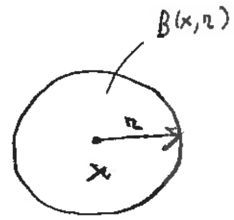
New thoughts on the Hardy-Littlewood maximal function

1. Basic properties

1.1. Definition. Let $f \in L^1_{loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \to [0,\infty]$ of f is

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy,$$

The centered maximal function



where

$$\fint_{B(x,r)} |f(y)|\,dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy.$$

1.2. Remark. (1) Mf is defined at every point $x \in \mathbb{R}^n$. If $f(x) = g(x)$ for almost every $x \in \mathbb{R}^n$, then

$$Mf(x) = Mg(x)$$

for every $x \in \mathbb{R}^n$.

(2) There are different definitions in the literature. For example,

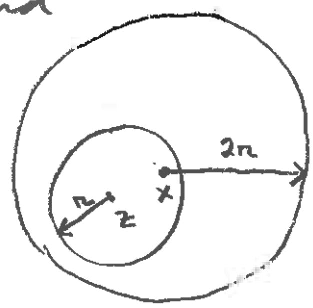
$$\widetilde{M}f(x) = \sup_{B \ni x} \fint_B |f(y)|\,dy,$$

The uncentered maximal function.

where the supremum is taken over all balls B that contain x. Clearly

$$Mf(x) \le \tilde{M} f(x).$$

On the other hand, if $B = B(z,r)$ is a ball that contains x, then $B(z,r) \subset B(x,2r)$ and



$$\fint_B |f(y)|\,dy \le \frac{|B(x,2r)|}{|B(z,r)|} \fint_{B(x,2r)} |f(y)|\,dy$$

$$= 2^n \fint_{B(x,2r)} |f(y)|\,dy \le 2^n Mf(x).$$

This implies that $\tilde{M} f(x) \le 2^n Mf(x)$ and consequently

$$Mf(x) \le \tilde{M} f(x) \le 2^n Mf(x)$$

for all $x \in \mathbb{R}^n$.

(3) By the Lebesgue differentiation theorem

$$|f(x)| = \lim_{r \to 0} \fint_{B(x,r)} |f(y)|\,dy$$

$$\le \sup_{r > 0} \fint_{B(x,r)} |f(y)|\,dy = Mf(x)$$

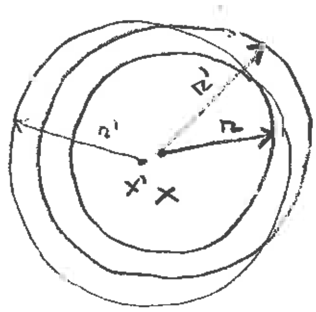
for almost every $x \in \mathbb{R}^n$.

(4) $M(f_1 + f_2)(x) \le Mf_1(x) + Mf_2(x)$ (sublinearity)

1.3. Lemma. Mf is lower semicontinuous and hence Lebesgue measurable.

Proof: $E_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$, $\lambda > 0$

$x \in E_\lambda \Rightarrow \exists\, r > 0$ s.t.



$$\fint_{B(x,r)} |f(y)|\,dy > \lambda.$$

Choose $r' > r$ s.t.

$$\frac{1}{|B(x,r')|} \int_{B(x,r)} |f(y)|\,dy > \lambda.$$

$|x - x'| < r' - r \Rightarrow B(x,r) \subset B(x',r')$

$(y \in B(x,r) \Rightarrow |y - x'| \le |y - x| + |x - x'| < r + r' - r = r')$

$$\Rightarrow \lambda < \frac{1}{|B(x,r')|} \int_{B(x,r)} |f(y)|\,dy \le \frac{1}{|B(x,r')|} \int_{B(x',r')} |f(y)|\,dy$$

$$= \fint_{B(x',r')} |f(y)|\,dy \le Mf(x') \quad \forall\, x' \in B(x, r'-r)$$

$\Rightarrow B(x, r'-r) \subset E_\lambda \Rightarrow E_\lambda$ is open. $\square$

1.4. **Lemma.** If $f \in L^\infty(\mathbb{R}^n)$, then $Mf \in L^\infty(\mathbb{R}^n)$ and

$$\|Mf\|_\infty \le \|f\|_\infty .$$

Proof:

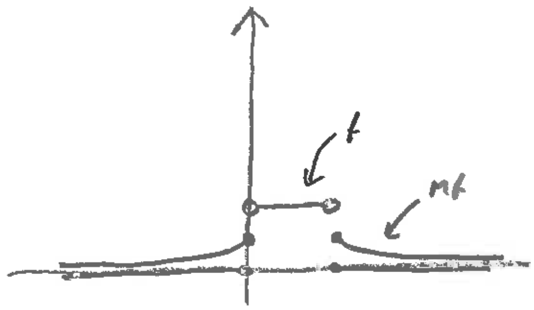
$$\fint_{B(x,r)} |f(y)|\,dy \le \|f\|_\infty \quad \forall\, x \in \mathbb{R}^n,\ r>0$$

$$\Rightarrow\ Mf(x) \le \|f\|_\infty \quad \forall\, x \in \mathbb{R}^n$$

$$\Rightarrow\ \|Mf\|_\infty \le \|f\|_\infty . \qquad \square$$

Example. $f: \mathbb{R} \to \mathbb{R}$, $f = \chi_{(0,1)}$

$$Mf(x) = \begin{cases} \dfrac{1}{2(1-x)}, & x \le 0, \\ 1, & 0 < x < 1, \\ \dfrac{1}{2x}, & x \ge 1 . \end{cases}$$



Observe: $f \in L^1(\mathbb{R}) \not\Rightarrow Mf \in L^1(\mathbb{R})$.

Remark. If $Mf \in L^1(\mathbb{R}^n)$, then $f = 0$. If $f \neq 0$, choose $r > 0$ so large that

$$\int_{B(0,r)} |f(y)|\,dy > 0 .$$

If $|x| > r$, then

$$Mf(x) \geq \fint_{B(x,2|x|)} |f(y)|\,dy \;\underset{\substack{\uparrow \\ B(0,r) \subset B(x,2|x|)}}{\geq}\; \frac{1}{|B(0,2|x|)|} \int_{B(0,r)} |f(y)|\,dy$$

$$= \frac{2^{-m}}{|x|^m} \int_{B(0,r)} |f(y)|\,dy = \frac{c}{|x|^m} \quad \forall\ |x| > r$$

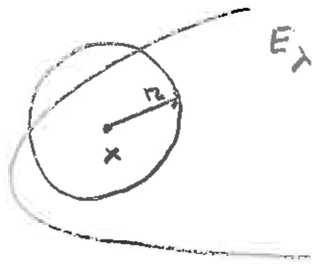
$$\Rightarrow \quad Mf \notin L^1(\mathbb{R}^m).$$

1.5. <u>Theorem</u>. (Hardy–Littlewood) If $f \in L^1(\mathbb{R}^m)$, then

$$|\{x \in \mathbb{R}^m : Mf(x) > \lambda\}| \leq \frac{3^m}{\lambda} \|f\|_1$$

for all $\lambda > 0$.

<u>Proof</u>: $E_\lambda = \{x \in \mathbb{R}^m : Mf(x) > \lambda\}$, $\lambda > 0$



$x \in E_\lambda \Rightarrow \exists\ r > 0$ s.t.

$$\fint_{B(x,r)} |f(y)|\,dy > \lambda$$

By a Vitali covering theorem there are pairwise disjoint balls $B(x_i, r_i)$, $i = 1, 2, \dots$, s.t.

$$E_\lambda \subset \bigcup_{i=1}^{\infty} B(x_i, 3r_i).$$

$$\Rightarrow |E_\lambda| \le \sum_{i=1}^{\infty} |B(x_i, 3r_i)| = 3^m \sum_{i=1}^{\infty} |B(x_i, r_i)|$$

$$\le \frac{3^m}{\lambda} \sum_{i=1}^{\infty} \int_{B(x_i, r_i)} |f(y)|\,dy$$

$$= \frac{3^m}{\lambda} \int_{\bigcup_{i=1}^{\infty} B(x_i, r_i)} |f(y)|\,dy \le \frac{3^m}{\lambda} \|f\|_1 . \qquad \square$$

1.6. <u>Theorem</u>. If $f \in L^p(\mathbb{R}^m)$, $1 < p \le \infty$, then $Mf \in L^p(\mathbb{R}^m)$ and there exists $c = c(m,p)$ s.t.

$$\|Mf\|_p \le c\, \|f\|_p .$$

<u>Proof</u>: $f = f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x), & |f(x)| > \frac{\lambda}{2}, \\ 0, & |f(x)| \le \frac{\lambda}{2}. \end{cases}$$

$$\int_{\mathbb{R}^m} |f_1(x)|\,dx = \int_{\mathbb{R}^m} |f_1(x)|^p |f_1(x)|^{1-p}\,dx \le \left(\frac{\lambda}{2}\right)^{1-p} \|f\|_1$$

$|f_1(x)| > \frac{\lambda}{2} \ \forall x \in \mathbb{R}^m$

$$\Rightarrow f_1 \in L^1(\mathbb{R}^m)$$

$$|f_2(x)| \le \frac{\lambda}{2} \ \forall x \in \mathbb{R}^m \Rightarrow \|f_2\|_\infty \le \frac{\lambda}{2} \Rightarrow f_2 \in L^\infty(\mathbb{R}^m)$$

M is sublinear

$$\Rightarrow \quad Mf(x) \le Mf_1(x) + Mf_2(x) \le Mf_1(x) + \frac{\lambda}{2}$$

lemma 1.4

$$\Rightarrow \quad |\{x \in \mathbb{R}^m : Mf(x) > \lambda\}|$$

$$\le |\{x \in \mathbb{R}^m : Mf_1(x) > \tfrac{\lambda}{2}\}|$$

$$\le \frac{2 \cdot 3^m}{\lambda} \|f_1\|_1$$

theorem 1.5

$$= \frac{2 \cdot 3^m}{\lambda} \int_{\{|f(x)| > \frac{\lambda}{2}\}} |f(x)|\, dx$$

$$\Rightarrow \quad \int_{\mathbb{R}^m} |Mf|^p\, dx = p \int_0^\infty \lambda^{p-1} |\{Mf(x) > \lambda\}|\, d\lambda$$

Cavalieri principle

$$\le p \cdot 2 \cdot 3^m \int_0^\infty \lambda^{p-2} \int_{\{|f(x)| > \frac{\lambda}{2}\}} |f(x)|\, dx\, d\lambda$$

$$= p \cdot 2^p \cdot 3^m \int_0^\infty \lambda^{p-2} \int_{\{|f(x)| > \lambda\}} |f(x)|\, dx\, d\lambda \quad \left(= p\, 2^p 3^m \int_{\mathbb{R}^m} |f(x)| \int_0^{|f(x)|} \lambda^{p-2} d\lambda\, dx\right)$$

change of variables

$$= \frac{p\, 2^p 3^m}{p-1} \int_{\mathbb{R}^m} |f(x)|^{p-1} |f(x)|\, dx.$$

Cavalieri principle

$\square$

2. Maximal function in Sobolev spaces

Let $1 \le p \le \infty$. The space $W^{1,p}(\mathbb{R}^n)$ consists of all functions $u \in L^p(\mathbb{R}^n)$ such that their first weak partial derivatives $\frac{\partial u}{\partial x_k}$, $k = 1, \dots, n$, exist and belong to $L^p(\mathbb{R}^n)$. We endow $W^{1,p}(\mathbb{R}^n)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \|Du\|_p ,$$

where $Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ is the weak gradient of u.

2.1. Theorem. Let $1 < p \le \infty$. If $u \in W^{1,p}(\mathbb{R}^n)$, then $Mu \in W^{1,p}(\mathbb{R}^n)$ and

$$\left| \frac{\partial Mu}{\partial x_k} \right| \le M\left(\frac{\partial u}{\partial x_k}\right), \quad k = 1, \dots, n,$$

almost everywhere in $\mathbb{R}^n$.

Proof: $\chi_r = \dfrac{\chi_{B(0,r)}}{|B(0,r)|}$, $r > 0$ $\quad (\|\chi_r\|_1 = 1 \quad \forall r > 0)$

$$\fint_{B(x,r)} |u(y)|\, dy = \frac{1}{|B(0,r)|} \int_{B(0,r)} |u(x-y)|\, dy$$

$$= \frac{1}{|B(0,r)|} \int_{\mathbb{R}^n} \chi_{B(0,r)}(y)\, |u(x-y)|\, dy$$

$$= \underbrace{|u| * \chi_r}_{\text{the convolution}}(x)$$

$$u \in W^{1,p}(\mathbb{R}^m) \Rightarrow |u| \in W^{1,p}(\mathbb{R}^m)$$

$$\Rightarrow |u| * \chi_r \in W^{1,p}(\mathbb{R}^m) \text{ and}$$

Young & Fubini!

$$\frac{\partial}{\partial x_k}(|u| * \chi_r) = \chi_r * \frac{\partial}{\partial x_k}(|u|), \quad k = 1, \dots, m$$

$$Mu(x) = \sup_j (|u| * \chi_{r_j})(x),$$

where r_j, $j = 1, 2, \dots$, is an enumeration of positive rational numbers

$$v_i = \max_{1 \le j \le i} (|u| * \chi_{r_j}), \quad i = 1, 2, \dots$$

$$\Rightarrow v_i \nearrow Mu \text{ as } i \to \infty$$

$$\left(\frac{\partial}{\partial x_k} \max(w_1, w_2) = \begin{cases} \frac{\partial w_1}{\partial x_k} & \text{a.e. in } \{w_1 > w_2\} \\ \frac{\partial w_2}{\partial x_k} & \text{a.e. in } \{w_2 \ge w_1\} \end{cases} \right)$$

$$\left| \frac{\partial v_i}{\partial x_k} \right| \le \max_{1 \le j \le i} \left| \frac{\partial}{\partial x_k}(|u| * \chi_{r_j}) \right|$$

$$= \max_{1 \le j \le i} \left| \chi_{r_j} * \frac{\partial}{\partial x_k}(|u|) \right|$$

$$\left| \frac{\partial |u|}{\partial x_k} \right| = \left| \frac{\partial u}{\partial x_k} \right| \text{ a.e.}$$

$$\le M\left(\frac{\partial}{\partial x_k}(|u|)\right) = M\left(\frac{\partial u}{\partial x_k}\right), \quad k = 1, \dots, m,$$

almost everywhere in $\mathbb{R}^m$.

$$\Rightarrow \| |Dv_i| \|_p \le \sum_{k=1}^m \left\| \frac{\partial v_i}{\partial x_k} \right\|_p \le \sum_{k=1}^m \left\| M\left(\frac{\partial u}{\partial x_k}\right) \right\|_p$$

$$\Rightarrow \|v_i\|_{1,p} = \|v_i\|_p + \||Dv_i|\|_p$$
$$\leq \|Mu\|_p + \sum_{k=1}^{m} \left\| M\left(\frac{\partial u}{\partial x_k}\right)\right\|_p$$
$$\leq c\left(\|u\|_p + \sum_{k=1}^{m} \left\|\frac{\partial u}{\partial x_k}\right\|_p\right) \quad \forall\, i = 1, 2, \dots$$

Theorem 1.6

$\Rightarrow$ (v_i) is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$,
$v_i \nearrow Mu$ pointwise as $i \to \infty$

The weak compactness of $W^{1,p}(\mathbb{R}^n)$
$v_i \to Mu$ weakly in $L^p(\mathbb{R}^n)$
$\Rightarrow$ $Mu \in W^{1,p}(\mathbb{R}^n)$, and $\dfrac{\partial v_i}{\partial x_k} \to \dfrac{\partial Mu}{\partial x_k}$, $k = 1, \dots, n$, weakly in $L^p(\mathbb{R}^n)$.

$$\left.\begin{array}{l} \dfrac{\partial v_i}{\partial x_k} \to \dfrac{Mu}{\partial x_k} \text{ weakly in } L^p(\mathbb{R}^n) \\[2ex] \left|\dfrac{\partial v_i}{\partial x_k}\right| \leq M\left(\dfrac{\partial u}{\partial x_k}\right),\ k = 1, \dots, n \end{array}\right\} \Rightarrow \left|\frac{\partial Mu}{\partial x_k}\right| \leq M\left(\frac{\partial u}{\partial x_k}\right),$$
$$k = 1, \dots, n$$

$\square$

<u>Remark.</u> By Theorem 1.6 $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is a bounded operator,
$$\|Mu\|_{1,p} \leq c\,\|u\|_{1,p}$$
for all $u \in W^{1,p}(\mathbb{R}^n)$. It is also continuous (H. Luiro).

Remark. $x, h \in \mathbb{R}^n$, $f_h(x) = f(x+h)$ (translated function)

$$(Mu)_h(x) = Mu(x+h) = \sup_{r>0} \fint_{B(x+h,r)} |u(y)|\,dy$$

$$\underset{y=z+h}{=} \sup_{r>0} \fint_{B(x,r)} |u(z+h)|\,dz = M(u_h)(x)$$

(M commutes with translation)

$$M(u_h) = M(u_h - u + u) \underset{\text{sublinear}}{\leq} M(u_h - u) + Mu$$

$$\Rightarrow M(u_h) - Mu \leq M(u_h - u)$$

Analogously $Mu - M(u_h) \leq M(u_h - u)$.

$$\Rightarrow |(Mu)_h - Mu| = |M(u_h) - Mu|$$

$$\leq M(u_h - u)$$

If $|u(x+h) - u(x)| \leq L|h|$ $\forall x, h \in \mathbb{R}^n$, then

$$|Mu(x+h) - Mu(x)| \leq M(u_h - u)(x)$$

$$= \sup_{r>0} \fint_{B(x,r)} |u(y+h) - u(y)|\,dy \leq L|h|.$$

Observe that this holds for Hölder continuous as well.

Open question: What happens if $p = 1$?

2.2. A capacity inequality

Let $1<p<\infty$. The Sobolev p-capacity of a set $E\subset\mathbb{R}^n$ is

$$C_p(E)=\inf_{u\in\mathcal{A}(E)}\int_{\mathbb{R}^n}(|u|^p+|Du|^p)\,dx,$$

where

$$\mathcal{A}(E)=\{u\in W^{1,p}(\mathbb{R}^n): u\geq 1 \text{ on a neighbourhood of } E\}.$$

capacity is an outer measure.

Let $u\in W^{1,p}(\mathbb{R}^n)$ and

$$E_\lambda=\{x\in\mathbb{R}^n: Mu(x)>\lambda\},\quad \lambda>0.$$

Lemma 1.3 $\Rightarrow$ E_λ is open

Theorem 2.2 $\Rightarrow$ $\dfrac{Mu}{\lambda}\in\mathcal{A}(E_\lambda)$

$$\Rightarrow C_p(E_\lambda)\leq\frac{1}{\lambda^p}\int_{\mathbb{R}^n}(|Mu|^p+|DMu|^p)\,dx$$

$$\leq\frac{c}{\lambda^p}\int_{\mathbb{R}^n}(|u|^p+|Du|^p)\,dx$$

$$\leq\frac{c}{\lambda^p}\|u\|_{1,p}^p.$$

This is a Sobolev space counterpart of the weak type estimate in Theorem 1.5.

Let $\Omega \subset \mathbb{R}^n$ be an open set and define

$$M_\Omega f(x) = \sup_{0<r<\delta(x)} \fint_{B(x,r)} |f(y)|\,dy\,, \quad \delta(x) = \operatorname{dist}(x, \partial\Omega).$$

2.3. <u>Theorem</u>. Let $1<p\le\infty$. If $u \in W^{1,p}(\Omega)$, then $M_\Omega u \in W^{1,p}(\Omega)$ and

$$|DM_\Omega u(x)| \le 2\, M_\Omega |Du|$$

almost everywhere in Ω.

<u>Proof</u>: Assume $u \in C^\infty(\Omega)$, $u \ge 0$.

$$u_t(x) = \fint_{B(x,t\delta(x))} u(y)\,dy\,, \quad 0<t\le 1.$$

Leibniz and chain rule

$$\Rightarrow \frac{\partial u_t}{\partial x_k}(x) = n\, \frac{\frac{\partial \delta}{\partial x_k}(x)}{\delta(x)} \Big[\fint_{\partial B(x,t\delta(x))} u(y)\,dH^{n-1}(y) - \fint_{B(x,t\delta(x))} u(y)\,dy \Big] + \fint_{B(x,t\delta(x))} \frac{\partial u}{\partial x_k}(y)\,dy\,,$$

$k = 1, \dots, n$

Green's first formula

$$\Rightarrow \quad \fint_{\partial B(x,r)} u(y)\, dH^{n-1}(y) - \fint_{B(x,r)} u(y)\, dy$$

$$= \frac{1}{n} \fint_{B(x,r)} Du(y) \cdot (y-x)\, dy$$

$$\leq \frac{r}{n} M_{\Omega} |Du|(x).$$

$r = t\,\delta(x)$

$$\Rightarrow \left| \frac{\partial u_t}{\partial x_k}(x) \right| \leq \left(t \left| \frac{\partial \delta}{\partial x_k}(x) \right| + 1 \right) M_{\Omega} |Du|(x)$$

$$\leq 2 M_{\Omega} |Du|(x), \quad k = 1, \dots, n.$$

$\Rightarrow \quad u_t \in W^{1,p}(\Omega), \quad 0 < t \leq 1$ and

$$|Du_t(x)| \leq 2 M_{\Omega} |Du|(x)$$

$$M_{\Omega} u(x) = \sup_{j} |u|_{t_j}(x),$$

where the supremum is taken over all rational numbers of the interval $(0,1)$.

The rest of the proof is similar to the proof of Theorem 2.2.

$\square$

3. Pointwise inequalities

Let $0<\beta<\infty$ and $R>0$. The fractional sharp maximal function of $f\in L^1_{loc}(\mathbb{R}^n)$ is

$$f^{\#}_{\beta,R}(x)=\sup_{0<r<R} r^{-\beta} \fint_{B(x,r)} |f(y)-f_{B(x,r)}|\,dy,$$

(Campanato spaces)

Fractional sharp maximal function of f.

where we denote

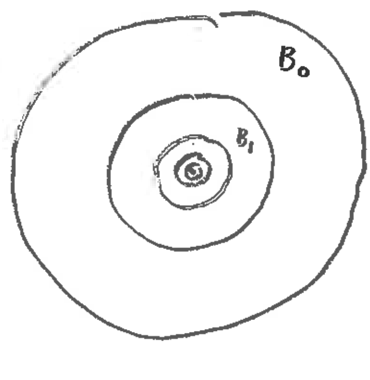
$$f_{B(x,r)}=\fint_{B(x,r)} f(y)\,dy=\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy.$$

3.1. Lemma. Suppose that $f\in L^1_{loc}(\mathbb{R}^n)$ and let $0<\beta<\infty$. Then there is $c=c(n,\beta)>0$ such that

$$|f(x)-f(y)|\le c\,|x-y|^{\beta}\left(f^{\#}_{\beta,4|x-y|}(x)+f^{\#}_{\beta,4|x-y|}(y)\right)$$

for every $x,y\in\mathbb{R}^n\setminus E$ with $|E|=0$.

