Variational problems with linear growth condition on metric measure spaces

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Collaboration

- Heikki Hakkarainen, Juha Kinnunen, Panu Lahti, Pekka Lehtelä, *Relaxation and integral representation for functionals of linear growth on metric measure spaces*, submitted.
- Heikka Hakkarainen, Juha Kinnunen and Panu Lahti, *Regularity of minimizers of the area functional in metric spaces*, Adv. Calc. Var., to appear.
- Juha Kinnunen, Riikka Korte, Nageswari Shanmugalingam and Heli Tuominen, *Pointwise properties of functions of bounded variation on metric spaces*, Rev. Mat. Complutense 27 (2014), 41–67.
- Juha Kinnunen, Riikka Korte, Andrew Lorent and Nageswari Shanmugalingam, *Regularity of sets with quasiminimal boundary surfaces in metric spaces*, J. Geom. Anal. 23 (2013), 1607–1640.
- Juha Kinnunen, Riikka Korte, Nageswari Shanmugalingam and Heli Tuominen, *A characterization of Newtonian functions with zero boundary values*, Calc. Var. Partial Differential Equations 43 (2012), no. 3–4, 507–528.

Plan of the talk

- **Goal:** The main goal is to develop theory of the calculus of variations with linear growth in metric measure spaces.
- **Questions:** Existence, regularity and integral representations.
- **Tool:** Functions of bounded variation (BV) on metric measure spaces.

Example

- The total variation

$$\|Du\| = \int |Du| = \sup \int \sum_{i=1}^{n} u \frac{\partial \psi_i}{\partial x_i} \, dx,$$

where the supremum is taken over all functions $\psi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\psi\|_\infty \leq 1$.

- The area functional

$$\int \sqrt{1 + |Du|^2} = \sup \int \left(\psi_{n+1} + \sum_{i=1}^{n} u \frac{\partial \psi_i}{\partial x_i} \right) dx,$$

where the supremum is taken over all functions $\psi \in C_0^1(\mathbb{R}^n; \mathbb{R}^{n+1})$ with $\|\psi\|_\infty \leq 1$.

Metric measure space

- (X, d, μ) is a complete metric measure space.
- μ is doubling, if there exists a uniform constant $c_D \geq 1$ such that
$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$
for all balls $B(x, r)$ in X.

Upper gradient

Definition (Heinonen and Koskela, 1998)

A nonnegative Borel function g on X is an upper gradient of a function u, if for all $x, y \in X$ and for all paths γ joining x and y in X,

$$|u(x) - u(y)| \leq \int_\gamma g \, ds.$$

Remarks

- If u has an upper gradient in $L^1(X)$, then there exists a minimal upper gradient g_u of u such that

$$g_u \le g \quad \mu\text{-almost everywhere in } X$$

  for all upper gradients $g \in L^1(X)$.
- Using upper gradients it is possible to define first order Sobolev spaces (Shanmugalingam, 2000) and functions of bounded variation (Ambrosio, Miranda Jr. and Pallara, 2003) on a metric measure space.

Poincaré inequality

Definition

The space X supports a Poincaré inequality, if there exist constants $c_P > 0$ and $\tau \geq 1$ such that for all balls $B(x, r)$, $u \in L^1_{\mathrm{loc}}(X)$ and for all upper gradients g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq c_P r \fint_{B(x,\tau r)} g \, d\mu,$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Doubling and Poincaré

Doubling condition and the Poincaré inequality imply Sobolev inequalities. This is important for partial differential equations and the calculus of variations. From now on we work under these assumptions.

BV-functions through relaxation

Definition (Ambrosio, 2001 and Miranda Jr., 2003)

Let $\Omega \subset X$ be an open set. The total variation of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is defined as

$$\|Du\|(\Omega) = \inf \Big\{ \liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu : \\ u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \Big\},$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i. We say that a function $u \in L^1(\Omega)$ is of bounded variation, $u \in BV(\Omega)$, if

$$\|Du\|(\Omega) < \infty.$$

Theory for BV-functions

- $u \in BV_{\mathrm{loc}}(X) \Longrightarrow \|Du\|(\cdot)$ is a Borel regular outer measure on X.
- Poincaré inequality, coarea formula and relative isoperimetric inequality are available.
- Lower semicontinuity of the total variation measure with respect to L^1-convergence and a compactness theorem hold true.
- If we consider L^p-integral with $p > 1$ instead of L^1 we obtain the Sobolev space.

Linear growth conditions

- Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex increasing function that satisfies the linear growth condition

$$mt \leq f(t) \leq M(1+t)$$

  for all $t > 0$, with some constants $0 < m \leq M < \infty$.
- **Examples:** $f(t) = |t|$, $f(t) = \sqrt{1+|t|^2}$.

