widetilde{g}(x) = a_Q^{-\frac{1}{p}} g(x),$$

for every $x \in Q \in \mathscr{W}$. Clearly $0 \leqslant \widetilde{f} \leqslant f$ and $0 \leqslant \widetilde{g} \leqslant g$ almost everywhere in Q_0, and thus $\widetilde{f}, \widetilde{g} \in L^p(Q_0)$.

Let $Q \in \mathscr{W}$. By (5.26), we have

$$\fint_Q \widetilde{f}(x)^p \, dx \leqslant t^p,$$

and Lemma 5.7 gives

$$\int_{\{x \in Q : \widetilde{f}(x) > t\}} \widetilde{f}(x)^p \, dx \leqslant c(n) t^p \left| \{ x \in Q : M_Q^p \widetilde{f}(x) > t \} \right|. \tag{5.27}$$

To estimate the right-hand side of (5.27), let $x \in Q$ and let $Q_x = Q(z_x, r_x) \subset Q$ be a subcube of Q containing x. Then the construction above guarantees that the doubled cube $Q(z_x, 2r_x)$ is contained in $Q_0 \subset \Omega$, and (5.22) implies

$$\begin{aligned} \left(\fint_{Q_x} \widetilde{f}(y)^p \, dy \right)^{\frac{1}{p}} &= a_Q^{-\frac{1}{p}} \left(\fint_{Q_x} f(y)^p \, dy \right)^{\frac{1}{p}} \\ &\leqslant c_1 a_Q^{-\frac{1}{p}} \left[\fint_{Q(z_x,2r_x)} f(y) dy + \left(\fint_{Q(z_x,2r_x)} g(y)^p \, dy \right)^{\frac{1}{p}} \right] \\ &\quad + \theta a_Q^{-\frac{1}{p}} \left(\fint_{Q(z_x,2r_x)} f(y)^p \, dy \right)^{\frac{1}{p}}. \end{aligned} \tag{5.28}$$

It is easy to see that the cube $Q(z_x, 2r_x)$ intersects at most those cubes in $\mathscr{W}$ which have a nonempty intersection with Q. In particular, there exists a cube $Q' = Q(x', r') \in \mathscr{W}$ which touches Q and satisfies

$$\widetilde{f}(y) \geqslant (a_{Q'})^{-\frac{1}{p}} f(y) \quad \text{and} \quad \widetilde{g}(y) \geqslant (a_{Q'})^{-\frac{1}{p}} g(y)$$

for almost every $y \in Q(z_x, 2r_x)$. Moreover, by the construction of the cubes, we have $Q \subset Q(x', 5r')$. This implies $|Q| \leqslant |Q(x', 5r')| = 5^n |Q'|$ and consequently $a_{Q'} \leqslant 5^n a_Q$. From this we obtain

$$\fint_{Q(z_x,2r_x)} f(y) dy \leqslant (5^n a_Q)^{\frac{1}{p}} \fint_{Q(z_x,2r_x)} \widetilde{f}(y) dy.$$

A similar reasoning shows that

$$\left(\fint_{Q(z_x,2r_x)} g(y)^p\,dy\right)^{\frac{1}{p}} \leqslant (5^n a_Q)^{\frac{1}{p}}\left(\fint_{Q(z_x,2r_x)} \widetilde{g}(y)^p\,dy\right)^{\frac{1}{p}}$$

and

$$\left(\fint_{Q(z_x,2r_x)} f(y)^p\,dy\right)^{\frac{1}{p}} \leqslant (5^n a_Q)^{\frac{1}{p}}\left(\fint_{Q(z_x,2r_x)} \widetilde{f}(y)^p\,dy\right)^{\frac{1}{p}}.$$

By substituting the estimates above to (5.28)

$$\begin{aligned}\left(\fint_{Q_x} \widetilde{f}(y)^p\,dy\right)^{\frac{1}{p}} &\leqslant 5^{\frac{n}{p}} c_1\left[\fint_{Q(z_x,2r_x)} \widetilde{f}(y)\,dy + \left(\fint_{Q(z_x,2r_x)} \widetilde{g}(y)^p\,dy\right)^{\frac{1}{p}}\right]\\ &\quad + 5^{\frac{n}{p}}\theta\left(\fint_{Q(z_x,2r_x)} \widetilde{f}(y)^p\,dy\right)^{\frac{1}{p}}\end{aligned}$$

and taking supremum over all cubes Q_x as above, we have

$$M_Q^p \widetilde{f}(x) \leqslant 5^{\frac{n}{p}} c_1 M_{Q_0}\widetilde{f}(x) + 5^{\frac{n}{p}} c_1 M_{Q_0}^p \widetilde{g}(x) + 5^{\frac{n}{p}}\theta M_{Q_0}^p \widetilde{f}(x)$$

