# Variational problems with linear growth on metric measure spaces

Research term on Analysis and Geometry in Metric Spaces, ICMAT, May 11-14, 2015

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**Abstract.** In this lecture series we discuss theory for minimizers of variational integrals with linear growth on metric measure spaces. In particular, we consider definitions, existence, regularity and integral representations for the minimizers. Somewhat unexpected feature is that the minimizers may have jump discontinuities inside the domain. Moreover, there is an new challenge related to the integral representations. Theory for Sobolev and BV functions on metric measure spaces is used throughout the course.

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# Variational problems with linear growth on metric measure spaces

## 1. BV functions

Euclidean case. $U \subset \mathbb{R}^n$ open. $u \in L^1(U)$ has bounded variation, $u \in BV(U)$, if

$$\sup\left\{ \int_U \sum_{i=1}^n u\, D_i \varphi_i \, dx : \varphi \in C_0^\infty(U;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty .$$

$u \in L^1_{loc}(U)$ has locally bounded variation in $U$, $u \in BV_{loc}(U)$, if $u \in BV(V)$ for every open $V \subset\subset U$.

Structure theorem. Let $u \in BV_{loc}(U)$. Then there exists a Radon measure $\|Du\|$ on $U$ and a $\|Du\|$-measurable function $\sigma : U \to \mathbb{R}^n$ such that

(i) $|\sigma(x)| = 1$ $\|Du\|$-almost everywhere in $U$ and

(ii) $$\int_U \sum_{i=1}^n u\, D_i \varphi_i \, dx = -\int_U \varphi \cdot \sigma \, d\|Du\| \quad \text{for every } \varphi \in C_0^\infty(U;\mathbb{R}^n)$$

$$\int_U \sum_{i=1}^n u\, D_i \varphi_i \, dx = \int_U u \operatorname{div} \varphi \, dx$$

$\|Du\|$ is called the total variation measure of $u$

The moral: Weak derivatives are Radon measures.

<u>Proof</u>: $L : C_0^\infty(U;\mathbb{R}^n) \to \mathbb{R}$, $L(\varphi) = -\int_U u \operatorname{div}\varphi \, dx$ is a bounded linear functional. The claim follows from the Riesz representation theorem. $\square$

The proof gives

$$\|Du\|(V) = \sup\left\{\int_V u \operatorname{div}\varphi \, dx : \varphi \in C_0^\infty(U;\mathbb{R}^n),\ |\varphi| \le 1\right\}$$

$$= \int_V |Du| \, dx$$

if $u$ is smooth enough

for every $V \subset U$.

The BV norm is $\|u\|_{BV(U)} = \|u\|_{L^1(U)} + \|Du\|(U)$.

With this norm $BV(U)$ is a Banach space, but this fact is rather useless since smooth functions are not dense and it is not separable.

<u>Examples</u>. (1) $u \in W^{1,1}_{loc}(U)$, $V \subset\subset U$, $\varphi \in C_0^\infty(V;\mathbb{R}^n)$, $|\varphi| \le 1$

$$\Rightarrow \int_U u \operatorname{div}\varphi \, dx = -\int_U Du \cdot \varphi \, dx \le \int_V |Du| \, dx < \infty$$

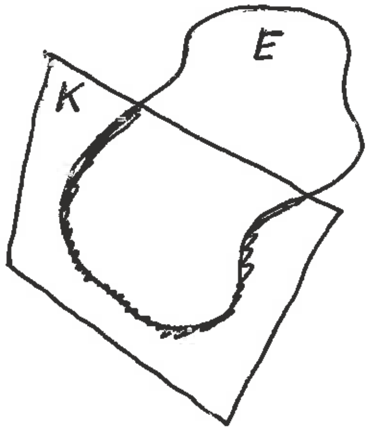
the weak gradient

$\Rightarrow u \in BV_{loc}(U)$

Moreover, $\|Du\|(V) = \int_V |Du| \, dx$, $V \subset U$.

$BV_{loc}(U) \not\subset W^{1,1}_{loc}(U)$, because the Heaviside function and the Cantor-Lebesgue function is not absolutely continuous.

(2) $E \subset \mathbb{R}^n$ smooth open set with $H^{n-1}(\partial E \cap K) < \infty$ for every compact $K \subset U$




$$\Rightarrow \int_E \operatorname{div} \varphi \, dx \underset{\text{Gauss}}{=} \int_{\partial E} \varphi \cdot \nu \, dH^{n-1},$$

where $\nu$ is the outward unit normal of $\partial E$

$$\Rightarrow \int_E \operatorname{div} \varphi \, dx = \int_{\partial E \cap V} \varphi \cdot \nu \, dH^{n-1} \leq H^{n-1}(\partial E \cap V) < \infty$$

$\Rightarrow \chi_E \in BV_{loc}(U)$, that is, $E$ has locally finite perimeter in $U$.

Moreover, $P(E, U) = \| D\chi_E \|(U) = H^{n-1}(\partial E \cap U)$.

Generally $P(E, U) = H^{n-1}(\partial^* E \cap U)$, where

$$\partial^* E = \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0, \ \limsup_{r \to 0} \frac{|B(x,r) \setminus E|}{|B(x,r)|} > 0 \right\}$$

is the measure theoretic boundary of $E$.

<u>Metric case</u>. $(X, d, \mu)$ complete metric measure space with a Borel regular outer measure $\mu$.

A nonnegative Borel function $g : X \to [0, \infty]$ is an upper gradient of $u : X \to [-\infty, \infty]$, if

$$|u(x) - u(y)| \leq \int_\gamma g \, ds$$


for every continuous path with end points $x$ and $y$.

The Sobolev space $N^{1,p}(X)$ consists of functions $u \in L^p(X)$ that have an upper gradient $g_u \in L^p(X)$. Here $X$ can be replaced with any $\mu$-measurable (typically open) $U \subset X$. In the Euclidean case with the Lebesgue measure we have

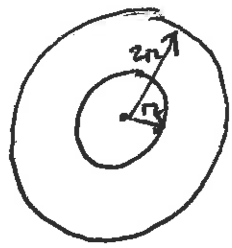
$$N^{1,p}(U) = W^{1,p}(U), \quad 1 \le p < \infty .$$

For every $u \in N^{1,1}_{loc}(U)$ there exists a minimal upper gradient $g_u$ satisfying $g_u \le g$ $\mu$-a.e. for any other upper gradient $g$.