Linear growth functional

Definition

Let $\Omega \subset X$ be an open. For $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the linear growth functional by relaxation as

$$\mathcal{F}(u,\Omega) = \inf \Big\{ \liminf_{i\to\infty} \int_\Omega f(g_{u_i})\, d\mu : \\ u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \Big\},$$

where g_{u_i} is the minimal upper gradient of u_i.

Observe: If $f(t) = |t|$, we obtain the total variation as in the definition of BV. In general,

$$m\|Du\|(\Omega) \leq \mathcal{F}(u,\Omega) \leq M(\mu(\Omega) + \|Du\|(\Omega)).$$

Measure properties

Goal: We want to use $\mathcal{F}(u, \cdot)$ as a measure.

Definition

We define $\mathcal{F}(u, A)$ for any set $A \subset X$ by

$$\mathcal{F}(u, A) = \inf\{\mathcal{F}(u, \Omega) : \Omega \text{ is open}, A \subset \Omega\}.$$

Theorem (Hakkarainen, Lahti, Lehtelä and K., 2014)

If $\Omega \subset X$ is open and $\mathcal{F}(u, \Omega) < \infty$, then $\mathcal{F}(u, \cdot)$ is a Borel regular outer measure on Ω.

Boundary values in BV

Definition (Hakkarainen, Lahti and K., 2013)

Let Ω and Ω^* be open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $h \in BV(\Omega^*)$. We define the space $BV_h(\Omega)$ as the space of functions $u \in BV(\Omega^*)$ such that $u = h$ μ-almost everywhere in $\Omega^* \setminus \Omega$.

Observe: (1) When $h = 0$, we get the BV space with zero boundary values $BV_0(\Omega)$. In particular, $u \in BV_h(\Omega)$ if and only if $u - h \in BV_0(\Omega)$.
(2) We could take $\Omega^* = X$ as the reference set, but this is not a big issue.

Boundary value problem

Definition (Hakkarainen, Lahti and K., 2013)

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $h \in BV(\Omega^*)$. A function $u \in BV_h(\Omega)$ is a minimizer with the boundary values h, if

$$\mathcal{F}(u, \Omega^*) = \inf \mathcal{F}(v, \Omega^*),$$

where the infimum is taken over all $v \in BV_h(\Omega)$.

Example

Let $f(t) = \sqrt{1+|t|^2}$. In the Euclidean case with Lebesgue measure we have the integral representation

$$\mathcal{F}(v,\Omega^*) = \int_\Omega \sqrt{1+|Dv|^2} + \int_{\partial\Omega} |v-h| \, d\mathcal{H}^{n-1} + \int_{\Omega^*\setminus\overline{\Omega}} \sqrt{1+|Dh|^2}.$$

for $v \in BV(\Omega^*)$. Minimizers do not depend on Ω^*, but the value of the generalized area functional does. However, if we are only interested in local regularity of the minimizers, the value of the area functional is irrelevant.

Observe: The penalty term takes care boundary values.

Existence of minimizers

Theorem (Hakkarainen, Lahti and K., 2013)

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$. Then for every $h \in BV(\Omega^*)$ there exists a minimizer $u \in BV_h(\Omega)$ of the linear growth functional with the boundary values h.

Proof.

Direct method in the calculus of variations: Growth conditions + Sobolev-Poincaré inequality + compactness result in BV + lower semicontinuity of $\mathcal{F}$ with respect to L^1-convergence. □

Remark: Solutions are not unique.

Local boundedness of minimizers

Theorem (Hakkarainen, Lahti and K., 2013)

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $h \in BV(\Omega^*)$. Let $u \in BV_h(\Omega)$ be a minimizer with the boundary values h. Assume that $B(x,R) \subset \Omega$ with $R > 0$, and let $k \in \mathbb{R}$. Then

$$\operatorname*{ess\,sup}_{B(x,R/2)} u \le k + c \fint_{B(x,R)} (u-k)_+ \, d\mu + R,$$

where the constant c depends only on the doubling constant and the constants in the Poincaré inequality.

Proof.

De Giorgi's method. □

De Giorgi condition

Theorem

For every $k \in \mathbb{R}$*, we have*

$$\|D(u-k)_+\|(B(x,r)) \le \frac{2}{R-r}\int_{B(x,R)} (u-k)_+ \, d\mu + \mu(A_{k,R}),$$

where $A_{k,R} = B(x,R) \cap \{u > k\}$.

Proof.

Minimizing sequence $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^*)$ + choose the test function $u_i - \eta(u_i - k)^+$ + the fact that u_i almost minimizes $\mathcal{F}$. □

Failure of interior continuity

Unexpected phenomenon: A minimizer of a linear growth functional may be discontinuous at interior point. In this sense, the previous local boundedness result is optimal and finer regularity theory fails to exist.