for every $x \in Q$. This implies

$$\{x \in Q : M_Q^p \widetilde{f}(x) > t\} \subset Q \cap (E \cup F \cup G),$$

where

$$\begin{aligned}E &= \left\{x \in Q_0 : M_{Q_0}\widetilde{f}(x) > \frac{1}{3}\frac{t}{5^{\frac{n}{p}} c_1}\right\},\\ F &= \left\{x \in Q_0 : M_{Q_0}^p \widetilde{g}(x) > \frac{1}{3}\frac{t}{5^{\frac{n}{p}} c_1}\right\},\\ G &= \left\{x \in Q_0 : M_{Q_0}^p \widetilde{f}(x) > \frac{1}{3}\frac{t}{5^{\frac{n}{p}}\theta}\right\}.\end{aligned}$$

Let $\gamma = \gamma(n,p) = 3 \cdot 5^{\frac{n}{p}}$. Lemma 5.2 implies

$$\begin{aligned}|E| &= \left|\left\{x \in Q_0 : M_{Q_0}\widetilde{f}(x) > \frac{t}{\gamma c_1}\right\}\right| \leqslant c(n)\frac{\gamma c_1}{t}\int_{\{x\in Q_0 : \widetilde{f}(x) > \frac{1}{2}\frac{t}{\gamma c_1}\}} \widetilde{f}(x)\,dx,\\ |F| &= \left|\left\{x \in Q_0 : M_{Q_0}^p \widetilde{g}(x) > \frac{t}{\gamma c_1}\right\}\right| \leqslant c(n)\left(\frac{\gamma c_1 t}{t}\right)^p \int_{\{x\in Q_0 : \widetilde{g}(x) > \frac{1}{2}\frac{t}{\gamma c_1}\}} \widetilde{g}(x)^p\,dx, \qquad (5.29)\\ |G| &= \left|\left\{x \in Q_0 : M_{Q_0}^p \widetilde{g}(x) > \frac{t}{\gamma\theta}\right\}\right| \leqslant c(n)\left(\frac{\gamma\theta}{t}\right)^p \int_{\{x\in Q_0 : \widetilde{g}(x) > \frac{1}{2}\frac{t}{\gamma\theta}\}} \widetilde{g}(x)^p\,dx.\end{aligned}$$

By (5.27) we have

$$\int_{\{x\in Q : \widetilde{f}(x) > t\}} \widetilde{f}(x)^p\,dx \leqslant c(n) t^p |Q \cap (E \cup F \cup G)| \qquad (5.30)$$

for every $W \in \mathscr{W}$. By summing (5.30) over all cubes $Q \in \mathscr{W}$ and applying (5.29), we obtain

$$
\begin{aligned}
\int_{\{x\in Q_0:\widetilde{f}(x)>t\}} \widetilde{f}(x)^p\,dx &\leqslant c(n)t^p|E\cup F\cup G| \leqslant c(n)t^p(|E|+|F|+|G|)\\
&\leqslant c(n,p,c_1)t^{p-1}\int_{\{x\in Q_0:\widetilde{f}(x)>\tau t\}} \widetilde{f}(x)\,dx\\
&\quad + c(n,p,c_1)\int_{\{x\in Q_0:\widetilde{g}(x)>\tau t\}} \widetilde{g}(x)^p\,dx\\
&\quad + c(n,p)\theta^p\int_{\{x\in Q_0:\widetilde{f}(x)>\tau t\}} \widetilde{f}(x)^p\,dx,
\end{aligned}
\tag{5.31}
$$

where

$$0<\tau=\tau(n,p,c_1)=\frac{1}{2}\max\left\{\frac{1}{\gamma(n,p)\theta},\frac{1}{\gamma(n,p)c_1}\right\}<1.$$

On the other hand, we have

$$
\begin{aligned}
\int_{\{x\in Q_0:\tau t<\widetilde{f}(x)\leqslant t\}} \widetilde{f}(x)^p\,dx &= \int_{\{x\in Q_0:\tau t<\widetilde{f}(x)\leqslant t\}} \widetilde{f}(x)^{p-1}\widetilde{f}(x)\,dx\\
&\leqslant t^{p-1}\int_{\{x\in Q_0:\widetilde{f}(x)>\tau t\}} \widetilde{f}(x)\,dx.
\end{aligned}
\tag{5.32}
$$

By adding (5.31) and (5.32) and reorganizing terms, we arrive at

$$
\begin{aligned}
&\big(1-c(n,p)\theta^p\big)\int_{\{x\in Q_0:\widetilde{f}(x)>\tau t\}} \widetilde{f}(x)^p\,dx\\
&\leqslant c(n,p,c_1)\left((\tau t)^{p-1}\int_{\{x\in Q_0:\widetilde{f}(x)>\tau t\}} \widetilde{f}(x)\,dx+\int_{\{x\in Q_0:\widetilde{g}(x)>\tau t\}} \widetilde{g}(x)^p\,dx\right).
\end{aligned}
$$

Recall that here $t \geqslant t_0$ was arbitrary. Also note that the term that is absorbed into the left-hand side is finite, since $\widetilde{f} \in L^p(Q_0)$.