Proof: Let E be the complement of the set of the Lebesgue points of f. By Lebesgue's differentiation theorem $|E|=0$. Let $x\in\mathbb{R}^n\setminus E$, $0<r<\infty$, and denote $B_i=B(x,2^{-i}r)$, $i=0,1,2,\dots$



Then

$$|f(x) - f_{B(x,r)}| \le \sum_{i=0}^{\infty} |f_{B_{i+1}} - f_{B_i}| \le \sum_{i=0}^{\infty} \fint_{B_{i+1}} |f(y) - f_{B_i}|\,dy$$

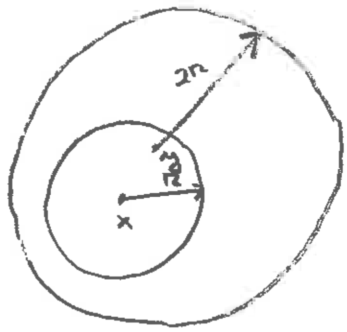
$$\le \sum_{i=0}^{\infty} \frac{|B_i|}{|B_{i+1}|} \fint_{B_i} |f(y) - f_{B_i}|\,dy$$

$$= 2^m \sum_{i=0}^{\infty} (2^{-i} r)^{\beta} (2^{-i} r)^{-\beta} \fint_{B_i} |f(y) - f_{B_i}|\,dy$$

$$\le 2^m f^{\#}_{\beta,r}(x)\, r^{\beta} \sum_{i=0}^{\infty} 2^{-i\beta}$$

$$= c(m,\beta)\, r^{\beta} f^{\#}_{\beta,r}(x).$$

If $y \in B(x,r) \setminus E$, then $B(x,r) \subset B(y,2r)$



and

$$|f(y) - f_{B(x,r)}|$$

$$\le |f(y) - f_{B(y,2r)}| + |f_{B(y,2r)} - f_{B(x,r)}|$$

$$\le c r^{\beta} f^{\#}_{\beta,2r}(y) + \fint_{B(x,r)} |f(z) - f_{B(y,2r)}|\,dz$$

$$\le c r^{\beta} f^{\#}_{\beta,2r}(y) + c \fint_{B(y,2r)} |f(z) - f_{B(y,2r)}|\,dz$$

$$\le c r^{\beta} f^{\#}_{\beta,2r}(y).$$

Finally, let $x, y \in \mathbb{R}^n \setminus E$, $x \neq y$ and $r = 2|x-y|$.

Then $x, y \in B(x,r)$ and

$$|f(x) - f(y)| \leq |f(x) - f_{B(x,r)}| + |f_{B(x,r)} - f(y)|$$

$$\leq c\, d(x,y)^\beta \left(f^\#_{\beta, 4|x-y|}(x) + f^\#_{\beta, 4|x-y|}(y) \right). \qquad \square$$

If $u \in W^{1,1}_{loc}(\mathbb{R}^n)$, then by Poincaré's inequality

$$\fint_{B(x,r)} |u(y) - u_{B(x,r)}|\, dy \leq c\, r \fint_{B(x,r)} |Du(y)|\, dy,$$

where $c = c(n)$ is independent of the ball $B(x,r) \subset \mathbb{R}^n$.

In particular, this implies that

$$u^\#_{1-\alpha, R}(x) \leq c\, M_{\alpha, R} |Du|(x),$$

where $0 \leq \alpha < 1$ and

$$M_{\alpha,R} f(x) = \sup_{0<r<R} r^\alpha \fint_{B(x,r)} |f(y)|\, dy$$

(Morrey spaces)

is the fractional maximal function of $f \in L^1_{loc}(\mathbb{R}^n)$.

3.2. Corollary. Suppose that $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ and let $0 \le \alpha < 1$. Then there is $c = c(n,\alpha) > 0$ such that

$$|u(x)-u(y)| \le c\,|x-y|^{1-\alpha}\left(M_{\alpha,4|x-y|}|Du|(x) + M_{\alpha,4|x-y|}|Du|(y)\right)$$

for every $x, y \in \mathbb{R}^n \setminus E$ with $|E| = 0$.

3.3. Remark. (1) If $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p \le \infty$, then

$$|u(x)-u(y)| \le c\,|x-y|\,(M|Du|(x) + M|Du|(y))$$

for every $x, y \in \mathbb{R}^n \setminus E$ with $|E| = 0$. Since $|Du| \in L^p(\mathbb{R}^n)$, by Theorem 1.6 $g = M|Du| \in L^p(\mathbb{R}^n)$ and

$$|u(x)-u(y)| \le c\,|x-y|\,(g(x)+g(y))$$

for every $x, y \in \mathbb{R}^n \setminus E$ with $|E| = 0$. In particular, if $p = \infty$, then

$$|u(x)-u(y)| \le c\,|x-y|$$

for every $x, y \in \mathbb{R}^n \setminus E$ with $|E| = 0$.

(2) $$\fint_{B(x,r)} |u(y)-u(x)|\,dy \le \fint_{B(x,r)} |u(y)-u_{B(x,r)}|\,dy + |u_{B(x,r)} - u(x)|$$

$$\le C\, r^{1-\alpha} u^{\#}_{1-\alpha, r}(x) \le C\, r^{1-\alpha} M_{\alpha,r}|Du|(x)$$

Poincaré! Lemma 3.1

If $M_\alpha |Du|(x) < \infty$, then x is a Lebesgue point of u. (*)

3.4. Theorem. Let $u \in L^p(\mathbb{R}^n)$, $1 < p \le \infty$, and suppose that there exists $g \in L^p(\mathbb{R}^n)$ such that $g \ge 0$ and

$$|u(x) - u(y)| \le |x-y|\,(g(x) + g(y))$$

for all $x, y \in \mathbb{R}^n \setminus E$ with $|E| = 0$. Then $u \in W^{1,p}(\mathbb{R}^n)$.

Proof: $|u(x+te_k) - u(x)| \le |te_k|\,(g(x+te_k) + g(x))$,

$$k = 1, \dots, n, \quad t \in \mathbb{R}$$

$$\Rightarrow \left(\int_{\mathbb{R}^n} \left(\frac{|u(x+te_k) - u(x)|}{|t|} \right)^p dx \right)^{\frac{1}{p}}$$

$$\le \left(\int_{\mathbb{R}^n} g(x+te_k)^p\, dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} g(x)^p\, dx \right)^{\frac{1}{p}}$$

$$= 2\,\|g\|_p < \infty$$

The characterization of $W^{1,p}(\mathbb{R}^n)$, $1 < p \le \infty$ with integrated difference quotients implies the claim. $\square$

Remark. The Poincaré inequality follows easily from the condition of Theorem 3.4. If $y \in B(x,r)$, then

$$\left| u(y) - \fint_{B(x,r)} u(z)\,dz \right| \le \fint_{B(x,r)} |u(y)-u(z)|\,dz$$

$$\le 2r \left(g(y) + \fint_{B(x,r)} g(z)\,dz \right)$$

$$\Rightarrow \quad \fint_{B(x,r)} |u(y) - u_{B(x,r)}|\,dy$$

$$\le 2r \left(\fint_{B(x,r)} g(y)\,dy + \fint_{B(x,r)} g(z)\,dz \right)$$

$$\le 4r \fint_{B(x,r)} g(y)\,dy .$$

Thus we have proved

3.5. Theorem. Let $1 < p \le \infty$. Then the following four conditions are equivalent.

(i) $u \in W^{1,p}(\mathbb{R}^n)$.

(ii) $u \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$, $g \ge 0$ s.t.

$$\fint_{B(x,r)} |u(y) - u_{B(x,r)}|\,dy \le r \fint_{B(x,r)} g(y)\,dy$$

for every $B(x,r) \subset \mathbb{R}^n$.

(iii) $u \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$, $g \geq 0$ s.t.

$$|u(x) - u(y)| \leq |x-y| \, (g(x) + g(y))$$

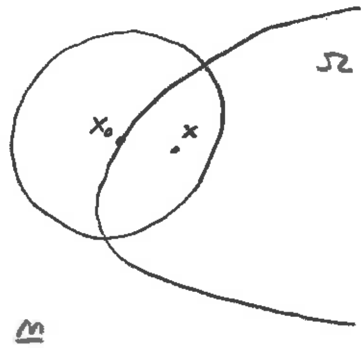
for all $x \in \mathbb{R}^n \setminus E$ with $|E| = 0$.