$\mu$ is doubling, if there is a constant $c_d$ s.t.

$$\mu(B(x,2r)) \le c_d \, \mu(B(x,r))$$

for all balls $B(x,r)$ in $X$.




The space supports a Poincaré inequality, if there are constants $c_p$ and $\lambda$ s.t.

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le c_p r \fint_{B(x,\lambda r)} g \, d\mu .$$

<u>Fact</u>: Doubling and Poincaré imply Sobolev inequalities. This is important in the calculus of variations. From now on we work under these assumptions.

For $u \in L^1_{loc}(X)$ we define the total variation as

$$\|Du\|(X) = \inf\Big\{ \liminf_{i\to\infty} \int_X g_{u_i}\, d\mu : u_i \in Lip_{loc}(X),$$
$$u_i \to u \text{ in } L^1_{loc}(X) \Big\},$$

where $g_{u_i}$ is the minimal upper gradient of $u_i$.

$u \in L^1(X)$ has bounded variation, $u \in BV(X)$, if $\|Du\|(X) < \infty$.

Instead of the whole space $X$, the total variation can be defined in any open $U \subset X$ in the same way. For an arbitrary set $A \subset X$ it can be defined as

$$\|Du\|(A) = \inf\{\|Du\|(U) : U \supset A,\ U \subset X \text{ open}\}.$$

<u>Fact</u>: If $u \in BV(X)$, then $\|Du\|$ is a finite Borel regular outer measure. (Miranda 2003)

<u>Note</u>: If we consider $L^p$-integral with $p > 1$ instead of $L^1$ in the definition above, we obtain the Sobolev space $N^{1,p}$.

<u>Claim</u>: In the Euclidean case with the Lebesgue measure, this definition gives the same class of functions than the classical definition.

<u>Proof</u>: $\boxed{\text{new} \subset \text{classical}}$ : $u \in BV(U)$

$\Rightarrow \exists\, u_i \in lip_{loc}(U)$, $i = 1, 2, \dots$, s.t. $u_i \to u$ in $L^1_{loc}(U)$ and

$$\lim_{i\to\infty} \int_U g_{u_i}\, \overset{dx}{d\mu} = \|Du\|(U) < \infty.$$

Lower semicontinuity of the total variation measure

$$\Rightarrow \|Du\|_{\text{classical}}(U) \le \liminf_{i\to\infty} \|Du_i\|_{\text{classical}}(U)$$

$\uparrow$ $\text{lip}_{loc}(U) \subset BV_{\text{classical}}(U)$

$\varphi \in C_0^\infty(U;\mathbb{R}^n)$, $|\varphi| \le 1$

$$\Rightarrow \int_U u_i \operatorname{div}\varphi\, dx = -\int_U Du_i \cdot \varphi\, dx \underset{|\varphi|\le 1}{\le} \int_U |Du_i|\, dx \underset{|Du_i| \le g_{u_i}}{\le} \int_U g_{u_i}\, dx$$

$$\Rightarrow \|Du\|_{\text{classical}}(U) \le \liminf_{i\to\infty} \int_U g_{u_i}\, dx = \|Du\|(U) < \infty$$

$$\Rightarrow u \in BV_{\text{classical}}(U)$$

$\boxed{\text{classical} \subset \text{new}}$ : $u \in BV_{\text{classical}}(U)$

$\Rightarrow \exists\, u_i \in BV_{\text{classical}}(U) \cap C^\infty(U)$, $i = 1, 2, \ldots$ s.t. $u_i \to u$ in $L^1(U)$ and

$\uparrow$ known

$$\|Du_i\|_{\text{classical}}(U) \to \|Du\|_{\text{classical}}(U) \text{ as } i\to\infty.$$

Note: We do not claim $\|D(u_i - u)\|_{\text{classical}}(U) \to 0$ which would imply $u \in W^{1,1}(U)$.

$$\Rightarrow \liminf_{i\to\infty} \int_U g_{u_i}\, dx \le \lim_{i\to\infty} \|Du_i\|_{\text{classical}}(U) = \|Du\|_{\text{classical}}(U) < \infty$$

$$\Rightarrow \|Du\|(U) = \inf\Big\{ \liminf_{i\to\infty} \int_U g_{u_i}\, dx : u_i \in \text{lip}_{loc}(U),\ u_i \to u \text{ in } L^1_{loc}(U) \Big\}$$

$$\le \|Du\|_{\text{classical}}(U) < \infty. \quad \square$$

<u>Lower semicontinuity</u>. $U \subset X$ open, $u \in L^1_{loc}(U)$, $u_i \in BV_{loc}(U)$ s.t. $u_i \to u$ in $L^1_{loc}(U)$

$$\Rightarrow \|Du\|(V) \le \liminf_{i\to\infty} \|Du_i\|(V), \quad V \subset U \text{ open.}$$

<u>Note</u>: $\sup_i \|Du_i\|(V) < \infty \ \forall\, V \subset\subset U \Rightarrow u \in BV_{loc}(U)$

<u>Proof</u>: $\forall\, i \ \exists\, u_{i,j} \in \mathrm{lip}_{loc}(U)$ s.t. $u_{i,j} \to u_i$ in $L^1_{loc}(V)$ as $j \to \infty$ and

$$\|Du_i\|(V) = \lim_{j\to\infty} \int_V g_{u_{i,j}}\, d\mu$$

$\Rightarrow \exists$ a subsequence $(v_i)$ of $(u_{i,j})$ s.t. $v_i \to u$ in $L^1_{loc}(V)$ and

$$\int_V g_{v_i}\, d\mu < \|Du_i\|(V) + \frac{1}{i}, \quad i = 1, 2, \dots$$

$$\Rightarrow \|Du\|(V) \le \liminf_{i\to\infty} \int_V g_{v_i}\, d\mu$$

$$\le \liminf_{i\to\infty} \left( \|Du_i\|(V) + \frac{1}{i} \right)$$

$$= \liminf_{i\to\infty} \|Du_i\|(V). \qquad \square$$

## 2. Area functional

Euclidean case. $U \subset \mathbb{R}^n$ open. The area functional can be defined for BV-functions as

$$\int_U \sqrt{1+|Du|^2} = \sup\left\{ \int_U \left( \varphi_{n+1} + \sum_{i=1}^n u\, D_i \varphi_i \right) dx \; : \; \varphi \in C_0^\infty(U; \mathbb{R}^{n+1}),\ |\varphi| \le 1 \right\}.$$

Note: This is the total variation of the vector measure $(-Du, \mathcal{L}^n)$.