Observe: Minimizers with superlinear growth are continuous (Shanmugalingam and K., 2001).

Failure of interior continuity

Example

Let $\mathbb{R}$ be equipped with the Euclidean distance, $\Omega = (-1, 1)$ and $d\mu = w(x)\,dx$ with

$$w(x) = \min\left\{\sqrt{2}, \sqrt{1 + x^{4/3}}\right\}.$$

Note that w is continuously differentiable and $1 \le w \le \sqrt{2}$ in Ω. Let u be a minimizer of the problem

$$\mathcal{F}(u, \Omega) = \int_{-1}^{1} \sqrt{1 + (u')^2}\, w\, dx,$$

with boundary values $u(-1) = -a$ and $u(1) = a$. By choosing a large enough ($a > 3$ will work), we obtain a jump discontinuity at the origin.

Failure of interior continuity

Example

This corresponds to minimizing the integral representation

$$\mathcal{F}(u) = \int_{-1}^{1} \sqrt{1 + (u')^2}\, w\, dx + \int_{-1}^{1} w\, d|(Du)^s| \\ + w(-1)|u(-1) + a| + w(1)|u(1) - a|,$$

where the boundary values are interpreted in the sense of traces and $(Du)^a = u'\, dx$ denotes the absolutely continuous part and $(Du)^s$ the singular part of the variation measure Du.

Failure of interior continuity

Example

- If $u \in W^{1,1}((-1,1))$, then u satisfies the weak form of the Euler-Lagrange equation

$$\frac{\partial}{\partial x}\left(\frac{u'(x)w(x)}{\sqrt{1+(u'(x))^2}}\right) = 0.$$

- This implies that

$$\left|u'(x)\right| = \left(\frac{w(x)^2}{C^2} - 1\right)^{-1/2}$$

for almost every $x \in (-1,1)$.

- $$a = \frac{1}{2}|u(1) - u(-1)| \leq \int_{-1}^{1} |u'(x)|\, dx \leq 3 < a.$$

The integral representation

Example

In the Euclidean case with Lebesgue measure we have an approach to the minimization problem via the decomposition of the measure

$$\mathcal{F}(u,\Omega) = \int_\Omega f(a)\,dx + f_\infty \|Du\|^s(\Omega),$$

where

$$d\,\|Du\| = a\,d\mu + d\,\|Du\|^s$$

is the decomposition of the variation measure into the absolutely continuous and singular parts.

Question: Is this true in the metric setting?
Answer: Yes, but in an unexpected form.

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The integral representation

Theorem (Hakkarainen, Lahti, Lehtelä and K., 2014)

Let Ω be an open set, and let $u \in L^1_{loc}(\Omega)$ with $\mathcal{F}(u,\Omega) < \infty$. Let

$$d\,\|Du\| = a\,d\mu + d\,\|Du\|^s$$

be the decomposition of the variation measure into the absolutely continuous and singular parts, where $a \in L^1(\Omega)$ and $\|Du\|^s$ is the singular part. Then we have

$$\begin{aligned}\int_\Omega f(a)\,d\mu + f_\infty\|Du\|^s(\Omega) &\le \mathcal{F}(u,\Omega)\\ &\le \int_\Omega f(Ca)\,d\mu + f_\infty\|Du\|^s(\Omega),\end{aligned}$$

where $f_\infty = \lim_{t\to\infty} \frac{f(t)}{t}$ is the recession factor.

Remarks

Let $\mathcal{F}^a(u,\cdot)$ and $\mathcal{F}^s(u,\cdot)$ be the absolutely continuous and singular parts of $\mathcal{F}(u,\cdot)$ with respect to μ. Assume that $\mathcal{F}(u,\Omega) < \infty$ and let $A \subset \Omega$ a μ-measurable subset of Ω.

- For the singular part, we obtain the integral representation

  $$\mathcal{F}^s(u,A) = f_\infty \|Du\|^s(A).$$

  This is analogous to the Euclidean case.
- For the absolutely continuous part we only get an integral representation up to a constant

  $$\int_A f(a)\, d\mu \le \mathcal{F}^a(u,A) \le \int_A f(Ca)\, d\mu.$$

- A counterexample shows that the constant cannot be dismissed already on the weighted real line.

Example

Let $\Omega = [0,1]$ equipped with the Euclidean distance. Take a fat Cantor set $\Delta \subset [0,1]$ with $\mathcal{L}^1(\Delta) = \frac{1}{2}$. Equip X with the weighted Lebesgue measure $d\mu = w\, d\mathcal{L}^1$, where $w = 2$ in Δ and $w = 1$ in $\Omega \setminus \Delta$.