(iv) $u \in L^p(\mathbb{R}^n)$ and $u_1^{\#} \in L^p(\mathbb{R}^n)$.

Remark. This works in a metric measure space. (Hajłasz -Kinnunen)

4. Hardy's inequality

Suppose that $n < q < p$, $0 \le \alpha < 1$ and let $\Omega \ne \mathbb{R}^n$ be an open set. Let $u \in C_0^\infty(\Omega)$ (extended as zero to $\mathbb{R}^n \setminus \Omega$). Let $x \in \Omega$ and take $x_0 \in \partial\Omega$ such that $|x - x_0| = \operatorname{dist}(x, \partial\Omega) = r$.



By the Sobolev imbedding

$$
\begin{aligned}
|u(x)| &= |u(x) - u(x_0)| \\
&\le c \left(\int_{B(x_0, 2r)} |Du(y)|^q \, dy \right)^{\frac{1}{q}} |x - x_0|^{1 - \frac{n}{q}} \\
&= c \left((2r)^\alpha \fint_{B(x_0, 2r)} |Du(y)|^q \, dy \right)^{\frac{1}{q}} r^{1 - \frac{\alpha}{q}} \\
&\le c \operatorname{dist}(x, \partial\Omega)^{1 - \frac{\alpha}{q}} \left(M_\alpha |Du|^q (x) \right)^{\frac{1}{q}}
\end{aligned}
$$

centered and non-centered maximal functions are equivalent

for every $x \in \Omega$. If $\alpha = 0$, then

$$
\begin{aligned}
\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx &\le c \int_\Omega \left(M |Du|^q (x) \right)^{\frac{p}{q}} dx \\
&\le c \int_\Omega |Du(x)|^p \, dx
\end{aligned}
$$

$\frac{p}{q} > 1$, Theorem 1.6

for every $u \in C_0^\infty(\Omega)$ with $c = c(n,p,q)$.

This is called Hardy's inequality.

4.1. Theorem. If $p > n$, then Hardy's inequality holds in every open set which has a boundary point.

Remark. By a density argument we may take $u \in W_0^{1,p}(\Omega)$.

If $1 < p \le n$, then Hardy's inequality does not hold without further assumptions on Ω. In fact, Hardy's inequality can be destroyed by removing a compact set of Hausdorff dimension zero.

4.1. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition if there is $\gamma \in (0,1]$ such that

$$|B(x,r) \cap (\mathbb{R}^n \setminus \Omega)| \ge \gamma \, |B(x,r)|$$

for all $x \in \mathbb{R}^n \setminus \Omega$ and $r > 0$.

4.2. Lemma. Suppose that $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition. Then for every $u \in C_0^\infty(\Omega)$ we have

$$\Big(\int_{B(x,r)} |u(y)|^p \, dy \Big)^{\frac{1}{p}} \le c r \Big(\int_{B(x,r)} |Du(y)|^p \, dy \Big)^{\frac{1}{p}}$$

for every $x \in \mathbb{R}^n \setminus \Omega$ and $r > 0$ with $c = c(n, p, \gamma)$.

Proof:

$$\Big(\int_{B(x,r)} |u|^p \, dy \Big)^{\frac{1}{p}}$$

$$\underset{\text{Minkowski}}{\le} \Big(\int_{B(x,r)} |u - u_{B(x,r)}|^p \, dy \Big)^{\frac{1}{p}} + |u_{B(x,r)}| \, |B(x,r)|^{\frac{1}{p}}$$

$$\le c r \Big(\int_{B(x,r)} |Du|^p \, dy \Big)^{\frac{1}{p}} + |B(x,r)|^{\frac{1}{p} - 1} \int_{B(x,r) \cap \{u \neq 0\}} |u| \, dy$$

where

$$\int_{B(x,r) \cap \{u \neq 0\}} |u| \, dy \underset{\text{Hölder}}{\le} \Big(\int_{B(x,r)} |u|^p \, dy \Big)^{\frac{1}{p}} |B(x,r) \cap \{u \neq 0\}|^{1 - \frac{1}{p}}$$

The measure density theorem gives $\dfrac{|B(x,r) \cap \{u \neq 0\}|}{|B(x,r)|} \le 1 - \gamma$.

$\square$

4.3. Theorem. Let $1 < p \le n$, $0 \le \alpha < 1$, and suppose that $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition. Then there is $c = c(n, p, \gamma)$ such that for every $u \in C_0^\infty(\Omega)$ we have

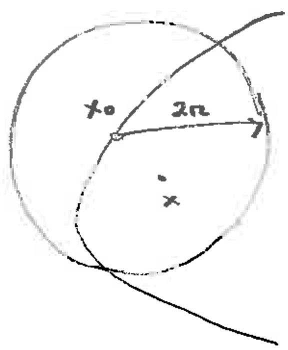
$$|u(x)| \le c \, \operatorname{dist}(x, \partial\Omega)^{1-\alpha} M_\alpha |Du|(x)$$

for all $x \in \Omega$.

Proof: Let $x \in \Omega$ and choose $x_0 \in \partial\Omega$ such that $|x - x_0| = \operatorname{dist}(x, \partial\Omega) = r$. Then

$$|u(x)| \le |u(x) - u_{B(x_0, 2r)}| + |u_{B(x_0,2r)}|$$

$$\le c r^{1-\alpha} M_\alpha |Du|(x) + \fint_{B(x_0,2r)} |u| \, dy.$$



By Lemma 4.2 with $p = 1$ we have

$$\fint_{B(x_0,2r)} |u| \, dy \le c r \fint_{B(x_0,2r)} |Du| \, dy$$

$$\le c r \fint_{B(x,4r)} |Du| \, dy \le c r^{1-\alpha} M_\alpha |Du|(x).$$

$\square$

4.4. Theorem. Hardy's inequality holds under the assumptions of Theorem 4.3.

Proof: By Theorem 4.3 with $\alpha = 0$ we have

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \le c \int_\Omega (M|Du|(x))^p \, dx$$

$$\le c \int_\Omega |Du(x)|^p \, dx. \qquad \square$$

Theorem 1.6.

4.5. Remark. The variational p-capacity of the condenser (K, Ω), where $\Omega \subset \mathbb{R}^n$ is open and $K \subset \Omega$ is compact, is defined as

$$C_p(K, \Omega) = \inf \int_\Omega |Du|^p \, dx,$$

where the infimum is taken over all functions $u \in C_0^\infty(\Omega)$ with $u = 1$ on K.

We say that $\mathbb{R}^n \setminus \Omega$ satisfies the p-capacity density condition if there is $\gamma > 0$ such that

$$C_p((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x, r), B(x, 2r)) \ge \gamma \, C_p(\overline{B}(x, r), B(x, 2r))$$

for every $x \in \mathbb{R}^n \setminus \Omega$ and $r > 0$.

It is enough to assume this condition in Theorem 4.4. Indeed, if $p = n$, then n-capacity density condition gives a characterization of Hardy's inequality.

Open question: What is the corresponding characterization in the case $1 < p < n$?

Very recently Juha Lehrbäck proved that the capacity density condition is a necessary and sufficient condition for a pointwise Hardy's inequality in Theorem 4.3.

Lewis showed in 1993 that capacity density condition is self improving: the p-capacity density implies q-capacity density condition for some $q < p$.

We show that Hardy's inequality is self improving.

