When do electromagnetic Gaussian beams propagate using Riemannian geometry

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Assumptions

- