INdAM Meeting "Singular and Degenerate Evolution Problems"
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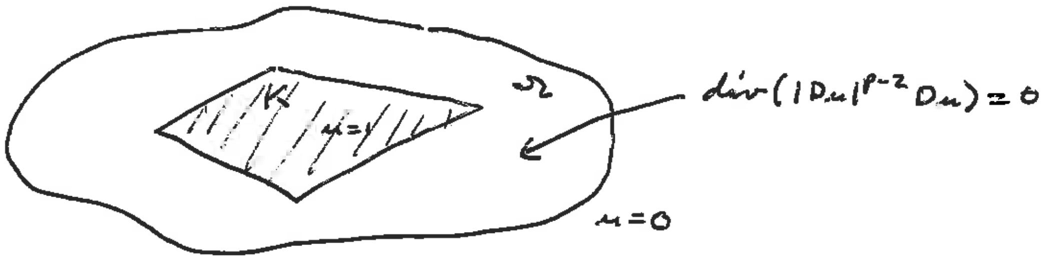
Nonlinear parabolic capacity theory

The elliptic case

$\Omega \subset \mathbb{R}^n$ open, $1 < p < \infty$

$\boxed{K \subset \Omega \text{ compact}}$

$$\mathrm{cap}_p(K,\Omega) = \inf\left\{ \int_\Omega |Du|^p\,dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ in } K \right\}$$



$\boxed{U \subset \Omega \text{ open}}$

$$\mathrm{cap}_p(U,\Omega) = \sup\{\mathrm{cap}_p(K,\Omega) : K \subset U \text{ compact}\}$$

$\boxed{E \subset \Omega \text{ arbitrary}}$

$$\mathrm{cap}_p(E,\Omega) = \inf\{\mathrm{cap}_p(U,\Omega) : U \supset E \text{ open}\}$$

The goal: Capacity is an outer measure, finer than Lebesgue measure, for which all Borel sets are capacitable, i.e.

$$\mathrm{cap}_p(E,\Omega) = \sup\{\mathrm{cap}_p(K,\Omega) : K \subset E \text{ compact}\}$$

$E \subset \Omega$ Borel

<u>Warning</u>: $\mathrm{cap}_p(K,\Omega) = \mathrm{cap}_p(\partial K,\Omega)$, $K \subset \Omega$ compact

$\Rightarrow$ no nontrivial measurable sets

$\Rightarrow$ measure theory is useless (no covering arguments, instead one has to construct test functions)

<u>Why</u>

(1) Exceptional sets for Sobolev functions

(2) Removability results

(3) Wiener's criterion: $x_0 \in \partial\Omega$ is regular for the Dirichlet problem

$$\begin{cases} \mathrm{div}(|Du|^{p-2}Du) = 0 \text{ in } \Omega \\ \lim\limits_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = f(x_0) \end{cases}$$

(holds for more general PDEs as well)

$f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, $u - f \in W^{1,p}_0(\Omega)$, $1 < p \le n$ ($p > n$ OK)

$$\Leftrightarrow \sum_{i=1}^{\infty} \left(\lambda^{i(p-1)} \mathrm{cap}_p(\Omega^c \cap \{\lambda^i < \Phi(x_0 - x) \le \lambda^{i+1}\}) \right)^{\frac{1}{p-1}} = \infty$$

$\forall \lambda > 1$

$$\Phi(x) = \begin{cases} |x|^{\frac{p-n}{p-1}}, & p < n, \\ -\log|x|, & p = n \end{cases}$$

(The fundamental solution)

$-\mathrm{div}(|D\Phi|^{p-2}D\Phi) = c\delta_0$

Second approach

$E \subset \Omega$ arbitrary

$$C_p(E,\Omega) = \sup\{\mu(\Omega) : \operatorname{supp}\mu \subset E,\ 0 \le u_\mu \le 1\}$$

Radon measure

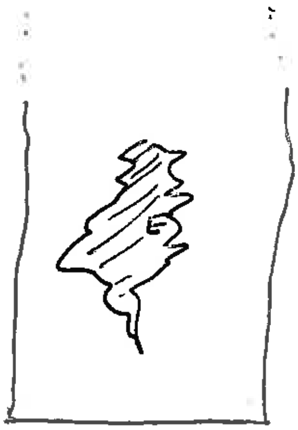
$$\begin{cases} -\operatorname{div}(|Du_\mu|^{p-2}Du_\mu) = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega \end{cases}$$

(The measure data problem)

Physical interpretation: The maximal total charge on E that induces potential bounded by one in the complement.