The Euler-Lagrange equation is

$$\sum_{i=1}^n D_i\left( \frac{D_i u}{\sqrt{1+|Du|^2}} \right) = 0, \quad \left( \sum_{i=1}^n \int_U \frac{D_i u\, D_i \varphi}{\sqrt{1+|Du|^2}}\, dx = 0 \ \ \forall\, \varphi \in C_0^\infty(U) \right)$$

which is called the minimal surface equation.

Existence theorem. Let $U \subset \mathbb{R}^n$ be a bounded open set with $C^2$-boundary of non-negative mean curvature and let $f \in C(\partial U)$. Then the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i=1}^n D_i\left( \frac{D_i u}{\sqrt{1+|Du|^2}} \right) = 0 & \text{in } U \\ u = f & \text{on } \partial U \end{cases}$$

has a unique solution $u \in C^2(U) \cap C(\overline{U})$.

<u>Metric case.</u> $U \subset X$ open. For $u \in L^1_{loc}(U)$ we define the area functional by

$$F(u,U) = \inf\Big\{ \liminf_{i\to\infty} \int_U \sqrt{1+g_{u_i}^2}\, d\mu : u_i \in \operatorname{lip}_{loc}(U), \\ u_i \to u \text{ in } L^1_{loc}(U) \Big\}.$$

<u>Note</u>: $t \le \sqrt{1+t^2} \le 1+t$, $t \ge 0$

$$\Rightarrow \|Du\|(U) \le F(u,U) \le \mu(U) + \|Du\|(U)$$

It is possible to define the variational integral with linear growth for more general $f: \mathbb{R}_+ \to \mathbb{R}_+$ which is a convex increasing function with

$$m t \le f(t) \le M(1+t), \quad t \ge 0,$$

with some constants $0 < m \le M < \infty$.

For an arbitrary set $A \subset X$ we define

$$F(u,A) = \inf\{ F(u,U) : U \supset A,\ U \subset X \text{ open} \}.$$

<u>Theorem.</u> (Hakkarainen, Lahti, Lehtelä, K, 2014)
If $U \subset X$ is open and $F(u,U) < \infty$, then $F(u,\cdot)$ is a Borel regular outer measure on $U$.

<u>Boundary values in BV</u>

Let $U$ and $U^*$ be open subsets of $X$ s.t. $U \subset\subset U^*$ and assume that $f \in BV(U^*)$. $BV_f(U)$ is the space of $u \in BV(U^*)$ such that $u = f$ $\mu$-almost everywhere in $U^* \setminus U$.

<u>Note</u>: We can take $U^* = X$.

<u>Boundary value problem.</u> $u \in BV_f(U)$ is a minimizer with the boundary values $f \in BV(U^*)$, if

$$\mathcal{F}(u, U^*) \le \mathcal{F}(v, U^*)$$

for every $v \in BV_f(U)$.

<u>Note:</u> Since $\mathcal{F}(u, U^*) = \mathcal{F}(u, \overline{U}) + \mathcal{F}(u, U^* \setminus \overline{U})$, the minimizers do not depend on $U^*$, but the value of the integral does.

<u>Existence.</u> Let $U$ and $U^*$ be bounded open subsets of $X$ with $U \subset\subset U^*$. Then for every $f \in BV(U^*)$ there exists a minimizer $u \in BV_f(U)$.

<u>Strategy:</u> Direct methods in the calculus of variations.
<u>Warning:</u> In general, the minimizer does not attain the boundary values.
<u>Proof:</u> $m = \inf\limits_{v \in BV_f(U)} \mathcal{F}(v, U^*)$

$\Rightarrow \exists\, u_i \in BV_f(U)$, $i = 1, 2, \dots$, s.t. $\mathcal{F}(u_i, U^*) \to m$ as $i \to \infty$

$$\|Du_i\|(U^*) \underset{\substack{\uparrow \\ t \le \sqrt{1+t^2},\, t \ge 0}}{\le} \mathcal{F}(u_i, U^*), \quad i = 1, 2, \dots \Rightarrow \sup_i \|Du_i\|(U^*) < \infty$$

Poincaré inequality

$$\Rightarrow \int_U |u_i - f| \, d\mu \underset{\substack{\uparrow \\ u_i - f \in BV_0(U)}}{\le} c \operatorname{diam}(U) \|D(u_i - f)\|(U^*)$$

$$\int_{U^*} |u_i|\, d\mu \le \int_{U^*} |f|\, d\mu + \underbrace{\int_{U^*} |u_i - f|\, d\mu}_{= \int_U |u_i - f|\, d\mu}$$

$$\le \int_{U^*} |f|\, d\mu + c\, \mathrm{diam}(U) \| D(u_i - f) \| (U^*)$$

Poincaré

$$\le \int_{U^*} |f|\, d\mu + c\, \mathrm{diam}(U) \big( \| Du_i \| (U^*) + \| Df \| (U^*) \big)$$

$\Rightarrow$ The sequence $(u_i)$ is bounded in $BV(U^*)$

Rellich type compactness theorem (Miranda, 2003)

$\Rightarrow$ $\exists$ a subsequence $(u_i)$ s.t. $u_i \to u$ in $L^1_{loc}(U^*)$ for some $u \in BV_{loc}(U^*)$. (Sobolev inequality is applied here)

