The mathematical formulation of the state evolution equation in electric process tomography

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Abstract

We consider the process tomography problem of imaging the concentration distribution of a given substance in a fluid moving in a pipeline based on electromagnetic measurements on the surface of the pipe. We view the problem as a state estimation problem. The concentration distribution is treated as a stochastic process satisfying a stochastic differential equation referred to as the state evolution equation. The measurements are described in terms of an observation equation containing the measurement noise. Our main interest is in the mathematical formulation of the state evolution equation. The time evolution is modelled by a stochastic convection-diffusion equation. We derive a discrete evolution equation for the concentration distribution by using the stochastic integration theory and the semigroup technique.

Keywords  process tomography, electrical impedance tomography, state estimation, stochastic differential equations

1 Introduction

We consider the process tomography problem of imaging the concentration distribution of a given substance in a fluid moving in a pipeline based on electromagnetic measurements on the surface of the pipe. In traditional electrical impedance tomography (EIT), it is assumed that the object remains stationary during the measurement process. A complete set of measurements, also called a frame, consists of all possible linearly independent injected current patterns and the corresponding set of voltage measurements. In process tomography, we cannot assume in general that the target remains unaltered during a full set of measurements. Thus conventional reconstruction methods cannot be used. The time evolution needs to be modelled properly. We view the problem as a state estimation problem. The concentration distribution is treated as a stochastic process, or a state of the system, that satisfies a stochastic differential equation referred to as the state evolution equation. The measurements are described in terms of an observation equation containing the measurement noise. This problem has been considered in the articles [Sep01a], [Sep01b] and [Sep01c]. Those articles concentrate on the numerical implementation of the problem. Our main interest is in the mathematical formulation of the state evolution equation. We refer to those articles and references in them concerning the observation equation. The rigorous knowledge of the stochastic nature of the state evolution equation is essential for solving the electric process tomography problem.

Our goal is to have a real-time monitoring for a flow in a pipeline. For that reason the computational time has to be minimized. Therefore, we use a simple model, the convection-diffusion equation, for the flow. It can be easily implemented and is fast to compute. Since
we cannot be sure that other features, such as turbulence, of the flow do not appear, we use stochastic modelling. So the randomness is due to the lack of information, not to the intrinsic randomness of the concentration.

The measurements are done in a part of the boundary of the pipe. We get enough information for an accurate computation only from a segment of the pipe. It would be natural to choose the domain of the model to be the segment of the pipe. If the domain is restricted to be a segment of the pipe, we have to use some boundary conditions in the input and the output end of the segment. The choice of boundary conditions has an effect to the solution. The most commonly used boundary conditions do not represent the actual circumstances in the pipe. Therefore, we do not do the domain restriction. We assume that the pipe is infinitely long. With the assumption we derive the state evolution model. The concentration distribution, which we are actually interested in, is the restriction of the solution of the evolution model to a segment of the pipe.

2 State estimation

Let $\Omega \in \mathbb{R}^n$ be a domain that corresponds to the object of interest. We denote by $X = X_t(x), x \in \Omega$, a distributed parameter describing the state of the object - the unknown distribution of a physical target - at time $t \geq 0$. We assume that we have a model for the time evolution of the parameter $X$. We assume that instead of a deterministic function $X$ is a stochastic process satisfying a stochastic differential equation. This allows us to incorporate phenomenon such as modelling uncertainties into the model.

Let $Y = Y_t$ denote a quantity that is directly observable at times $t \in I$, $I = \{t_k : t_k < t_{k+1}, \ k \in \mathbb{N}\}$. We assume that the dependence of $Y$ upon the state $X$ is known up to observation noise and modelling errors. The system consists of a pair of equations

\begin{align*}
    dX_t &= (\mathcal{L}X_t + H_t)dt + dW_t, \tag{1} \\
    Y_t &= F(t, X, V), \quad t \in I. \tag{2}
\end{align*}

Equation (1) is called the state evolution equation and is to be interpreted as a stochastic differential equation in which $\mathcal{L}$ is a linear differential operator and $H = H_t$ and $W = W_t$ are stochastic processes. The process $W$ is called the state noise. The function $F$ is the observation model function and $V = V_t$ is a stochastic process, the observation noise. Equation (2) is called the observation equation.

We may solve Equation (1) to obtain a discrete evolution equation

\begin{equation}
    X_{k+1} = U_k X_k - U_k H_k + W_k, \tag{3}
\end{equation}

where we have used the notation $X_k = X_{t_k}$ and $H_k = H_{t_k}$. The operator $U_k$ is related to the differential operator $\mathcal{L}$ and the process $W_k$ to the state noise $W$ and the process $H$. The state estimation problem can be formulated as follows: Estimate the state $X_k$ satisfying an evolution equation of the type (3) based on the observed values of $Y_k$, when $t$ is in a given subset of $I$.

We consider the special case in which observations are obtained by EIT measurements and in which the physical target can be described by the convection-diffusion equation.
3 Mathematical formulation of the state evolution equation

We assume that the concentration distribution \( C(t) \) is a stochastic process satisfying the stochastic differential equation

\[
dC(t) = [\mathcal{L}C(t) + f(t)]dt + BdW(t)
\]

for every \( t > 0 \) with the initial value

\[
C(0) = C_0.
\]

The operator \( \mathcal{L} \) is the deterministic convection-diffusion operator

\[
\mathcal{L} : \mathcal{D}(\mathcal{L}) \to L^2(\Omega)
\]

\[
c \mapsto \nabla \cdot (\kappa \nabla c) - \mathbf{v} \cdot \nabla c
\]

with the domain

\[
\mathcal{D}(\mathcal{L}) = \left\{ c \in H^2(\Omega) : \frac{\partial c}{\partial n} \bigg|_{\partial \Omega} = 0 \right\}
\]

where \( \Omega \subset \mathbb{R}^n \) is an infinitely long pipe and \( \kappa = \kappa(x) \) is the diffusion coefficient and \( \mathbf{v} = \mathbf{v}(x) \) is the velocity of the flow. We model with \( f(t) \) a possible control of the system. Since the control term is known, if the state of the system is known, we may assume that \( f(t) \) is a predictable process. The term \( BdW(t) \) is a source term representing possible modelling errors, where \( B \) is a linear operator and \( W(t) \) is a Wiener process.

We use the semigroup technique to solve the stochastic convection-diffusion equation (4). Under certain assumptions of the diffusion coefficient and the velocity of the flow the convection-diffusion operator \( \mathcal{L} \) generates an analytic semigroup \( \mathcal{U}(t) \). Consequently, the deterministic version of Equation (4)

\[
\begin{cases}
\frac{\partial c(t)}{\partial t} = \mathcal{L}c(t) + f(t) \\
c(0) = c_0
\end{cases}
\]

has a weak solution

\[
c(t) = \mathcal{U}(t)c_0 + \int_0^t \mathcal{U}(t-s)f(s) \, ds.
\]

Under certain assumption of the operator \( B \) and the Wiener process \( W \), the stochastic convection-diffusion equation (4) with the initial value (5) has exactly one weak solution given by the formula

\[
C(t) = \mathcal{U}(t)C_0 + \int_0^t \mathcal{U}(t-s)f(s) \, ds + \int_0^t \mathcal{U}(t-s)BdW(s).
\]

The measurements are done in a discrete set of times \( t_k \). We use the notation \( C_k = C(t_k) \), \( f_k = f(t_k) \) and \( \Delta_k = t_{k+1} - t_k \). Then

\[
C_{k+1} = \mathcal{U}(\Delta_k)C_k + \int_{t_k}^{t_{k+1}} \mathcal{U}(t_{k+1} - s)f(s) \, ds + \int_{t_k}^{t_{k+1}} \mathcal{U}(t_{k+1} - s)BdW(s).
\]

The system is controlled only at the measurement times. Thereby,

\[
C_{k+1} = \mathcal{U}(\Delta_k)C_k - \mathcal{U}(\Delta_k)f_k + f_{k+1} + \int_{t_k}^{t_{k+1}} \mathcal{U}(t_{k+1} - s)BdW(s).
\]
Hence a discrete evolution model for the concentration distribution is

\[ C_{k+1} = U(\Delta_k)C_k - U(\Delta_k)f_k + W_k, \]

where

\[ W_k = f_{k+1} + \int_{t_k}^{t_{k+1}} U(t_{k+1} - s)BdW(s). \]

The term \( W_k \) can be seen as a state noise.

A detailed mathematical analysis of the problem will be published in [Pik].

References