There is a Lipschitz function u with $Du = 2\chi_\Delta$ ($a = \chi_\Delta$ and $g_u = 2\chi_\Delta$) and a functional $\mathcal{F}(\cdot, \Omega)$ for which

$$\int_\Omega |Du|\, d\mu > \|Du\|([0,1]) \quad \text{and} \quad \mathcal{F}(u,\Omega) > \int_\Omega f(Du)\, d\mu.$$

The main phenomenon is that the derivatives of an approximating sequence live in the cheaper set $[0,1] \setminus \Delta$ with weight 1, but Du lives in Δ, where it costs more with weight 2. However, $\mathcal{F}(u,\Omega)$ can be constructed so that it places extra weight in $\Omega \setminus \Delta$.

The lower bound

- Take a minimizing sequence $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega f(g_{u_i})\, d\mu \to \mathcal{F}(u, \Omega) \quad \text{as} \quad i \to \infty.$$

- By the growth conditions, we have a subsequence $g_{u_i}\, d\mu \rightharpoonup d\nu$ weakly, where ν is a Radon measure with finite mass in Ω.
- By the definition of the variation measure, we have $\nu \geq \|Du\|$.
- As a nonnegative nondecreasing convex function, f can be presented as

$$f(t) = \sup_{j \in \mathbb{N}} (d_j t + e_j), \quad t \geq 0,$$

for some sequences $d_j, e_j \in \mathbb{R}$, with $d_j \geq 0$, $j = 1, 2, \ldots$, and furthermore $\sup_j d_j = f_\infty$. The result follows from the weak convergence.

The upper bound

- Since the functional $\mathcal{F}(u,\cdot)$ is a Radon measure, we decompose it into the absolutely continuous and singular parts as $\mathcal{F}(u,\cdot) = \mathcal{F}^a(u,\cdot) + \mathcal{F}^s(u,\cdot)$.
- The estimate for $\mathcal{F}^s(u,\cdot)$ follows rather directly by choosing a minimizing sequence and using $f(t) \leq f(0) + tf_\infty$.
- For $\mathcal{F}^a(u,\cdot)$ we approximate u by the discrete convolutions related to Whitney type coverings and partitions of unities.
- Decompose the upper grandients of the approximation as $g_i^a + g_i^s$, show that g_i^a are equi-integrable, extract a weakly converging subsequence g_i^a converges weakly in $L^1(G)$ to a function $\check{a} \leq Ca$, with $C = C(c_d,\lambda)$, and use Mazur's lemma to estimate $\mathcal{F}^a(u,\cdot)$.

A by-product

As a by-product , we have that a BV function is a Sobolev function in a set where the variation measure is absolutely continuous.

Theorem (Hakkarainen, Lahti and K., 2014)

Let $u \in BV(\Omega)$, and let

$$d\,\|Du\| = a\,d\mu + d\,\|Du\|^s$$

be the decomposition of the variation measure, where $a \in L^1(\Omega)$ and $\|Du\|^s$ is the singular part. Let $F \subset \Omega$ be a μ-measurable set for which $\|Du\|^s(F) = 0$. Then, by modifying u on a set of μ-measure zero if necessary, we have

$$u|_F \in N^{1,1}(F) \quad \text{and} \quad g_u \le Ca$$

μ-almost everywhere in F, with $C = C(c_d, c_P, \lambda)$.

Remark. Our previous example shows that the constant cannot be dismissed.

Remarks

- Our previous example shows that the constant cannot be dismissed.
- If $\|Du\|$ is absolutely continuous on the whole of Ω, then $u \in N^{1,1}(\Omega)$ we also have the inequality

$$\int_{\Omega} g_u \, d\mu \leq C\|Du\|(\Omega)$$

with $C = C(c_d, c_P, \lambda)$.

Further developments

Under suitable conditions on the space and the domain, we can establish equivalence between the above minimization problem and minimizing the functional

$$\mathcal{F}(u,\Omega) + f_\infty \int_{\partial\Omega} |T_\Omega u - T_{X\setminus\Omega} h| \theta_\Omega \, d\mathcal{H}$$

over all $u \in BV(\Omega^*)$. Here $T_\Omega u$ and $T_{X\setminus\Omega} u$ are boundary traces and θ_Ω is a strictly positive density function.

Observe: The penalty term takes care boundary values.

P. Lahti, *Extensions and traces of functions of bounded variation on metric spaces*, J. Math. Anal. Appl. (to appear).

Summary

It is possible to develop theory for variational problems with linear growth conditions in the metric setting, but some unexpected phenomena occur already in the weighted Euclidean case.