4.6. Theorem. Suppose that Hardy's inequality holds for some p in Ω. Then there exists $\varepsilon > 0$ so that Hardy's inequality holds in Ω for every q with

$$p - \varepsilon < q \leq p.$$

<u>Proof</u>: Let u be a Lipschitz function that vanishes in $\mathbb{R}^n \setminus \Omega$ and extend u to be zero in $\mathbb{R}^n \setminus \Omega$. Define

$$F_\lambda = \{ x \in \Omega : |u(x)| \le \lambda \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$$
$$\text{and } M|Du|(x) \le \lambda \}, \quad \lambda > 0.$$

We claim that $u|_{F_\lambda \cup (\mathbb{R}^n \setminus \Omega)}$ is $c\lambda$-Lipschitz with $c = c(n)$. If $x, y \in F_\lambda$, then by Corollary 3.2

$$|u(x) - u(y)| \le c|x-y| \left(M|Du|(x) + M|Du|(y) \right)$$
$$\le c\, 2\lambda |x-y|.$$

If $x \in F_\lambda$, $y \in \mathbb{R}^n \setminus \Omega$, then

$$|u(x) - u(y)| = |u(x)| \le \lambda \operatorname{dist}(x, \partial\Omega) \le \lambda |x-y|.$$

Thus the claim is true. By the classical McShane extension

$$\tilde{u}(x) = \inf_{y \in F_\lambda \cup (\mathbb{R}^n \setminus \Omega)} \{ u(y) + c\lambda |x-y| \}$$

we extend $u|_{F_\lambda \cup (\mathbb{R}^n \setminus \Omega)}$ to $c\lambda$-Lipschitz function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$.

Let

$$G_\lambda = \{x \in \Omega : |u(x)| \le \lambda \operatorname{dist}(x, \partial\Omega)\}$$

and

$$E_\lambda = \{x \in \Omega : M|Du|(x) \le \lambda\}$$

so that $F_\lambda = G_\lambda \cap E_\lambda$. We notice that

$$|D\tilde{u}(x)| \le |Du(x)|\chi_{F_\lambda}(x) + c\lambda\chi_{\Omega \setminus F_\lambda}(x)$$

for a.e. $x \in \mathbb{R}^n$. By Hardy's inequality

$$\int_{F_\lambda} \left(\frac{|\tilde{u}(x)|}{\operatorname{dist}(x, \partial\Omega)}\right)^p dx \le c \int_{F_\lambda} |Du(x)|^p dx + c\lambda^p |\Omega \setminus F_\lambda|$$

and

$$\int_{G_\lambda} \left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)}\right)^p dx = \int_{G_\lambda \setminus E_\lambda} \dots dx + \int_{F_\lambda} \dots dx$$

$$\le c \int_{F_\lambda} |Du(x)|^p dx + c\lambda^p |\Omega \setminus F_\lambda|$$

$$+ \int_{G_\lambda \setminus E_\lambda} \underbrace{\left(\frac{|u(x)|}{\operatorname{dist}(x, \partial\Omega)}\right)^p}_{\le \lambda^p} dx$$

$$\le c \int_{E_\lambda} |Du(x)|^p dx + c\lambda^p \left(|\Omega \setminus G_\lambda| + |\Omega \setminus E_\lambda|\right)$$

$$\Omega \setminus F_\lambda = \Omega \setminus (E_\lambda \cap G_\lambda) = (\Omega \setminus E_\lambda) \cup (\Omega \setminus G_\lambda)$$

This implies

$$\int_0^\infty \lambda^{-\varepsilon-1} \int_{G_\lambda} \left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)}\right)^p dx\, d\lambda$$

$$\leq C \int_0^\infty \lambda^{-\varepsilon-1} \int_{E_\lambda} |Du(x)|^p \, dx\, d\lambda$$

$$+ C \int_0^\infty \lambda^{p-\varepsilon-1} |\Omega \setminus F_\lambda| \, d\lambda + C \int_0^\infty \lambda^{p-\varepsilon-1} |\Omega \setminus E_\lambda| \, d\lambda,$$

where

$$\int_0^\infty \lambda^{-\varepsilon-1} \int_{G_\lambda} \left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)}\right)^p dx\, d\lambda = \frac{1}{\varepsilon} \int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)}\right)^{p-\varepsilon} dx,$$

$$\int_0^\infty \lambda^{-\varepsilon-1} \int_{E_\lambda} |Du(x)|^p \, dx\, d\lambda \leq \int_0^\infty \lambda^{-\varepsilon-1} \int_{\{x\in\Omega:\, |Du(x)|\leq\lambda\}} |Du(x)|^p \, dx\, d\lambda$$

$$= \frac{1}{\varepsilon} \int_\Omega |Du(x)|^{p-\varepsilon} dx,$$

$$\int_0^\infty \lambda^{p-\varepsilon-1} |\Omega \setminus E_\lambda| \, d\lambda = \frac{1}{p-\varepsilon} \int_\Omega (M|Du|(x))^{p-\varepsilon} dx$$

$$\leq \frac{C}{p-\varepsilon} \int_\Omega |Du(x)|^{p-\varepsilon} dx$$

Theorem 1.6, $p-\varepsilon > 1$

and

$$\int_0^\infty \lambda^{p-\varepsilon-1} |\Omega \setminus G_\lambda| \, d\lambda = \frac{1}{p-\varepsilon} \int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)} \right)^{p-\varepsilon} dx.$$

Consequently

$$\frac{1}{\varepsilon} \int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)} \right)^{p-\varepsilon} dx \leq \frac{1}{\varepsilon} \int_\Omega |Du(x)|^{p-\varepsilon} dx$$

$$+ \frac{c}{p-\varepsilon} \int_\Omega |Du(x)|^{p-\varepsilon} dx + \frac{c}{p-\varepsilon} \int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)} \right)^{p-\varepsilon} dx.$$

The claim follows if we choose $\varepsilon > 0$ small enough. $\square$

<u>Remark</u>. It is easy to show that Hardy's inequality also holds for q with $p \leq q < p+\varepsilon$.

4.7. Theorem. Suppose that $u \in W^{1,p}(\Omega)$. If

$$\frac{u(x)}{\operatorname{dist}(x,\partial\Omega)}$$

belongs to the weak $L^p(\Omega)$, then $u \in W_0^{1,p}(\Omega)$.

Proof: $u_i(x) = \min\left(|u(x)|, i \operatorname{dist}(x,\partial\Omega)\right)$, $i = 1,2,\dots$

$$\int_\Omega |u_i(x)|^p\,dx \le \int_\Omega |u(x)|^p\,dx, \quad i=1,2,\dots$$

$$F_i = \{x \in \Omega : |u(x)| > i \operatorname{dist}(x,\partial\Omega)\}, \quad i = 1,2,\dots$$

$$\int_\Omega |Du_i(x)|^p\,dx$$

$$= \int_{\Omega\setminus F_i} |Du(x)|^p\,dx + i^p \int_{F_i} \underbrace{|D \operatorname{dist}(x,\partial\Omega)|}_{\le 1}{}^p\,dx$$

$$\le \int_\Omega |Du(x)|^p\,dx + i^p\,|F_i|$$

$$= \int_\Omega |Du(x)|^p\,dx + \underbrace{i^p \left|\left\{x \in \Omega : \frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)} > i\right\}\right|}_{\le C}$$

$$\left.\begin{array}{l} u_i \in W_0^{1,p}(\Omega), \; i=1,2,\dots \\ \|u_i\|_{1,p} \le C < \infty, \; i=1,2,\dots \\ u_i \to u \end{array}\right\} \Rightarrow u \in W_0^{1,p}(\Omega)$$

weak compactness

$\square$

<u>Remark</u>. (1) $p>n$, $\Omega \neq \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$. Then

$$u \in W_0^{1,p}(\Omega) \iff u \text{ satisfies Hardy's inequality.}$$

(2) $1<p\le n$, $\mathbb{R}^n \setminus \Omega$ satisfies the p-capacity (or measure) density condition. Then

$$u \in W_0^{1,p}(\Omega) \iff u \text{ satisfies Hardy's inequality.}$$

4.8. <u>Theorem</u>. Let $\Omega \subset \mathbb{R}^n$ be an open set and suppose that $u \in W^{1,p}(\Omega)$ with $p>1$. Then

$$|u| - M_\Omega u \in W_0^{1,p}(\Omega).$$

<u>Proof</u>: $\overset{u\ge 0}{u \in W^{1,p}(\Omega)}$, $\delta(x) = \operatorname{dist}(x,\partial\Omega)$

$$u_t(x) = \fint_{B(x,t\delta(x))} u(y)\,dy, \quad 0<t\le 1$$

Let $w_t(x) = u(x) - u_t(x)$. Then

$$|w_t(x)| = \left| u(x) - \fint_{B(x, t\delta(x))} u(y)\,dy \right|$$

$$\leq c\, t\, \delta(x)\, M_\Omega |Du|(x)$$

Lemma 3.1

$$\Rightarrow \int_\Omega \left(\frac{|w_t(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^p dx \leq c \int_\Omega (M_\Omega |Du|(x))^p\, dx$$

$$\leq c \int_\Omega |Du(x)|^p\, dx < \infty$$

Theorem 1.6

Theorem 4.7 $\Rightarrow w_t \in W_0^{1,p}(\Omega)$, $0 < t \leq 1$

The rest of the proof goes along the lines of the proof of Theorem 2.3. Let t_j, $j = 1, 2, \ldots$, be an enumeration of rationals between 0 and 1.