Let $0<\theta=\theta(n,p)<1$ be so small that

$$1-c(n,p)\theta^p \geqslant \frac{1}{2}.$$

Lemma 5.11, applied for the functions $\widetilde{f},\widetilde{g} \in L^p(Q_0)$, and the estimates $\widetilde{f} \leqslant f$, $\widetilde{g} \leqslant g$ imply the existence of $q=q(n,p,c_1)>p$ such that

$$
\begin{aligned}
\int_{Q_0} \widetilde{f}(x)^q\,dx &\leqslant c\left((\tau t_0)^{q-p}\int_{Q_0} f(x)^p\,dx+\int_{Q_0} g(x)^q\,dx\right)\\
&= ct_0^{q-p}\int_{Q_0} f(x)^p\,dx + c\int_{Q_0} g(x)^q\,dx
\end{aligned}
$$

with $c=c(n,p,c_1)$. Here we used the fact that $\tau=\tau(n,p,c_1)$. By (5.25) and (5.22), we obtain

$$
\begin{aligned}
\int_{Q_0} \widetilde{f}(x)^q\,dx &\leqslant c\left(\fint_{Q_0} f(x)^p\,dx\right)^{\frac{q}{p}-1}\int_{Q_0} f(x)^p\,dx + c\int_{Q_0} g(x)^q\,dx\\
&\leqslant c\left(\fint_{Q_0} f(x)^p\,dx\right)^{\frac{q}{p}}|Q_0|\left(\int_{Q_0} f(x)^p\,dx\right)^{-1}\int_{Q_0} f(x)^p\,dx + c\int_{Q_0} g(x)^q\,dx\\
&= c|Q_0|\left(\fint_{Q_0} f(x)^p\,dx\right)^{\frac{q}{p}} + c\int_{Q_0} g(x)^q\,dx,
\end{aligned}
$$

which gives

$$\left(\fint_{Q_0} \widetilde{f}(x)^q \, dx\right)^{\frac{1}{q}} \leqslant c\left(\fint_{Q_0} f(x)^p \, dx\right)^{\frac{1}{p}} + c\left(\fint_{Q_0} g(x)^q \, dx\right)^{\frac{1}{q}}$$

with $c = c(n,p,c_1)$.

Finally, it follows from the construction of $\mathscr{W}$ that $Q_1 = Q(x_0, \frac{l_0}{2})$ is divided to 2^n cubes $Q \in \mathscr{W}$ with the side length $\frac{l_0}{2}$. Since $a_Q = \frac{|Q_0|}{|Q|} = 4^n$ for these cubes, it holds by the definition of $\widetilde{f}$ that

$$\widetilde{f}(x) = 4^{-\frac{n}{p}} f(x) \quad \text{for every} \quad x \in Q_1.$$

Hence we conclude that

$$\left(\fint_{Q_1} f(x)^q \, dx\right)^{\frac{1}{q}} \leqslant c\left(\fint_{Q_0} f(x)^p \, dx\right)^{\frac{1}{p}} + c\left(\fint_{Q_0} g(x)^q \, dx\right)^{\frac{1}{q}}$$

with $c = c(n,p,c_1)$. This is the desired inequality for the cube $Q(z,r) = Q_1$, and the proof is complete. □

Remark 5.33. By covering cubes in (5.22) with balls by the Vitali $5r$-covering lemma, and covering balls in the final inequality by dyadic cubes, we obtain the following variant of Theorem 5.21. Let $1 < p < \infty$, $c_1 > 0$, $1 < \tau_1, \tau_2 < \infty$, and let $\Omega \subset \mathbb{R}^n$ be an open set. There exist $\theta = \theta(n,p,\tau_1) > 0$, $q = q(n,p,c_1,\tau_1) > p$ and $c = c(n,p,c_1,\tau_1,\tau_2) \geqslant 1$ such that if $f, g \in L^p_{\text{loc}}(\Omega)$ are nonnegative functions satisfying