$\Rightarrow$ $\exists$ a subsequence $(u_i)$ s.t. $u_i \to u$ $\mu$-a.e. in $U^*$

$\Rightarrow$ $|u(x) - f(x)| \le |u(x) - u_i(x)| + \underbrace{|u_i(x) - f(x)|}_{= 0} \to 0$ as $i \to \infty$

for $\mu$-a.e. $x \in U^* \setminus U$

$\Rightarrow$ $u = f$ $\mu$-a.e. in $U^* \setminus U$

$\Rightarrow$ $u \in BV(U^*)$ and $u_i \to u$ in $L^1(U^*)$ as $i \to \infty$

$\Rightarrow$ $u \in BV_f(U)$

Lower semicontinuity

$\Rightarrow$ $m \le F(u, U^*) \le \liminf_{i \to \infty} F(u_i, U^*) = m$. $\square$

## 3. Regularity results

**Euclidean case.** Solutions to the minimal surface equation are smooth.

**Metric case.**

**Strategy:** We apply the De Giorgi method to show that minimizers are locally bounded.

**De Giorgi estimate.** For every $k \in \mathbb{R}$ we have

$$\| D(u-k)_+ \| (B(x,r)) \le \frac{2}{R-r} \int_{B(x,R)} (u-k)_+ \, d\mu + \mu(A_{k,R}),$$


where $A_{k,R} = B(x,R) \cap \{u > k\}$. We assume that $u$ is a minimizer.

**Proof:** $u_i \in \mathrm{Lip}_{loc}(U^*)$, $i = 1, 2, \dots$, s.t. $u_i \to u$ in $L^1_{loc}(U^*)$ and

$$F(u, U^*) = \lim_{i \to \infty} \int_{U^*} \sqrt{1 + g_{u_i}^2} \, d\mu$$

$A_{k,R,i} = B(x,R) \cap \{u_i > k\}$, $y \in B(x,R)$

$$\int_{-\infty}^{\infty} |\chi_{A_{k,R}}(y) - \chi_{A_{k,R,i}}(y)| \, dk \underset{(*)}{=} \int_{\min\{u(y), u_i(y)\}}^{\max\{u(y), u_i(y)\}} 1 \, dk = |u(y) - u_i(y)|$$

$$\begin{cases} \max\{a,b\} = \frac{a+b}{2} + \frac{1}{2}|a-b| \\ \min\{a,b\} = \frac{a+b}{2} - \frac{1}{2}|a-b| \end{cases}$$

$$(*)\quad |\chi_{A_{k,R}}(y)-\chi_{A_{k,R,i}}(y)| = \begin{cases} 0, & u\le k,\ u_i\le k \Leftrightarrow \max\{u,u_i\}\le k \text{ or} \\ & u>k,\ u_i>k \Leftrightarrow \min\{u,u_i\}>k \\ 1 & \text{otherwise} \end{cases}$$

$$\Rightarrow \underbrace{\int_{B(x,R)} |u-u_i|\,d\mu}_{\to 0} = \int_{-\infty}^{\infty}\left(\int_{B(x,R)} |\chi_{A_{k,R}}-\chi_{A_{k,R,i}}|\,d\mu\right)dk$$

$\Rightarrow \exists$ a subsequence $(u_i)$ s.t. $\chi_{A_{k,R,i}} \to \chi_{A_{k,R}}$ in $L^1(B(x,R))$ as $i\to\infty$ for $\mathcal{L}^1$-a.e. $k\in\mathbb{R}$

$\Rightarrow \mu(A_{k,R,i}) \to \mu(A_{k,R})$ as $i\to\infty$ for $\mathcal{L}^1$-a.e. $k\in\mathbb{R}$

Let $k\in\mathbb{R}$ s.t. the convergence occurs.

Choose $\eta\in \mathrm{Lip}_0(B(x,R))$, $0\le\eta\le 1$, $\eta=1$ on $B(x,r)$ s.t. $g_\eta \le \frac{2}{R-r}$.

Let $\varphi_i = -\eta(u_i-k)_+$ and $\varphi=-\eta(u-k)_+$.

$u+\varphi\in BV_f(U) \Rightarrow F(u,U^*)\le F(u+\varphi,U^*)$

$\varepsilon>0$

$u_i+\varphi_i \to u+\varphi$ in $L^1_{loc}(U^*)$ as $i\to\infty$

$\Rightarrow \exists N_\varepsilon$ s.t. $\int_{U^*}\sqrt{1+g_{u_i}^2}\,d\mu < F(u,U^*)+\frac{\varepsilon}{2}$ and

$$F(u+\varphi,U^*) < \int_{U^*}\sqrt{1+(g_{u_i+\varphi_i})^2}\,d\mu + \frac{\varepsilon}{2} \quad \text{for } i\ge N_\varepsilon$$

(for a subsequence)

$$\Rightarrow \int_{U^*} \sqrt{1+g_{u_i}^2}\,d\mu < \int_{U^*} \sqrt{1+(g_{u_i+\varphi_i})^2}\,d\mu + \varepsilon\,,\quad i \geq N_\varepsilon$$

$$u_i = u_i + \underbrace{\varphi_i}_{=0} \text{ in } U^* \setminus A_{k,R,i} \Rightarrow g_{u_i} = g_{u_i+\varphi_i}\ \mu\text{-a.e. in } U^* \setminus A_{k,R,i}$$

$$\Rightarrow \int_{A_{k,R,i}} \sqrt{1+g_{u_i}^2}\,d\mu < \int_{A_{k,R,i}} \sqrt{1+(g_{u_i+\varphi_i})^2}\,d\mu + \varepsilon\,,\quad i \geq N_\varepsilon$$

$$g_{(u_i-k)_+} = g_{u_i} \text{ in } A_{k,R,i}$$

$$g_{u_i+\varphi_i} = g_{u_i - \eta(u_i-k)_+} \underset{\text{in } A_{k,R,i}}{=} g_{u_i(1-\eta)+\eta k}$$