Define

$$v_i(x) = \max_{1 \leq j \leq i} u_{t_j}(x), \quad i = 1, 2, \ldots$$

Then (v_i) is a bounded sequence in $W^{1,p}(\Omega)$, $v_i \uparrow M_\Omega u$ as $i \to \infty$ and $u - v_i \in W_0^{1,p}(\Omega)$.

A weak compactness argument implies that $u - M_\Omega u \in W_0^{1,p}(\Omega)$

□

5. Approximation of Sobolev functions

5.1. Theorem. Suppose that $u \in W^{1,p}(\mathbb{R}^n)$. For every $0 \le \alpha < 1$ and $\lambda > 0$ there is an open set $U_\lambda \subset \mathbb{R}^n$ and a function $u_\lambda \in W^{1,p}(\mathbb{R}^n)$ such that

(i) $u(x) = u_\lambda(x)$ for every $x \in \mathbb{R}^n \setminus U_\lambda$,

(ii) $\| u - u_\lambda \|_{1,p} \to 0$ as $\lambda \to \infty$

(iii) u_λ is $(1-\alpha)$-Hölder continuous and

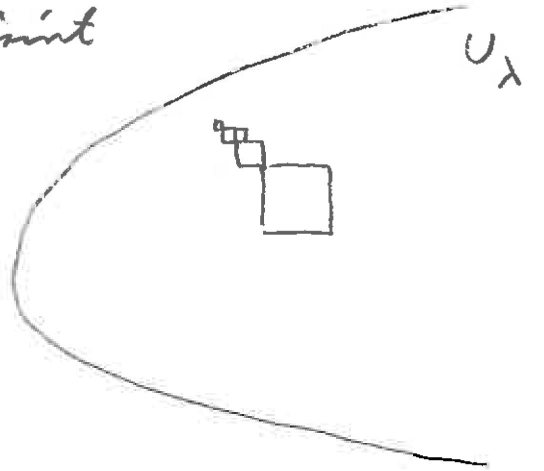
(iv) $C_{\alpha p}(U_\lambda) \to 0$ as $\lambda \to \infty$.
$\left(H^{n-\alpha p}_\infty (U_\lambda) \to 0 \quad \text{as } \lambda \to \infty \right)$

Proof: Corollary 3.2 $\Rightarrow$

$$|u(x) - u(y)| \le c\,|x-y|^{1-\alpha} \left(M_\alpha |Du|(x) + M_\alpha |Du|(y) \right)$$
$$\forall\, x, y \in \mathbb{R}^n \setminus E, \quad |E| = 0$$

$U_\lambda = \{ x \in \mathbb{R}^n : M_\alpha |Du|(x) > \lambda \}$, $\lambda > 0$, is open (see Lemma 1.3)

Let Q_i, $i=1,2,\dots$, be a Whitney decomposition of U_λ with the following properties:



(i) cubes Q_i, $i=1,2,\dots$, are disjoint

(ii) $U_\lambda = \bigcup_{i=1}^{\infty} \overline{Q_i}$,

(iii) $\sum_{i=1}^{\infty} \chi_{2Q_i} \le N < \infty$,

(iv) $4Q_i \subset U_\lambda$, $i=1,2,\dots$

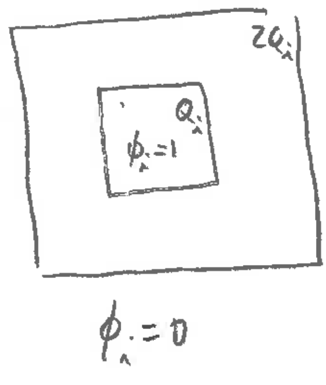
(v) $c_1 \operatorname{dist}(Q_i, \mathbb{R}^n \setminus U_\lambda) \le \operatorname{diam}(Q_j) \le c \operatorname{dist}(Q_i, \mathbb{R}^n \setminus U_\lambda)$.

We construct a partition of unity associated with the covering $2Q_i$, $i=1,2,\dots$

First, let $\phi_i \in C_0^\infty(2Q_i)$ s.t. $0 \le \phi_i \le 1$,

$\phi_i = 1$ on Q_i and

$$|D\phi_i| \le \frac{c}{\operatorname{diam}(Q_i)}.$$



Then we define

$$\varphi_i(x) = \frac{\phi_i(x)}{\sum_{j=1}^{\infty} \phi_j(x)}, \quad i=1,2,\dots$$

observe that the sum is over finitely many terms only since $\phi_j \in C_0^\infty(2Q_j)$ and $\sum_{j=1}^{\infty} \chi_{2Q_j} \le N < \infty$.

The functions φ_i have the property

$$\sum_{i=1}^{\infty} \varphi_i(x) = \chi_{U_\lambda}(x)$$

for every $x \in \mathbb{R}^m$. Then we define u_λ by

$$u_\lambda(x) = \begin{cases} u(x), & x \in \mathbb{R}^m \setminus U_\lambda, \\ \sum_{i=1}^{\infty} \varphi_i(x) u_{2Q_i}, & x \in U_\lambda . \end{cases}$$

This function is a Whitney type extension of u to the set U_λ.

<u>Claim I</u>: $\|u_\lambda\|_{1,p,U_\lambda} \leq c\, \|u\|_{1,p,U_\lambda}$

<u>Reason</u>:

$$\int_{U_\lambda} |u_\lambda(x)|^p\,dx = \int_{\mathbb{R}^m} \Big| \sum_{i=1}^{\infty} \varphi_i(x) u_{2Q_i} \Big|^p dx$$

$$\leq c \sum_{i=1}^{\infty} \int_{2Q_i} |u_{2Q_i}|^p\,dx$$

$2Q_i$ are of bounded overlap

$$\leq c \sum_{i=1}^{\infty} |2Q_i| \fint_{2Q_i} |u(x)|^p\,dx$$

Hölder

$$\leq c \int_{U_\lambda} |u(x)|^p\,dx \implies \|u_\lambda\|_{p,U_\lambda} \leq c\,\|u\|_{p,U_\lambda}$$

$2Q_i$ are of bounded overlap

Then we turn to the gradient estimate. We recall that

$$\psi(x) = \sum_{i=1}^{\infty} \varphi_i(x) = 1 \quad \forall\, x \in U_\lambda$$

and $\psi \in C^\infty(U_\lambda)$. Since the cubes $2Q_i$, $i = 1, 2, \dots$, are of bounded overlap, we have

$$D\psi(x) = \sum_{i=1}^{\infty} D\varphi_i(x) = 0$$

for every $x \in U_\lambda$. Hence

$$|Du_\lambda(x)| = \left| \sum_{i=1}^{\infty} D\varphi_i(x) u_{2Q_i} \right|$$

$$= \left| \sum_{i=1}^{\infty} D\varphi_i(x) \left(u(x) - u_{2Q_i} \right) \right|$$

$$\sum_{i=1}^{\infty} D\varphi_i(x) = 0$$

$$\leq c \sum_{i=1}^{\infty} \operatorname{diam}(Q_i)^{-1} |u(x) - u_{2Q_i}| \chi_{2Q_i}(x)$$

and

$$|Du_\lambda(x)|^p \leq c \sum_{i=1}^{\infty} \operatorname{diam}(Q_i)^{-p} |u(x) - u_{2Q_i}|^p \chi_{2Q_i}(x),$$

since $2Q_i$, $i = 1, 2, \dots$, are of bounded overlap.