$$\begin{aligned}\left(\fint_{B(z,r)} f(x)^p \, dx\right)^{\frac{1}{p}} \leqslant c_1 &\left[\fint_{B(z,\tau_1 r)} f(x)\, dx + \left(\fint_{B(z,\tau_1 r)} g(x)^p \, dx\right)^{\frac{1}{p}}\right] \\ &+ \theta\left(\fint_{B(z,\tau_1 r)} f(x)^p \, dx\right)^{\frac{1}{p}},\end{aligned}$$

for every ball $B(z,r)$, with $B(z,\tau_1 r) \subset \Omega$, then

$$\left(\fint_{B(z,r)} f(x)^q \, dx\right)^{\frac{1}{q}} \leqslant c\left(\fint_{B(z,\tau_2 r)} f(x)^p \, dx\right)^{\frac{1}{p}} + c\left(\fint_{B(z,\tau_2 r)} g(x)^q \, dx\right)^{\frac{1}{q}}$$

for every ball $B(z,r)$ with $B(z,\tau_2 r) \subset \Omega$.

5.4 Local higher integrability of the gradient

We begin with a local higher integrability result for the gradient of a weak solution, showing that a weak solution $u \in W^{1,2}_{\text{loc}}(\Omega)$ to an elliptic partial differential belongs to a slightly higher Sobolev space, that is, $u \in W^{1,2+\delta}_{\text{loc}}(\Omega)$ for some $\delta > 0$. The proof is based on the energy estimate in Lemma 4.7 and a Sobolev–Poincaré inequality in Theorem 1.40, which give a reverse Hölder inequality for the weak gradient. Local higher integrability then follows from the self-improvement property of reverse Hölder inequalities, see Theorem 5.21.

Theorem 5.34. Assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to (4.1) in Ω. There exists $\delta = \delta(n,\lambda,\Lambda) > 0$ such that $|Du| \in L^{p+\delta}_{\mathrm{loc}}(\Omega)$. Moreover, there exists a constant $c = c(n,\lambda,\Lambda)$ such that

$$\left(\fint_{B(z,r)} |Du|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}} \leqslant c\left(\fint_{B(z,2r)} |Du|^2\,dx\right)^{\frac{1}{2}}$$

whenever $B(z,2r) \Subset \Omega$.

T H E M O R A L : We assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and show that $Du \in L^{2+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta > 0$ and, consequently Du is integrable to a higher power than assumed in the beginning. Moreover, there is a uniform weak reverse Hölder type estimate.

Proof. Let $q = \frac{2n}{n+2}$. Observe that $1 \leqslant q < 2$ for $n \geqslant 2$. Since $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \subset W^{1,q}_{\mathrm{loc}}(\Omega)$, by Theorem 1.40, we have

$$\left(\fint_{B(z,2r)} |u - u_{B(z,2r)}|^2\,dx\right)^{\frac{1}{2}} \leqslant c(n)r\left(\fint_{B(z,2r)} |Du|^q\,dx\right)^{\frac{1}{q}} \tag{5.35}$$

whenever $B(z,2r) \Subset \Omega$. By Lemma 4.7 and (5.35), we obtain

$$\begin{aligned}\left(\fint_{B(z,r)} |Du|^2\,dx\right)^{\frac{1}{2}} &\leqslant \frac{c(\lambda,\Lambda)}{r}\left(\fint_{B(z,2r)} |u - u_{B(z,2r)}|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant c(n,\lambda,\Lambda)\left(\fint_{B(z,2r)} |Du|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

Let $f = |Du|^q$. The estimate above can be rewritten as

$$\left(\fint_{B(z,r)} f(x)^{\frac{2}{q}}\,dx\right)^{\frac{q}{2}} \leqslant c(n,\lambda,\Lambda)\fint_{B(z,2r)} f(x)\,dx$$

for every $B(z,2r) \Subset \Omega$. Remark 5.33, see also Theorem 5.21, asserts that there exists $\delta = \delta(n,\lambda,\Lambda) > 0$ and $c = c(n,\lambda,\Lambda)$ such that

$$\begin{aligned}\left(\fint_{B(z,r)} |Du|^{2+\delta}\,dx\right)^{\frac{q}{2+\delta}} &= \left(\fint_{B(z,r)} f(x)^{\frac{2+\delta}{q}}\,dx\right)^{\frac{q}{2+\delta}}\\ &\leqslant c\fint_{B(z,2r)} f(x)\,dx\\ &= c\fint_{B(z,2r)} |Du|^q\,dx\end{aligned}$$

and, since $1 \leqslant q < 2$, we conclude that

$$\left(\fint_{B(z,r)} |Du|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}} \leqslant c\left(\fint_{B(z,2r)} |Du|^q\,dx\right)^{\frac{1}{q}} \leqslant c\left(\fint_{B(z,2r)} |Du|^2\,dx\right)^{\frac{1}{2}} \qquad \square$$

for every $B(z,2r) \Subset \Omega$.