$$\leq (1-\eta)\,g_{u_i} + g_\eta\,|u_i - k|$$

$$= (1-\eta)\,g_{(u_i-k)_+} + g_\eta\,(u_i-k)_+ \text{ in } A_{k,R,i}$$

$$\Rightarrow \int_{A_{k,R,i}} g_{(u_i-k)_+}\,d\mu = \int_{A_{k,R,i}} g_{u_i}\,d\mu \underset{t \leq \sqrt{1+t^2},\ t\geq 0}{\leq} \int_{A_{k,R,i}} \sqrt{1+g_{u_i}^2}\,d\mu$$

$$\leq \int_{A_{k,R,i}} \sqrt{1+(g_{u_i+\varphi_i})^2}\,d\mu + \varepsilon$$

$$\underset{\sqrt{1+t^2} \leq 1+t,\ t\geq 0}{\leq} \int_{A_{k,R,i}} (1+g_{u_i+\varphi_i})\,d\mu + \varepsilon$$

$$\leq \int_{A_{k,R,i}} (1-\eta)\, g_{(u_i-k)_+}\, d\mu + \int_{A_{k,R,i}} g_\eta (u_i-k)_+\, d\mu + \mu(A_{k,R,i}) + \varepsilon, \quad i \geq N_\varepsilon$$

$$\Longrightarrow \int_{A_{k,R,i}} \eta\, g_{(u_i-k)_+}\, d\mu \leq \frac{2}{R-r} \int_{A_{k,R,i}} (u_i-k)_+\, d\mu + \mu(A_{k,R,i}) + \varepsilon$$

$\vee|$ $\longleftarrow$ $\eta = 1$ on $B(x,r)$ $\quad \wedge|$

$$\int_{B(x,r)} g_{(u_i-k)_+}\, d\mu \qquad \frac{2}{R-r} \int_{B(x,R)} (u_i-k)_+\, d\mu + \mu(A_{k,R,i}) + \varepsilon, \quad i \geq N_\varepsilon$$

$$\|D(u-k)_+\|(B(x,r)) \leq \liminf_{i\to\infty} \int_{B(x,r)} g_{(u_i-k)_+}\, d\mu$$

$\uparrow$ $(u_i-k)_+ \to (u-k)_+$ in $L^1_{loc}(U^*)$, $i \to \infty$

$$\leq \frac{2}{R-r} \int_{B(x,R)} (u-k)_+\, d\mu + \underbrace{\lim_{i\to\infty} \mu(A_{k,R,i})}_{=\mu(A_{k,R})} + \varepsilon$$

Let $\varepsilon \to 0$. This holds for $\mathcal{L}^1$-a.e. $k \in \mathbb{R}$. For an arbitrary $k \in \mathbb{R}$, take $k_i \searrow k$ for which the claim holds.

$$\Longrightarrow \|D(u-k)_+\|(B(x,r)) \leq \liminf \|D(u-k_i)_+\|(B(x,r))$$

$\uparrow$ lower semicontinuity

$$\leq \liminf_{i\to\infty} \left( \frac{2}{R-r} \int_{B(x,R)} (u-k_i)_+\, d\mu + \mu(A_{k_i,R}) \right)$$

$$\leq \frac{2}{R-r} \int_{B(x,R)} (u-k)_+\, d\mu + \mu(A_{k,R}). \qquad \square$$

## Local boundedness of the minimizers

$$\operatorname*{ess\,sup}_{B(x,\frac{R}{2})} u \le k_0 + C \fint_{B(x,R)} (u-k_0)_+ \, d\mu + R, \quad C = C(c_d, c_p).$$

**Proof:**

$$d = c(c_d, c_p) \fint_{B(x,R)} (u-k_0)_+ \, d\mu + R$$

There are sequences $r_i \searrow \frac{R}{2}$ and $k_i \nearrow k_0 + d$ such that the De Giorgi condition and Sobolev inequality imply

$$\frac{1}{d} \fint_{B(x,r_i)} (u-k_i)_+ \, d\mu \to 0 \quad \text{as } i \to \infty.$$

$$\Rightarrow \fint_{B(x,\frac{R}{2})} (u-k_0-d)_+ \, d\mu \le \lim_{i\to\infty} \int_{B(x,r_i)} (u-k_i)_+ \, d\mu = 0$$

$$B(x, \tfrac{R}{2}) \subset B(x, r_i).$$

$$(u-k_0-d)_+ \le (u-k_i)_+$$

$$\Rightarrow \operatorname*{ess\,sup}_{B(x,\frac{R}{2})} u \le k_0 + d = k_0 + C \fint_{B(x,R)} (u-k_0)_+ \, d\mu + R. \quad \square$$

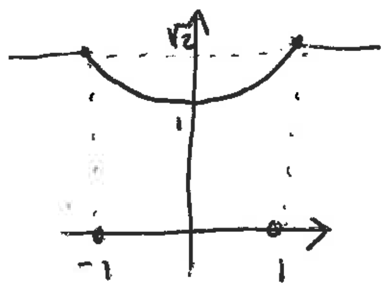
## Failure of interior continuity

Unexpected phenomenon: A minimizer of a linear growth functional may fail to be continuous at an interior point. In this sense, the previous local boundedness result is optimal.

Note: Minimizers with $p$-growth, $p>1$, are continuous. (Shanmugalingam, K., 2001)

<u>Example.</u> Let $\mathbb{R}$ be with the Euclidean distance, $U=(-1,1)$ and $d\mu = w(x)\,dx$ with

$$w(x) = \min\{\sqrt{2}, \sqrt{1+x^{4/3}}\}.$$



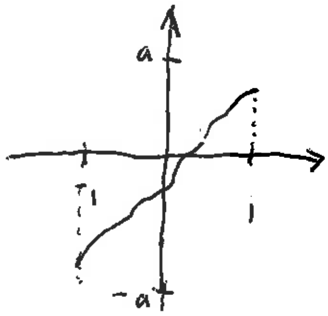

Note that $w \in C^1(U)$ and $1 \le w \le \sqrt{2}$ in $U$.