Fact: $\operatorname{cap}_p(E,\Omega) = C_p(E,\Omega)$ (Kilpeläinen and Malý)

Parabolic (thermal) capacity



Ω

$E \subset \Omega_\infty = \Omega \times (0,\infty)$

$\Omega \subset \mathbb{R}^n$ regular, $2 < p < \infty$

$\operatorname{cap}_p(E,\Omega_\infty)$

$$= \sup\{\mu(\Omega_\infty) : \operatorname{supp}\mu \subset E,\ 0 \le u_\mu \le 1\}$$

Radon measure on $\mathbb{R}^{n+1}$

$$\begin{cases} \dfrac{\partial u_\mu}{\partial t} - \operatorname{div}(|Du_\mu|^{p-2}Du_\mu) = \mu & \text{in } \Omega_\infty \\ u_\mu = 0 & \text{on } \partial_p\Omega_\infty = (\Omega \times \{t=0\}) \cup (\partial\Omega \times (0,\infty)) \end{cases}$$

Remark. The solution exists if μ is a bounded Radon measure.
(Lukkari, Paronetto and K.)
JFA (2010)

Measure data problem

μ Radon measure on $\mathbb{R}^{n+1}$

$u \in L^p(0,\infty; W^{1,p}(\Omega))$ is a weak solution of

locally

$$\frac{\partial u}{\partial t} - \operatorname{div}(|Du|^{p-2}Du) = \mu \text{ in } \Omega_\infty$$

$$\Longleftrightarrow \int_{\Omega_\infty} \left(|Du|^{p-2}Du \cdot D\varphi - u\frac{\partial \varphi}{\partial t}\right) dz = \int_{\Omega_\infty} \varphi \, d\mu \quad \forall \varphi \in C_0^\infty(\Omega_\infty)$$

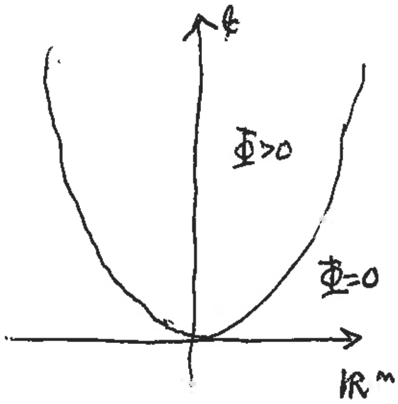
$\boxed{\mu = 0}$: Weak solutions

$\boxed{\mu \geq 0}$: Weak supersolutions

Remark. Every supersolution is a solution to a measure data problem by the Riesz representation theorem. Conversely, every solution of the measure data problem is a supersolution.

Example (Barenblatt's solution)

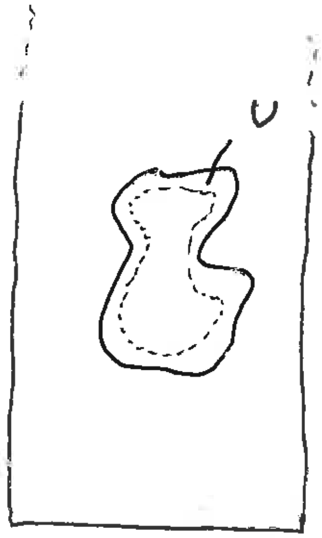
$$\Phi(x,t) = \begin{cases} t^{-\frac{n}{\lambda}} \left[C - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}, & t > 0 \\ 0, \ t \leq 0 \end{cases}$$



$\lambda = n(p-2)+p$, $p > 2$, $\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1$

$$\frac{\partial \Phi}{\partial t} - \operatorname{div}(|D\Phi|^{p-2}D\Phi) = \delta_0$$

Warning: $\Phi \notin L^p_{loc}(-\infty,\infty; W^{1,p}_{loc}(\mathbb{R}^n))$



Remark: $U \subset\subset \Omega_\infty$ smooth

$\varphi_\varepsilon \in C_0^\infty(U)$

$\varphi_\varepsilon = 1$ on $\{x \in U : dist(x, \partial U) > \varepsilon\}$

$\uparrow$ topological boundary

$$\mu_u(U) = \int_U 1 \, d\mu_u$$

$$= \lim_{\varepsilon \to 0} \int_U \varphi_\varepsilon \, d\mu_u$$

$$= \lim_{\varepsilon \to 0} \int_{U \atop {D\varphi_\varepsilon \neq 0 \atop \frac{\partial \varphi_\varepsilon}{\partial t} \neq 0}} \left(|Du|^{p-2} Du \cdot D\varphi_\varepsilon - u \frac{\partial \varphi_\varepsilon}{\partial t} \right) dz$$

Observation: $\mu_u(U)$ depends only on values of u near the boundary of U.

Boundary values

$u = 0$ on $\partial_p \Omega_\infty \iff u \in L^p(0,T; W_0^{1,p}(\Omega))$ and

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \int_\Omega |u|^2 \, dx\, dt = 0$$

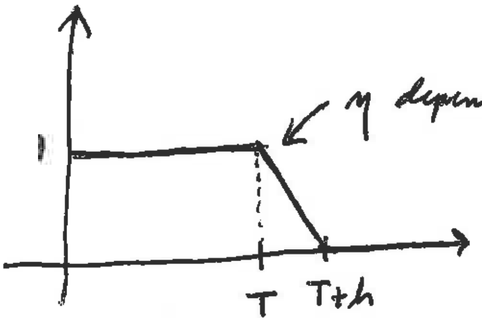
Remark. supp μ compact $\Rightarrow$ $u \to 0$ uniformly as $t \to \infty$

compare to the Barenblatt

A comparison result

u, v solution to the measure data problem with zero boundary values and $\mu_v \le \mu_u \Rightarrow v \le u$ in Ω_∞.

Proof:



Choose $\eta(v-u)_+$ as a test function (zero boundary values)

$$0 \underset{\mu_v \le \mu_u}{\le} \int_{\Omega_\infty} \eta (v-u)_+ \, d\mu_u - \int_{\Omega_\infty} \eta (v-u)_+ \, d\mu_v$$

$$= \int_{\Omega_\infty} \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv \right) \cdot D\left(\eta (v-u)_+\right) dz$$

$$- \int_{\Omega_\infty} (u-v) \frac{\partial}{\partial t} \left(\eta (v-u)_+\right) dz$$

$$= \underbrace{- \int_{\Omega_\infty \cap \{v \geq u\}} \eta \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv \right) \cdot (Du - Dv)\, dz}_{\leq 0 \text{ (monotonicity)}}$$

$$+ \underbrace{\int_{\Omega_\infty} \eta (v-u)_+ \frac{\partial}{\partial t}(u-v)\, dz}$$

$$= - \int_{\Omega_\infty} \tfrac{1}{2} \eta \frac{\partial}{\partial t}\left[(v-u)_+^2 \right] dz$$

$$= \frac{1}{2} \int_{\Omega_\infty} \frac{\partial \eta}{\partial t} (v-u)_+^2\, dz$$

$$= -\frac{1}{2h} \int_T^{T+h} \int_\Omega (v-u)_+^2\, dx\, dt$$

$$\underset{h \to 0}{\Rightarrow} \int_\Omega \underbrace{(v-u)_+^2 (x,T)}_{\geq 0}\, dx \leq 0 \quad \text{for a.e. } T > 0$$

$$\Rightarrow (v-u)_+ = 0 \text{ a.e. in } \Omega_\infty. \quad \square$$

$\mathrm{cap}_p(\bigcup_{i=1}^{\infty} E_i; \Omega_\infty)$
$\shortparallel$

<u>Theorem</u>. $E_i \subset \Omega_\infty$, $i=1,2,\dots \Rightarrow \mathrm{cap}_p(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} \mathrm{cap}_p(E_i)$
(countable subadditivity)

<u>Conclusion</u>: Capacity is an outer measure.