Remark 5.36. We shall consider the example in Section 1.5 in the two-dimensional case, that is, $n = 2$ and $0 < \alpha < 1$. See also Chen and Wu [1, p. 189] and Giaquinta [5, p. 157]. In Section 1.5 we showed that the function $u : B(0,1) \to \mathbb{R}$,

$$u(x) = u(x_1, x_2) = x_1 |x|^{-\alpha}$$

is a weak solution to

$$-\sum_{i,j=1}^{2} D_j(a_{ij} D_i u) = 0$$

in $B(0,1)$, where

$$a_{ij}(x) = \delta_{ij} + \frac{\alpha(2-\alpha)}{(1-\alpha)^2} \frac{x_i x_j}{|x|^2}, \quad i,j = 1,2.$$

Estimate in (1.32) implies that the uniform ellipticity condition in Definition 1.7 is satisfied with

$$\lambda = 1 \quad \text{and} \quad \Lambda = 1 + \frac{\alpha(2-\alpha)}{(1-\alpha)^2}.$$

By solving the equation above with respect to α, we obtain

$$\alpha = \frac{\Lambda^2 - 1}{\Lambda^2}$$

Observe that $\alpha < 1$ can be made arbitrarily close to one by choosing $\Lambda > 1$ large enough.

By (1.29), we have

$$D_i u(x) = \delta_{i1} |x|^{-\alpha} - \alpha x_1 x_i |x|^{-\alpha-2}, \quad i = 1,2,$$

where $D_i u$, $i = 1,2$, is the weak partial derivative of u. A similar computation as in Section 1.5 shows that

$$\int_{B(0,1)} |Du|^p \, dx < \infty$$

for $2 \leqslant p < \frac{2}{\alpha}$ and

$$\int_{B(0,1)} |Du|^{\frac{2}{\alpha}} \, dx = \infty.$$

The exponent $\frac{2}{\alpha} > 2$ can be made as close to two as we wish by choosing $\alpha < 1$ close enough to one, or equivalently, choosing $\Lambda > 1$ large enough.

THE MORAL: The previous example shows that the higher integrability exponent δ in Theorem 5.34 is not very large and depends on the ellipticity constants. In particular, for every $\delta > 0$, there exists a solution to an elliptic equation, with a large enough ellipticity constant $\Lambda > 1$, such that $|Du| \notin L^{2+\delta}_{\text{loc}}(\Omega)$. In this sense, Theorem 5.34 is sharp.

Corollary 5.37. Assume that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to (4.1) in Ω. There exists $\delta = \delta(n, \lambda, \Lambda) > 0$ such that $u \in W^{1,2+\delta}_{\text{loc}}(\Omega)$.

THE MORAL: We assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and show that $u \in W^{1,2+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta > 0$ and, consequently u belongs to a higher Sobolev space than assumed in the beginning.

Proof. By Theorem 1.40, we obtain $q = q(n) > 2$ such that

$$\left(\fint_{B(z,r)} |u - u_{B(z,r)}|^q \, dx\right)^{\frac{1}{q}} \leqslant c(n) r \left(\fint_{B(z,r)} |Du|^2 \, dx\right)^{\frac{1}{2}}$$

for every $B(z,r) \Subset \Omega$, which implies

$$\begin{aligned}\left(\fint_{B(z,r)} |u|^q \, dx\right)^{\frac{1}{q}} &\leqslant \left(\fint_{B(z,r)} |u - u_{B(z,r)}|^q \, dx\right)^{\frac{1}{q}} + |u_{B(z,r)}| \\ &\leqslant c(n) r \left(\fint_{B(z,r)} |Du|^2 \, dx\right)^{\frac{1}{2}} + |u|_{B(z,r)} < \infty.\end{aligned}$$

Since $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we have $|u|_{B(z,r)} < \infty$. This shows that $u \in L^q_{\mathrm{loc}}(\Omega)$ for some $q = q(n) > 2$. Together with Theorem 5.34 this implies that $u \in W^{1,2+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta = \delta(n, \lambda, \Lambda) > 0$. □

Remark 5.38. Corollary 4.22 asserts that a weak solution to (4.1) is locally bounded and Theorem 4.36 asserts that a weak solution is continuous. Both facts imply $u \in L^\infty_{\mathrm{loc}}(\Omega)$. This fact together with Theorem 5.34 can be used to give an alternative proof of the previous corollary.

5.5 Higher integrability up to the boundary

Next we consider a global higher integrability result over the entire open set Ω. In the argument we need the following variant of the energy estimate given in Theorem 4.6.