Let $U^* \supset\supset U$ be a bounded open set and let $u \in BV(U^*)$ be a minimizer of

$$\mathcal{F}(u, U^*) = \int_{-1}^{1} \sqrt{1+(u')^2}\, w\, dx = F(u)$$

with boundary values $u(-1) = -a$ and $u(1) = a$, where the parameter $a > 0$ will be chosen later. This corresponds to minimizing the integral representation

$$F(u) = \int_{-1}^{1} \sqrt{1+(u')^2}\, w\, dx + \int_{-1}^{1} w\, d|(Du)^s|$$
$$+ w(-1)|u(-1)+a| + w(1)|u(1)-a|,$$




where the boundary values are interpreted in the sense of traces, $(Du)^a = u'\,dx$ denotes the absolutely continuous part and $(Du)^s$ the singular part of the total variation measure $Du$.

First we note that $u$ attains the correct boundary values $u(-1) = -a$ and $u(1) = a$. If this were not the case, consider

$$v(x) = \begin{cases} u(x) - u(-1) - a, & -1 \le x < 0, \\ u(x) - u(1) + a, & 0 \le x \le 1, \end{cases}$$

Moving jump to the origin gives a better candidate.

and obtain $F(v) < F(u)$, which contradicts the fact that $u$ is a minimizer. $\lightning$

Any minimizer $u \in W^{1,1}((-1,1))$ satisfies the weak form of the corresponding Euler–Lagrange equation

$$\frac{\partial}{\partial x}\left(\frac{u'(x)\,w(x)}{\sqrt{1+(u'(x))^2}}\right) = 0,$$

that is,

$$\int_{-1}^{1} \frac{u'(x)\,w(x)}{\sqrt{1+(u'(x))^2}}\,\eta'(x)\,dx = 0 \quad \forall\ \eta \in C_0^\infty((-1,1)).$$

$$\Rightarrow \frac{u'(x)\,w(x)}{\sqrt{1+(u'(x))^2}} = c \qquad 0<|c|\le 1$$

DuBois-Reymond lemma

for a.e. $x \in (-1,1)$

$$\Rightarrow |u'(x)| = \underbrace{\frac{c}{\sqrt{w(x)^2-c^2}}}_{=\left(\frac{w(x)^2}{c^2}-1\right)^{-\frac{1}{2}}} \quad \text{a.e. } x\in(-1,1).$$

$$a = \tfrac{1}{2}|u(1)-u(-1)| \le \tfrac{1}{2}\int_{-1}^{1}|u'(x)|\,dx = \tfrac{1}{2}\int_{-1}^{1}\left(\frac{w(x)^2}{c^2}-1\right)^{-\frac{1}{2}}dx$$

$$\le \tfrac{1}{2}\int_{-1}^{1}\left(w(x)^2-1\right)^{-\frac{1}{2}}dx = \tfrac{1}{2}\int_{-1}^{1} x^{-\frac{2}{3}}\,dx = 3 < a \quad ⚡$$

choose such a!

Thus $u \notin W^{1,1}((-1,1))$, because the minimizer will attain ~~thus~~ the correct boundary values.

Finally we show that $(Du)^s$ has a nontrivial point mass at $x=0$. To this end, define a function $v$ such that

$$v(-1)=-a,\quad v'=u' \quad\text{and}\quad (Dv)^s=\left(\int_{-1}^{1} d(Du)^s\right)\delta_0 .$$

Dirac's $\delta$ at $x=0$

In other words, we collect the singular part of $u$ to $x=0$. It follows that $v(1)=a$. Moreover,

$$F(v)=\int_{(-1,1)\setminus\{0\}}\sqrt{1+(u')^2}\,w\,dx+\underbrace{w(0)}_{=1}\left|\int_{-1}^{1} d(Du)^s\right|$$

$$\le\int_{(-1,1)\setminus\{0\}}\sqrt{1+(u')^2}\,w\,dx+\int_{-1}^{1} d\,|(Du)^s| .$$

Since $u$ is a minimizer, $F(u)\le F(v)$.

$$\underbrace{\cancel{\int_{-1}^{1}\sqrt{1+(u')^2}\,w\,dx}+\int_{-1}^{1} w\,d\,|(Du)^s|}_{=F(u)}\le\cancel{\int_{(-1,1)\setminus\{0\}}\sqrt{1+(u')^2}\,w\,dx}+\int_{-1}^{1} d\,|(Du)^s|$$

$$\Rightarrow\int_{-1}^{1} w\,d\,|(Du)^s|\le\int_{-1}^{1} d\,|(Du)^s|$$

This can happen only if $\operatorname{supp}(Du)^s\subset\{0\}$, since $w(x)>1$ for $x\neq 0$. Since $u\notin W^{1,1}((-1,1))$ it is not an absolutely continuous function and hence $(Du)^s\neq 0$. Thus $u$ has a jump at $x=0$.

## 4. Integral representations

The Euclidean case. In the Euclidean case with the Lebesgue measure we have

$$F(u) = \int_U \sqrt{1+|Du|^2}\,dx + \|Du\|^s(U),$$

where $d\|Du\| = |Du|\,dx + d\|Du\|^s$ is the decomposition of the total variation measure into the absolutely continuous and singular parts.

Let $U^* \supset\supset U$ be a bounded set and assume that $U$ is smooth and $f$ is a smooth boundary function.

$$F(u, U^*) = F(u, \bar U) + F(f, U \setminus \bar U)$$

$$= \int_U \sqrt{1+|Du|^2} + \int_{\partial U} |(Du)^s| + \int_{U^* \setminus \bar U} \sqrt{1+|Df|^2}$$

$$= \underbrace{\int_U \sqrt{1+|Du|^2} + \int_{\partial U} |u-f|\,dH^{n-1}}_{\text{it is enough to minimize this}} + \int_{U^* \setminus \bar U} \sqrt{1+|Df|^2}$$

the penalty term takes care of the boundary values

fixed

The functional

$$\int_U \sqrt{1+|Du|^2} + \int_{\partial U} |u-f| \, dH^{n-1}$$

always has a ~~...~~ minimizer. Moreover, if $u, v$ are two minimizers, then $u - v = \text{constant}$. [GM], [GMS, Ch 6]
Vol 2.