This implies

$$\int_{U_\lambda} |Du_\lambda|^p\,dx$$

$$\le C \int_{U_\lambda} \Big(\sum_{i=1}^\infty \operatorname{diam}(Q_i)^{-p} |u(x) - u_{2Q_i}|^p \chi_{2Q_i}(x)\Big)\,dx$$

$$\le C \sum_{i=1}^\infty \int_{2Q_i} \operatorname{diam}(Q_i)^{-p} |u(x) - u_{2Q_i}|^p\,dx$$

$$\le C \sum_{i=1}^\infty \int_{2Q_i} |Du|^p\,dx$$

Poincaré

$$\le C \int_{U_\lambda} |Du|^p\,dx \Rightarrow \|Du_\lambda\|_{p,U_\lambda} \le C\|Du\|_{p,U_\lambda}$$

$2Q_i$ are of bdd overlap

From this we conclude that

$$\|u_\lambda\|_{1,p,U_\lambda} \le C\|u\|_{1,p,U_\lambda}.$$

<u>Claim II</u>: $u_\lambda \in C^{0,1-\alpha}(\mathbb{R}^n)$

Clearly $u|_{\mathbb{R}^n \setminus U_\lambda}$ is $(1-\alpha)$-Hölder continuous.

This is an exercise.

Claim III : $u_\lambda \in W^{1,p}(\mathbb{R}^n)$

Reason : $u_\lambda \in C^{0,1-\alpha}(\mathbb{R}^n) \cap W^{1,p}(U_\lambda)$,

$u = v$ in $\mathbb{R}^n \setminus U_\lambda$, $u \in W^{1,p}(\mathbb{R}^n)$

$\Rightarrow w = u - v \in W^{1,p}(U_\lambda)$ and $w = 0$ in $\mathbb{R}^n \setminus U_\lambda$

By the ACL-property w is absolutely continuous on almost every line parallel to the coordinate axes. Take any such line. Now w is absolutely continuous on the part of the line which intersects U_λ. On the other hand, $w = 0$ in the complement of U_λ and hence u_λ is absolutely continuous on the whole line. This completes the proof of Claim III.

Now

$$\| u - u_\lambda \|_{1,p} \underset{\substack{\uparrow \\ u = u_\lambda \text{ on } \mathbb{R}^n \setminus U_\lambda}}{=} \| u - u_\lambda \|_{1,p,U_\lambda}$$

$$\le \| u \|_{1,p,U_\lambda} + \| u_\lambda \|_{1,p,U_\lambda}$$

$$\underset{\substack{\uparrow \\ \text{claim I}}}{\le} c \| u \|_{1,p,U_\lambda} \underset{\substack{\uparrow \\ |U_\lambda| \to 0}}{\longrightarrow} 0 \quad \text{as } \lambda \to \infty$$

A similar argument as in the proof of Theorem 1.5 implies that

$$C_{\alpha p}(\{x \in \mathbb{R}^n : M_{\alpha p}|Du|^p(x) > \lambda^p\}) \leq \frac{c}{\lambda^p} \int_{\mathbb{R}^n} |Du(x)|^p \, dx$$

for all $\lambda > 0$. Here we also use the fact that

$$C_{\alpha p}(B(x,r)) = c\, r^{n-\alpha p}.$$

$\square$

6. Very weak solutions

A function $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the p-Laplace equation

$$\operatorname{div} |Du|^{p-2} Du = 0$$

if

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \qquad (6.1)$$

for all $\varphi \in C_0^\infty(\Omega)$.

The function $u \in W^{1,q}_{loc}(\Omega)$, $q \geq \max\{1, p-1\}$, which satisfies (6.1) is a very weak solution.

Question: Is a very weak solution a weak solution?

6.2. Theorem. There is $q < p$ such that if $u \in W^{1,q}_{loc}(\Omega)$ satisfies (6.1), then $u \in W^{1,p}_{loc}(\Omega)$.

Proof: We consider the following global version of the problem: $u \in W^{1,q}(\mathbb{R}^n)$ satisfies (6.1) $\Rightarrow u = 0$.

$E_\lambda = \{ x \in \mathbb{R}^n : M|Du|(x) > \lambda \}$, $\lambda > 0$

$u|_{\mathbb{R}^n \setminus E_\lambda}$ is $c\lambda$-Lipschitz

$\Rightarrow \exists$ $c\lambda$-Lipschitz $u_\lambda : \mathbb{R}^n \to \mathbb{R}$ s.t.

Theorem 5.1 $\qquad u_\lambda = u$ in $\mathbb{R}^n \setminus E_\lambda$.

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot Du_\lambda \, dx = 0$$

$$\Rightarrow \int_{\mathbb{R}^n \setminus E_\lambda} |Du|^p \, dx = - \int_{E_\lambda} |Du|^{p-2} Du \cdot Du_\lambda \, dx$$

$$\leq c\lambda \int_{E_\lambda} |Du|^{p-1} \, dx \quad \forall \lambda > 0$$

$$\Rightarrow \underbrace{\int_0^\infty \lambda^{-\varepsilon-1} \int_{\{M|Du| \leq \lambda\}} |Du|^p \, dx \, d\lambda}_{= \frac{1}{\varepsilon} \int_{\mathbb{R}^n} |Du|^p (M|Du|)^{-\varepsilon} dx} \leq c \underbrace{\int_0^\infty \lambda^{-\varepsilon} \int_{\{M|Du| > \lambda\}} |Du|^{p-1} \, dx \, d\lambda}_{= \frac{1}{1-\varepsilon} \int_{\mathbb{R}^n} |Du|^{p-1} (M|Du|)^{1-\varepsilon} dx}$$

$$\Rightarrow \int_{\mathbb{R}^n} |Du|^p (M|Du|)^{-\varepsilon} \, dx \leq c \frac{\varepsilon}{1-\varepsilon} \int_{\mathbb{R}^n} |Du|^{p-1} (M|Du|)^{1-\varepsilon} \, dx$$

$$\Rightarrow \int_{\mathbb{R}^n} (M|Du|)^{p-\varepsilon} \, dx \leq c \int_{\mathbb{R}^n} |Du|^{p-\varepsilon} \, dx$$

Theorem 1.6

$$=: c \int_{\{|Du| \leq \delta M|Du|\}} |Du|^{p-\varepsilon} \, dx + c \int_{\{|Du| > \delta M|Du|\}} |Du|^{p-\varepsilon} \, dx$$

$$\leq c\,\delta^{p-\varepsilon} \int_{\mathbb{R}^m} (M|Du|)^{p-\varepsilon}\,dx + c\,\delta^{-\varepsilon} \int_{\mathbb{R}^m} |Du|^p (M|Du|)^{-\varepsilon}\,dx$$

choosing $\delta > 0$ small enough we have

$$\int_{\mathbb{R}^m} (M|Du|)^{p-\varepsilon}\,dx \leq c \int_{\mathbb{R}^m} |Du|^p (M|Du|)^{-\varepsilon}\,dx$$

$$\leq c\,\frac{\varepsilon}{1-\varepsilon} \int_{\mathbb{R}^m} |Du|^{p-1} (M|Du|)^{1-\varepsilon}\,dx$$

$$\leq c\,\frac{\varepsilon}{1-\varepsilon} \int_{\mathbb{R}^m} (M|Du|)^{p-1} (M|Du|)^{1-\varepsilon}\,dx$$

$$= c\,\frac{\varepsilon}{1-\varepsilon} \int_{\mathbb{R}^m} (M|Du|)^{p-\varepsilon}\,dx$$

$\Rightarrow$ $Du = 0$

choose $\varepsilon > 0$ small enough

$\Rightarrow$ $u = 0$.

$u \in W^{1,q}(\mathbb{R}^m)$