Theorem 5.39. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and let $g \in W^{1,2}(\Omega)$. Let $u \in W^{1,2}(\Omega)$ is a weak solution of (4.1) in Ω with $u - g \in W^{1,2}_0(\Omega)$. There exists a constant $c = c(\lambda, \Lambda)$ such that

$$\int_\Omega \varphi^2 |Du|^2 \, dx \leqslant c \int_\Omega |u - g|^2 |D\varphi|^2 \, dx + \int_\Omega \varphi^2 |Dg|^2 \, dx$$

for every $\varphi \in C^\infty_0(\mathbb{R}^n)$.

THE MORAL: Observe that the support of the test function φ in the following lemma need not be a compact subset of Ω. This gives us estimates up to the boundary.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and define $v = \varphi^2(u-g)$. Since $u-g \in W_0^{1,2}(\Omega)$, we have $v \in W_0^{1,2}(\Omega)$. Moreover,

$$D_j v = \varphi^2(D_j u - D_j g) + 2\varphi(u-g)D_j\varphi, \quad j = 1,\dots,n,$$

almost everywhere in Ω. Since u is a weak solution and $v \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned}
0 &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j v\,dx \\
&= \int_\Omega \varphi^2 \sum_{i,j=1}^n a_{ij} D_i u (D_j u - D_j g)\,dx + 2\int_\Omega \varphi(u-g) \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi\,dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_\Omega \varphi^2 \sum_{i,j=1}^n a_{ij} D_i u D_j u\,dx \\
&\leqslant 2\left|\int_\Omega \varphi(u-g) \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi\,dx\right| + \left|\int_\Omega \varphi^2 \sum_{i,j=1}^n a_{ij} D_i u D_j g\,dx\right| \\
&\leqslant 2\int_\Omega |\varphi||u-g| \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} |D_i u||D_j \varphi|\,dx + \int_\Omega \varphi^2 \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} |D_i u||D_j g|\,dx \\
&\leqslant c\left(\int_\Omega |\varphi||u-g||Du||D\varphi|\,dx + \int_\Omega \varphi^2 |Du||Dg|\,dx\right).
\end{aligned}$$

Next we first apply the uniform ellipticity condition to the previous estimate, and then we use Young's inequality with epsilon to have

$$\begin{aligned}
\lambda \int_\Omega \varphi^2 |Du|^2\,dx &\leqslant c\int_\Omega |\varphi||u-g||Du||D\varphi|\,dx + \int_\Omega \varphi^2 |Du||Dg|\,dx \\
&\leqslant \frac{\lambda}{2}\int_\Omega \varphi^2|Du|^2\,dx + c\int_\Omega |u-g|^2|D\varphi|^2\,dx + c\int_\Omega \varphi^2 |Dg|^2\,dx.
\end{aligned}$$

Both terms on the right-hand side are finite, since $u \in W_{\mathrm{loc}}^{1,2}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. The claim follows by absorbing the first term on the right-hand side. □

We state a global higher integrability result on open sets whose complement satisfies the following measure density condition. This is a relatively standard regularity assumption on the domain in the theory of PDEs.

Definition 5.40. A set $E \subset \mathbb{R}^n$ satisfies the measure density condition, if there exists a constant γ, with $0 < \gamma \leqslant 1$, such that

$$|E \cap B(x,r)| \geqslant \gamma |B(x,r)| \tag{5.41}$$

for every $x \in E$ and $r > 0$.

T H E M O R A L : A set satisfying the measure density condition is thick in the sense that every ball centered in the set contains at least certain percentage points of the set.

Remarks 5.42:

(1) If $\Omega \subset \mathbb{R}^n$ is an open set with a smooth boundary, that is, the boundary is locally represented by a graph of a smooth function, then $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition. The same holds true for Lipchitz boundaries. One advantage of the measure density condition is that it also applies to sets whose boundary is not represented by a graph of a function.

(2) If $\Omega \subset \mathbb{R}^n$ is an open set such that $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition, by the Lebesgue differentiation theorem the boundary of Ω has Lebesgue measure zero.

The next result is a global version of Theorem 5.34.

Theorem 5.43. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that the complement $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition with a constant γ. Assume that $g \in W^{1,s}(\Omega)$ for some $s > 2$ and that $u \in W^{1,2}(\Omega)$ is a weak solution to (4.1) in Ω with $u - g \in W_0^{1,2}(\Omega)$. There exists $\delta = \delta(n, \Lambda, \lambda, s, \gamma) > 0$, with $2 + \delta \leqslant s$, such that $|Du| \in L^{2+\delta}(\Omega)$. Moreover, there exists a constant $c = c(n, s, \gamma)$ such that

$$\left(\int_\Omega |Du|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}} \leqslant c\left[\left(\int_\Omega |Du|^2\,dx\right)^{\frac{1}{2}} + \left(\int_\Omega |Dg|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}}\right].$$

T H E M O R A L : If the domain and the boundary value function are smooth enough, then the gradient of the solution to the Dirichlet problem is integrable to a higher power over the entire domain than assumed in the beginning. Moreover, this result comes with a weak reverse Hölder type estimate.