<u>Proof</u>: $E = \bigcup_{i=1}^{\infty} E_i$

$\boxed{\mathrm{cap}_p(E) < \infty}$: $\varepsilon > 0$

$\exists\, \mu$ s.t. $\mathrm{supp}\,\mu \subset E$, $0 \le u_\mu \le 1$ and $\mu(\Omega_\infty) \ge \mathrm{cap}_p(E) - \varepsilon$

$\mu_i = \mu \llcorner E_i$, $i = 1,2,\dots$,

Comparison result $\Rightarrow 0 \le u_{\mu_i} \le 1$, $\mathrm{supp}\,\mu_i \subset E$, $\mu_i(\Omega_\infty) \le \mathrm{cap}_p(E_i)$
$\mu_i \le \mu$, existence of u_{μ_i} okay

$$\mathrm{cap}_p(E) \le \mu(\Omega_\infty) + \varepsilon \le \sum_{i=1}^{\infty} \mu_i(\Omega_\infty) + \varepsilon \le \sum_{i=1}^{\infty} \mathrm{cap}_p(E_i) + \varepsilon$$
$\mathrm{supp}\,\mu \subset E$

$\varepsilon \to 0$ $\blacksquare$

$\boxed{\mathrm{cap}_p(E) = \infty}$: $M > 0$ large

$\exists\, \mu$ s.t. $\mathrm{supp}\,\mu \subset E$, $0 \le u_\mu \le 1$ and $\mu(\Omega_\infty) \ge M$

$$M \le \mu(\Omega_\infty) \le \sum_{i=1}^{\infty} \mu_i(\Omega_\infty) \le \sum_{i=1}^{\infty} \mathrm{cap}_p(E_i)$$

$M \to \infty$ $\square$

<u>Other properties with similar proofs</u>:

(i) $E_1 \subset E_2 \subset \dots \subset \Omega_\infty \Rightarrow \lim_{i\to\infty} \mathrm{cap}_p(E_i) = \mathrm{cap}_p(\bigcup E_i)$

(ii) $E \subset \Omega_\infty$ Borel $\Rightarrow \mathrm{cap}_p(E) = \sup\{\mathrm{cap}(K) : K \subset E,\ K \text{ compact}\}$

Capacitary potentials

T. Kuusi, Diff. Int. Eq. (2009)

$K \subset \Omega_\infty$ compact

$R_K = \inf\{u : u \text{ lsc supersolution},\ u \geq 1 \text{ on } K\}$

$$\hat{R}_K(x,t) = \liminf_{(y,s)\to(x,t)} R_K(y,s)$$ (Lower semicontinuous regularization)

$$= \lim_{r\to 0} \operatorname{ess\,inf}_{B(x,r)\times(t-r^p,\,t+r^p)} R_K$$

Korte, Kuusi, Siljander, J. Diff. Eq. (2009)

Properties: Lindqvist, Parviainen, JFA (2012)

(i) $\hat{R}_K = R_K$ a.e. in Ω_∞ (q.e.?)

(ii) $\hat{R}_K$ is a weak supersolution in Ω_∞ ← $\hat{R}_K$ has Riesz measure μ_K

(iii) $\hat{R}_K$ is a weak solution in $\Omega_\infty \setminus K$ ← $\hat{R}_K$ is continuous in $\Omega_\infty \setminus K$

(iv) $\hat{R}_K = 0$ on $\partial_p \Omega_\infty$ ← Ω smooth, $\hat{R}_K$ takes the boundary values continuously

Remark. $\hat{R}_K$ is the solution of the obstacle problem with χ_K. (Applies to other obstacles as well)

Theorem. $\operatorname{cap}_p(K) = \mu_K(\Omega_\infty)$

Moral: The capacitary potential of K is the smallest supersolution above χ_K.

Proof: $\boxed{\text{"}\geq\text{"}}$: $\hat{R}_K$ weak supersolution, $0 \leq \hat{R}_K \leq 1$

$\Rightarrow \mu_K(\Omega_\infty) \leq \operatorname{cap}_p(K)$

$\boxed{\text{"}\leq\text{"}}$: $\psi_i \in C_0^\infty(\Omega_\infty)$, $\psi_i \searrow \chi_K$ as $i \to \infty$,

$\psi_i = 1 + \varepsilon_i$ on K,

$\psi_i = 0$ in $\Omega_\infty \setminus K'$, $K \subset K' \subset\subset \Omega_\infty$, K' compact

u_i is a solution of the obstacle problem with ψ_i

$\operatorname{supp}\mu_v \subset K$

Let v be a weak supersolution in Ω_∞ with $0 \le v \le 1$.