The metric case

Let $U \subset X$ be an open set, $u \in L^1_{loc}(U)$ with $F(u, U) < \infty$. Let $d\|Du\| = a\, d\mu + d\|Du\|^s$ be the decomposition of $\|Du\|$ into the absolutely continuous and singular parts, where $a \in L^1(U)$ and $\|Du\|^s$ is the singular part. Then

$$\int_U \sqrt{1+a^2}\, d\mu + \|Du\|^s(U) \le F(u, U)$$

$$\le \int_U \sqrt{1+c a^2}\, d\mu + \|Du\|^s(U),$$

where $c = C(c_d, c_p, \ldots)$.
doubling Poincaré

Unexpected constant that does not appear in the classical case

Remarks: Let $F^a(u, \cdot)$ and $F^s(u, \cdot)$ be the absolutely continuous and singular parts of $F(u, \cdot)$ with respect to $\mu$. Assume that $F(u, U) < \infty$ and let $A \subset U$ be a $\mu$-measurable set.

Then

(1) $\mathcal{F}^a(u,A) = \|Du\|^a(A)$.

(2) $\int_A \sqrt{1+a^2}\,d\mu \le \mathcal{F}^a(u,A) \le \int_A \sqrt{1+ca^2}\,d\mu$

(3) The constant $c$ cannot be dismissed already on the weighted $\mathbb{R}$.

A sketch of the proof:

$\boxed{\text{The lower bound}}$: Take a minimizing sequence $u_i \in \mathrm{Lip}_{loc}(U)$ s.t. $u_i \to u$ in $L^1_{loc}(U)$ and

$$\int_U \sqrt{1+g_{u_i}^2}\,d\mu \to \mathcal{F}(u,U) \quad \text{as } i \to \infty.$$

There is a subsequence $g_{u_i}\,d\mu \to d\nu$ weakly in $U$, where $\nu$ is a Radon measure with finite mass in $U$.

By the definition of the total variation measure $\nu \ge \|Du\|$.
(requires a small argument)

Any nonnegative nondecreasing convex function $f$ can be represented as

$$f(t) = \sup_{j \in \mathbb{N}} (d_j t + e_j), \quad t \ge 0, \qquad (f(t) = \sqrt{1+t^2})$$

for some sequences $d_j, e_j \in \mathbb{R}$ with $d_j \ge 0$, $j = 1, 2, \dots$ and $\sup_{j \in \mathbb{N}} d_j = f_\infty$. [AFP] $\left(f_\infty = \lim_{t\to\infty} \frac{f(t)}{t}\right)$
$\qquad\qquad\; \overset{\|}{1}$

$A_1, \dots, A_k \subset U$ open, $\varphi_j \in C_0(A_j)$, $0 \le \varphi_j \le 1$ ← disjoint

$$\Rightarrow \int_{A_j} (d_j g_{u_i} + e_j)\varphi_j \, d\mu \le \int_{A_j} \sqrt{1+g_{u_i}^2}\, d\mu, \quad j = 1, \dots, k$$

$$\Rightarrow \sum_{j=1}^{k} \Big( \underbrace{\int_{A_j} d_j \varphi_j \, d\nu}_{\substack{\nu \ge \|Du\| \\ \int_{A_j} d_j \varphi \, d\|Du\|}} + \int_{A_j} e_j \varphi_j \, d\mu \Big) \le \underbrace{\liminf_{i\to\infty} \int_U \sqrt{1+g_{u_i}^2}\, d\mu}_{= F(u_i; U)}$$

weak convergence $i \to \infty$

$d\|Du\| = a\, d\mu + d\|Du\|^s$ gives the result. (after some calculations)

The key is to use the weak convergence to conclude something meaningful for $\sqrt{1+g_{u_i}^2}\, d\mu$.

$\boxed{\text{The upper bound}}$: Since $F(u, \cdot)$ is a Radon measure, we decompose $F(u, \cdot) = F^a(u, \cdot) + F^s(u, \cdot)$.

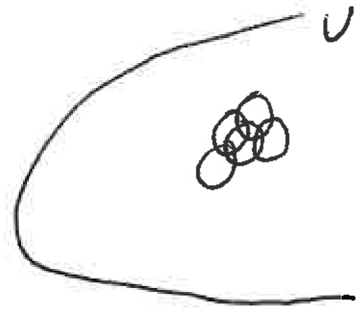
The estimate for $F^s(u, \cdot)$ follows rather directly by choosing a minimizing sequence and using $\sqrt{1+t^2} \le 1+t$.

For $F^a(u, \cdot)$ we approximate $u$ by the discrete convolutions related to Whitney type coverings and partitions of unities.

For every $i \in \mathbb{N}$ take a Whitney covering $\{B_j^i\}_{j=1}^\infty$ of $U$ s.t. $\mathrm{rad}(B_j^i) \le \frac{1}{i}$ for every $j=1,2,\dots$, $5\lambda B_j^i \subset U$, $j=1,2,\dots$, and $5\lambda B_j^i$, $j=1,2,\dots$, are of bounded overlap.

Take a partition of unity $\{\varphi_j^i\}_{j=1}^\infty$ subordinate to the Whitney cover $\{B_j^i\}_{j=1}^\infty$ s.t. $0 \le \varphi_j^i \le 1$, $\mathrm{lip}\,\varphi_j^i \le \frac{c}{\mathrm{rad}(B_j^i)}$ and $\mathrm{supp}(\varphi_j^i) \subset 2B_j^i$, $j=1,2,\dots$

Define the discrete convolutions




$$u_i = \sum_{j=1}^\infty u_{B_j^i}\varphi_j^i, \quad i=1,2,\dots$$

$\mathrm{rad}(B_j^i) \le \frac{1}{i}$, $i=1,2,\dots$ $\Rightarrow$ $u_i \to u$ in $L^1(U)$, $i \to \infty$.

$\mathrm{lip}\, u_i = ?$

Let $x, y \in B_k^i$. Then

$$|u_i(x)-u_i(y)| \le \sum_{j=1}^\infty |u_{B_k^i} - u_{B_j^i}|\,|\varphi_j^i(x) - \varphi_j^i(y)|$$

$$\sum u_{B_j^i}\varphi_j^i(x) - \sum u_{B_j^i}\varphi_j^i(y) + \underbrace{\sum_{j=1}^\infty u_{B_k^i}\varphi_j^i(x)}_{=u_{B_k^i}} - \underbrace{\sum_{j=1}^\infty u_{B_k^i}\varphi_j^i(y)}_{=u_{B_k^i}}$$

$$\le C \fint_{5B_k^i} |u - u_{5B_k^i}|\,d\mu\, \frac{d(x,y)}{\mathrm{rad}(B_k^i)}$$

Bounded overlap

$$\le C\, \frac{d(x,y)}{\cancel{\mathrm{rad}(B_k^i)}}\, 5\,\cancel{\mathrm{rad}(B_k^i)}\, \frac{\|Du\|(5\lambda B_k^i)}{\mu(5\lambda B_k^i)}$$

$$\Rightarrow g_i = c \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\|Du\|(5\lambda B_j^i)}{\mu(B_j^i)} \quad \text{is an upper gradient of } u_i$$

$c = c(c_D, C_P)$

$$= \underbrace{c \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\int_{5\lambda B_j^i} a \, d\mu}{\mu(B_j^i)}}_{= g_i^a} + \underbrace{c \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\|Du\|^s(5\lambda B_j^i)}{\mu(B_j^i)}}_{= g_i^s}$$

The sequence $\{g_i^a\}$ is equi-integrable (by a nontrivial reason) and there exists (Dunford–Pettis) a weakly converging subsequence in $L^1(U)$ to a function $\check{a} \le Ca$.

By Mazur's lemma there are convex combinations that converge strongly in $L^1(U)$. The claim follows from this.

The next example shows that the constant is necessary. It is not just an artifact from the discrete convolution.

<u>Example.</u>

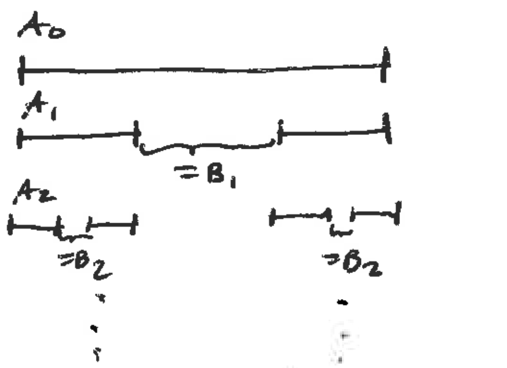
$X = [0,1]$ with the Euclidean distance. Next we construct a measure.

Fat Cantor set:

$A = \bigcap_{i=1}^{\infty} A_i$, $\mathcal{L}^1(A_i) = \alpha_i$, $\mathcal{L}^1(A) = \frac{1}{2}$

$w = 2\chi_A + \chi_{X \setminus A}$, $d\mu = w \, d\mathcal{L}^1$

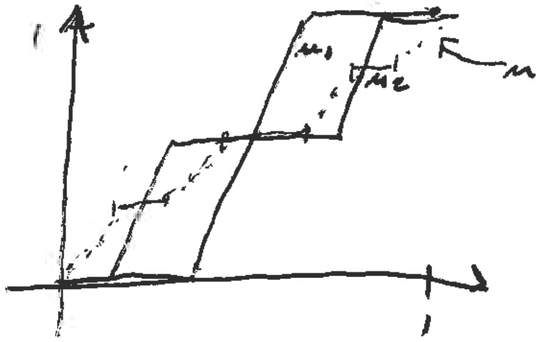
Note: $1 \le w \le 2 \Rightarrow w \approx \mathcal{L}^1$




$\mathcal{L}^1(A_{i-1}) = \mathcal{L}^1(B_i) + \mathcal{L}^1(A_i)$

$g = 2\chi_A$, $g_i = \dfrac{1}{\alpha_{i-1}-\alpha_i}\chi_{B_i}$, $i=1,2,\dots$, $\int_0^1 g_i \, d\mathcal{L}^1 = 1$, $i=1,2,\dots$

$u(x) = \int_0^x g \, d\mathcal{L}^1$, $u_i(x) = \int_0^x g_i \, d\mathcal{L}^1$




$u, u_i$ Lipschitz, $u_i \to u$ in $L^1(X)$ (even uniformly)

$g$ is a minimal upper gradient of $u$

$$\int_0^1 g \, d\mu = 2 > 1 = \lim_{i\to\infty} \int_0^1 g_i \, d\mu \geq \|Du\|([0,1]) (= 1)$$

$d\|Du\| = \chi_A \, d\mu$ (this requires work)

<u>Note</u>: This also gives $\|Du\|([0,1]) < \int_0^1 g_u \, d\mu$.
even for $u \in \mathrm{Lip}$.
↑ the minimal upper gradient

<u>The moral</u>: The derivatives of the approximating sequence live in the cheaper set $[0,1]\setminus A$ with weight 1, but $g = Du$ lives in $A$ where it costs more with weight 2.

Finally we can construct a variational integral $F(u,\Omega)$ so that it places extra weight in $X \setminus A$ and

$$\underset{\substack{\shortparallel \\ F(u,X)}}{F^a(u,X)} > \int_X f(a) \, d\mu .$$

<u>Discussion</u>: In the classical case we may use the standard convolution mollification and the constant does not appear, because the derivation commutes with the convolution. If we use the discrete convolution instead, then we loose a multiplicative constant. The previous example shows that the constant cannot be discarded. Observe that the discrete convolution gives an approximation in $L^1$, but not $L^1$ convergence for the derivatives. For the derivatives, we get only weak convergence and this is not enough.

$\boxed{\text{THE END}}$