Proof. $\boxed{(1)}$ Let $B(z,r)$ be a ball with $\overline{B(z,2r)} \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$. Let $\varphi \in C_0^\infty(B(z,2r))$ be a cutoff function such that $\varphi = 1$ in $B(z,r)$, $0 \leqslant \varphi \leqslant 1$ and $|D\varphi| \leqslant \frac{C}{r}$. Since Ω is bounded, we have $g \in W^{1,s}(\Omega) \subset W^{1,2}(\Omega)$. By Lemma 5.39, there exists a constant $c = c(\lambda, \Lambda)$ such that

$$\begin{aligned}\int_{B(z,r)\cap\Omega} |Du|^2\,dx &\leqslant \int_\Omega \varphi^p |Du|^2\,dx \\ &\leqslant \frac{c}{r^2}\int_{B(z,2r)\cap\Omega} |u-g|^2\,dx + c\int_{B(z,2r)\cap\Omega} |Dg|^2\,dx.\end{aligned} \tag{5.44}$$

As in the proof of Theorem 5.34, let $q = \frac{2n}{n+2}$. Then $1 \leqslant q < 2$ for $n \geqslant 2$. Hölder's inequality implies $u - g \in W_0^{1,2}(\Omega) \subset W_0^{1,q}(\Omega)$. By considering the zero extension $v \in W^{1,q}(\mathbb{R}^n)$ of $u - g \in W_0^{1,q}(\Omega)$, defined by

$$v = \begin{cases} u - g, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

we have $v = 0$ almost everywhere in $\mathbb{R}^n \setminus \Omega$. Since $B(z,2r) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$, there exists a point $y \in B(z,2r) \cap (\mathbb{R}^n \setminus \Omega)$. By the fact that $v = 0$ almost everywhere in

$\mathbb{R}^n \setminus \Omega$ and the measure density property, we obtain

$$|\{x \in B(y,4r) : v(x) = 0\}| \geqslant |B(y,4r) \cap (\mathbb{R}^n \setminus \Omega)| \geqslant \gamma |B(y,4r)|.$$

Since $v \in W^{1,q}(\mathbb{R}^n)$, by Theorem 1.47, there exists a constant $c = c(n,\gamma)$ such that

$$\left(\fint_{B(y,4r)} |v|^2 \, dx \right)^{\frac{1}{2}} \leqslant cr \left(\fint_{B(y,4r)} |Dv|^q \, dx \right)^{\frac{1}{q}}.$$

Since $v = u - g$ almost everywhere in Ω, $v = 0$ almost everywhere in $\mathbb{R}^n \setminus \Omega$ and $B(y,4r) \subset B(z,6r)$, there exists a constant $c = c(n,\gamma)$, such that

$$\begin{aligned}
&\frac{1}{r} \left(\frac{1}{|B(z,r)|} \int_{B(z,2r)\cap\Omega} |u-g|^2 \, dx \right)^{\frac{1}{2}} \leqslant \frac{c}{r} \left(\fint_{B(y,4r)} |v|^2 \, dx \right)^{\frac{1}{2}} \\
&\leqslant c \left(\fint_{B(y,4r)} |Dv|^q \, dx \right)^{\frac{1}{q}} = c \left(\frac{1}{|B(y,4r)|} \int_{B(y,4r)\cap\Omega} |Du - Dg|^q \, dx \right)^{\frac{1}{q}} \\
&\leqslant c \left[\left(\frac{1}{|B(z,6r)|} \int_{B(z,6r)\cap\Omega} |Du|^q \, dx \right)^{\frac{1}{q}} + \left(\frac{1}{|B(z,6r)|} \int_{B(z,6r)\cap\Omega} |Dg|^2 \, dx \right)^{\frac{1}{2}} \right].
\end{aligned}$$

By (5.44) and the estimates above, there exists a constant $c = c(n,\Lambda,\lambda,\gamma)$ such that

$$\begin{aligned}
&\left(\frac{1}{|B(z,r)|} \int_{B(z,r)\cap\Omega} |Du|^2 \, dx \right)^{\frac{1}{2}} \\
&\leqslant \frac{c}{r} \left(\frac{1}{|B(z,r)|} \int_{B(z,2r)\cap\Omega} |u-g|^2 \, dx \right)^{\frac{1}{2}} + c \left(\frac{1}{|B(z,r)|} \int_{B(z,2r)\cap\Omega} |Dg|^2 \, dx \right)^{\frac{1}{2}} \\
&\leqslant c \left[\left(\frac{1}{|B(z,6r)|} \int_{B(z,6r)\cap\Omega} |Du|^q \, dx \right)^{\frac{1}{q}} + \left(\frac{1}{|B(z,6r)|} \int_{B(z,6r)\cap\Omega} |Dg|^2 \, dx \right)^{\frac{1}{2}} \right]
\end{aligned} \tag{5.45}$$

whenever $\overline{B(z,2r)} \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$.

$\boxed{(2)}$ Assume then that $B(z,r)$ is a ball with $\overline{B(z,2r)} \cap (\mathbb{R}^n \setminus \Omega) = \emptyset$, that is, $B(z,2r) \Subset \Omega$. By the proof of Theorem 5.34, we have

$$\left(\fint_{B(z,r)} |Du|^2 \, dx \right)^{\frac{1}{2}} \leqslant c \left(\fint_{B(z,2r)} |Du|^q \, dx \right)^{\frac{1}{q}}.$$

with $c = c(n,\lambda,\Lambda)$. From this we conclude that (5.45) is also valid when $\overline{B(z,2r)} \cap (\mathbb{R}^n \setminus \Omega) = \emptyset$. This implies that (5.45) holds for every ball $B(z,r) \subset \mathbb{R}^n$.

$\boxed{(3)}$ Let $f = |Du|^q \chi_\Omega$ and $h = |Dg|^q \chi_\Omega$. By (5.45) there exists a constant $c = c(n,\Lambda,\lambda,\gamma)$ such that

$$\left(\fint_{B(z,r)} f^{\frac{2}{q}} \, dx \right)^{\frac{q}{2}} \leqslant c \left[\fint_{B(z,6r)} f \, dx + \left(\fint_{B(z,6r)} h^{\frac{2}{q}} \, dx \right)^{\frac{q}{2}} \right]$$

for every $B(z,r) \subset \mathbb{R}^n$. By Remark 5.33, there exists $\delta = \delta(n,\Lambda,\lambda,s,\gamma)$, with $2+\delta \leqslant s$, and $c = c(n,\Lambda,\lambda,\gamma)$ such that

$$\begin{aligned}&\left(\frac{1}{|B(z,r)|}\int_{B(z,r)\cap\Omega}|Du|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}}\\&\leqslant c\left[\left(\frac{1}{|B(z,2r)|}\int_{B(z,2r)\cap\Omega}|Du|^{2}\,dx\right)^{\frac{1}{p}}+\left(\frac{1}{|B(z,2r)|}\int_{B(z,2r)\cap\Omega}|Dg|^{2+\delta}\,dx\right)^{\frac{1}{2+\delta}}\right]\end{aligned}$$

for every $B(z,r)\subset\mathbb{R}^n$. The claim follows by considering a ball $B(z,r)$ with $z\in\partial\Omega$ and $r = \operatorname{diam}(\Omega)$. $\square$

Corollary 5.46. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set such that the complement $\mathbb{R}^n\setminus\Omega$ satisfies the measure density condition with a constant γ. Assume that $g\in W^{1,s}(\Omega)$ for some $s>2$ and that $u\in W^{1,2}(\Omega)$ is a weak solution to (4.1) in Ω with $u-g\in W_0^{1,2}(\Omega)$. There exists $\delta=\delta(n,\Lambda,\lambda,s,\gamma)>0$, with $2+\delta\leqslant s$, such that $u\in W^{1,2+\delta}(\Omega)$.

Proof. Since $u-g\in W_0^{1,2}(\Omega)$, Corollary 1.37 implies that there exists $q=q(n,s)>2$, with $q\leqslant s$, such that

$$\left(\int_{\Omega}|u-g|^{q}\,dx\right)^{\frac{1}{q}}\leqslant c(n,p,s,\Omega)\left(\int_{\Omega}|Du-Dg|^{2}\,dx\right)^{\frac{1}{2}}<\infty.$$

It follows that

$$\left(\int_{\Omega}|u|^{q}\,dx\right)^{\frac{1}{q}}\leqslant\left(\int_{\Omega}|u-g|^{q}\,dx\right)^{\frac{1}{q}}+\left(\int_{\Omega}|g|^{q}\,dx\right)^{\frac{1}{q}}<\infty$$

and thus $u\in L^{2+\delta}(\Omega)$ for some $\delta=\delta(n,s)>0$. Together with Theorem 5.43 this implies that $u\in W^{1,2+\delta}(\Omega)$ for some $\delta=\delta(n,\Lambda,\lambda,s,\gamma)>0$. $\square$

THE END

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