<u>Facts:</u> $\mu_v(K) \le \mu_{u_i}(\Omega_\infty) = \mu_{u_i}(K')$

u_i is a weak solution in $\Omega \setminus K'$

$u_i \to \hat{R}_K$ a.e. in Ω_∞

$\mu_{u_i} \to \mu_K$ weakly

$$\Rightarrow \mu_v(K) \underset{\text{above}}{\le} \limsup_{i\to\infty} \mu_{u_i}(K')$$

$$\underset{\text{weak convergence}}{\le} \mu_K(K') = \mu_K(\Omega_\infty)$$

$\hat{R}_K$ is a solution in $\Omega \setminus K'$

$$\Rightarrow \operatorname{cap}_p(K) \le \mu_K(\Omega_\infty). \quad \square$$

<u>Theorem.</u> $\Omega_\infty \supset K_1 \supset K_2 \supset \dots$, K_i compact $\Rightarrow$

$$\lim_{i\to\infty} \operatorname{cap}_p(K_i) = \operatorname{cap}\left(\bigcap_{i=1}^{\infty} K_i\right)$$

<u>Proof:</u> $K = \bigcap_{i=1}^{\infty} K_i$

$\hat{R}_{K_i}$ is a bounded and decreasing sequence of weak supersolutions in Ω_∞,

$\hat{R}_{K_i} \to \hat{R}_K$ a.e. in Ω_∞ and

$\mu_{K_i} \to \mu_K$ weakly as $i \to \infty$.

$\varphi \in C_0^\infty(\Omega_\infty)$, $\varphi = 1$ on K_1

$$\lim_{i\to\infty} \mathrm{cap}_p(K_i) = \lim_{i\to\infty} \mu_{K_i}(K_1)$$

capacitary potential

$$\le \lim_{i\to\infty} \int_{\Omega_\infty} \varphi \, d\mu_{K_i}$$

$$= \int_{\Omega_\infty} \varphi \, d\mu_K = \mu_K(K_1) = \mathrm{cap}_p(K)$$

$\varphi = 1$ on K_1, $\operatorname{supp} \mu_K \subset K$

weak convergence

$$\mathrm{cap}_p(K_i) \ge \mathrm{cap}_p(K) \Rightarrow \lim_{i\to\infty} \mathrm{cap}_p(K_i) \ge \mathrm{cap}_p(K) \qquad \square$$

<u>An open question:</u>

$$\mathrm{cap}_p(K_1 \cup K_2) \le \mathrm{cap}_p(K_1) + \mathrm{cap}_p(K_2) - \mathrm{cap}_p(K_1 \cap K_2)$$

(Strong subadditivity)

<u>Variational approach</u>

M. Pierre, SIAM J. Math. Anal. 1983

J. Droniou, A. Porretta and A. Prignet, Potential Anal. 2003

B. Avelin, T. Kuusi and M. Parviainen, Variational parabolic capacity 2014

$p > 2$

$$V(\Omega_\infty) = L^p(0,T; W_0^{1,p}(\Omega)), \quad V'(\Omega_\infty) = L^p(0,\infty; W_0^{1,p}(\Omega))'$$

$$= L^{p'}(0,\infty; W_0^{1,p}(\Omega)')$$

$$\|u\|_{V(\Omega_\infty)} = \left(\int_{\Omega_\infty} |Du|^p \, dz\right)^{\frac{1}{p}}$$

$$\|u\|_{V'(\Omega_\infty)} = \sup\left\{\left|\int_{\Omega_\infty} u\varphi \, dz\right| : \varphi \in C_0^\infty(\Omega_\infty), \ \|\varphi\|_{V(\Omega_\infty)} \le 1\right\}$$

$$W(\Omega_\infty) = \left\{u \in V(\Omega_\infty) : \frac{\partial u}{\partial t} \in V'(\Omega_\infty)\right\}$$

with the norm $\|u\|_{V(\Omega_\infty)} + \left\|\frac{\partial u}{\partial t}\right\|_{V'(\Omega_\infty)}$

$$\|u\|_{W(\Omega_\infty)} = \|u\|^p_{V(\Omega_\infty)} + \left\|\frac{\partial u}{\partial t}\right\|^{p'}_{V'(\Omega_\infty)}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

nonhomogeneous quantity

$K \subset \Omega_\infty$ compact

$$\operatorname{cap}_{var}(K, \Omega_\infty) = \inf\left\{\|u\|_{W(\Omega_\infty)} : u \in C_0^\infty(\Omega_\infty), \ u \ge 1 \text{ on } K\right\}$$

Extend to more general sets as before.

<u>Theorem.</u> $c_1 \operatorname{cap}_{var}(K, \Omega_\infty) \le \operatorname{cap}(K, \Omega_\infty) \le c_2 \operatorname{cap}_{var}(K, \Omega_\infty)$, $K \subset \Omega_\infty$ compact, $c_i = c_i(n,p)$, $i = 1,2$.

<u>Warning</u>: Because of inhomogeneity, this capacity is dangerous. For example it is not subadditive without a multiplicative constant. DPP use the norm instead and this is subadditive. These capacities have the same zero sets, but they are not equivalent, in general.

<u>Remark.</u> This capacity does not see the PDE and hence the parabolic capacity is independent of the operator.

Capacitary estimates

$E \subset \mathbb{R}^{n+1}$

$\Pi_t(E) = \{ x \in \mathbb{R}^n : (x,t) \in E \}$ (the time slice)

Lemma. $K \subset \Omega_\infty$ compact $\Rightarrow$

$$\int_0^\infty \mathrm{cap}_p(\Pi_t(K), \Omega)\, dt \le \mathrm{cap}_{var}(K, \Omega_\infty)$$

the elliptic capacity

Proof: $u \in C_0^\infty(\Omega_\infty)$, $u \ge 1$ on K s.t.

$$\|u\|_{W(\Omega_\infty)} \le \mathrm{cap}_{var}(K) + \varepsilon$$

$x \mapsto u(x,t)$ is admissible for the elliptic capacity of $\Pi_t(E) \Rightarrow$

$$\mathrm{cap}_p(\Pi_t(K), \Omega) \le \int_\Omega |Du(x,t)|^p\, dx$$

$$\Rightarrow \int_0^\infty \mathrm{cap}_p(\Pi_t(K), \Omega)\, dt \le \int_0^\infty \int_\Omega |Du(x,t)|^p\, dx\, dt$$

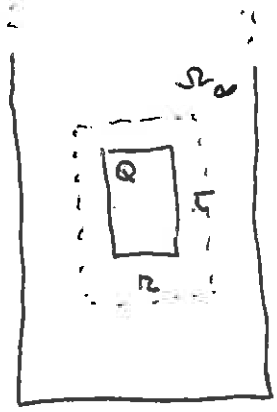
$$\le \|u\|_{W(\Omega_\infty)} \le \mathrm{cap}_{var}(K) + \varepsilon$$

$\square$

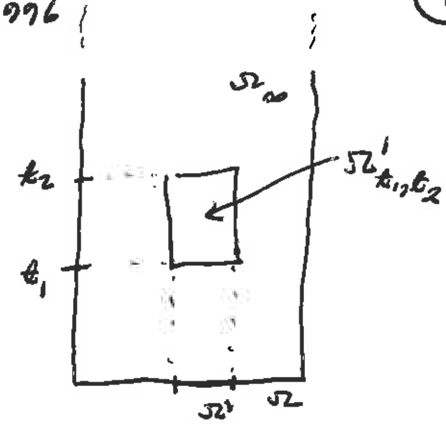
Example. $1 < p < n$

$$\mathrm{cap}_p(Q, \Omega_\infty) \gtrsim \tau r^{n-p} + r^n$$

The reverse estimate holds as well.



Kilpeläinen and Lindqvist, SIAM J. Math. Anal. 1996



Superparabolic functions

$u:\Omega_\infty \to (-\infty,\infty]$ is p-superparabolic if

(i) u is lsc,

(ii) $u<\infty$ in a dense subset and

(iii) $h \in C(\overline{\Omega'_{t_1,t_2}})$ is a solution, $h\le u$ on $\partial_p \Omega'_{t_1,t_2}$

$\Rightarrow h\le u$ in Ω'_{t_1,t_2} (Comparison principle)

Remarks. Lindqvist and K., AMPA 2006, ASNSP 2005
Korte, Kuusi, Parviainen, J. Evol. Eq. 2010

(1) Lsc representative of a supersolution is superparabolic.

(2) Locally bounded superparabolic functions are supersolutions.

(3) $u\in L^{p-2+\varepsilon}_{loc}(\Omega_\infty)$ superparabolic ($\Leftrightarrow$ the infinity set of a given time slice is of measure zero) Lindqvist, Kuusi, Parviainen in preparation

$\Rightarrow \exists\ Du \in L^q_{loc}(\Omega_\infty)$, $q<p-1+\frac{1}{n+1}$

$u\in L^{p-1+\frac{p}{n}-\varepsilon}_{loc}(\Omega_\infty)$

$\Rightarrow \exists$ Riesz measure μ_u

$\Omega=\mathbb{R}^n$ is always OK

(4) $\mu(\Omega_\infty)<\infty \Rightarrow \exists$ superparabolic u with $\mu_u=\mu$.

Boccardo, Gallouët, JFA ~~1997~~ 1989
Boccardo, Dall'Aglio, Gallouët, Orsina, JFA 1997
Lukkari, Parviainen and K., JFA 2010, JFPTA 2013

Obs: No regularity in time when $\mu \notin L^{p}(0,\infty; W^{1,p}_0(\Omega))'$.

Examples. (1) The Barenblatt solution ($u\in L^{p-2+\varepsilon}_{loc}$)

(2) Unfriendly giant by separation of variables ($u\notin L^{p-2+\varepsilon}_{loc}$ for any $\varepsilon>0$) at $t=0$.

(3) Blow up solutions at $t=T$ (DiBenedetto)

Infinity sets of superharmonic functions

Theorem: $u \in L^{p-2+\varepsilon}_{loc}(\Omega_\infty)$ is p-superharmonic in Ω_∞, $K \subset \Omega_\infty$ compact $\Rightarrow$

$$\mathrm{cap}_p(\{u \ge \lambda\} \cap K) \le C \mu_{\hat{R}_{u\chi_K}}(\Omega_\infty)\left(\lambda^{1-p} + \lambda^{\frac{1}{1-p}}\right), \quad \lambda > 1.$$

Proof: In the elliptic case if u_1 is a solution of the obstacle problem with χ_K, then λu_1 is the solution of the obstacle problem with $\lambda \chi_K$. Since we cannot scale solutions in the parabolic case, we derive estimates for solutions of the scaled obstacle problems instead.

u_1 is a solution of the obstacle problem with $\chi_{\{\hat{R}_{u\chi_K} \ge \lambda\}}$,

u_λ —— " —— $\lambda \chi_{\{\hat{R}_{u\chi_K} \ge \lambda\}}$

$$\mathrm{cap}_p(\{u \ge \lambda\} \cap K) \le \mathrm{cap}_p(\{\hat{R}_{u\chi_K} \ge \lambda\})$$

$\hat{R}_{u\chi_K} \ge u$ in K

$$= \mu_{u_1}(\Omega_\infty)$$

capacitary potential theorem

$$\le C\left(\lambda^{1-p} + \lambda^{\frac{1}{1-p}}\right)\mu_\lambda(\Omega_\infty)$$

the main estimate, requires work

$$\le C\left(\lambda^{1-p} + \lambda^{\frac{1}{1-p}}\right)\mu_{\hat{R}_{u\chi_K}}(\Omega_\infty)$$

$u_\lambda \le \hat{R}_{u\chi_K}$ and a comparison result

$\square$

Remark. $\lambda \to \infty \Rightarrow \mathrm{cap}_p(\{u = \infty\}) = 0$

Wiener's criterion in the parabolic case

$z = (x,t)$

$\boxed{p=2}$: $\Phi(z) = \begin{cases} \dfrac{1}{(4\pi t)^{m/2}} \exp\left(-\dfrac{|x|^2}{4t}\right), & t>0, \\ 0, & t \le 0 \end{cases}$

A point $z_0 \in \partial\Omega \subset \mathbb{R}^{m+1}$ is regular for the heat equation

$$\Leftrightarrow \sum_{i=1}^{\infty} \lambda^i \operatorname{cap}_p\left(\Omega^c \cap \{\lambda^i < \Phi(z_0 - z) \le \lambda^{i+1}\}\right) = \infty, \quad \lambda > 1$$

Lanconelli 1973, Evans and Gariepy 1982, Garofalo and Lanconelli, ...

$\boxed{p>2}$: Φ = Barenblatt's solution

$$\sum_{i=1}^{\infty} \lambda^i \operatorname{cap}_p\left(\Omega^c \cap \{\lambda^i < \Phi(z_0 - z) \le \lambda^{i+1}\}\right)^{\frac{1}{p-1}} = \infty \quad ?$$

Skrypnik

Open questions

(1) $1 < p < 2$?

(2) Removability results for p-superparabolic functions ?

(3) Other operators (PME, DNE) ?

$\boxed{\text{THE END}}$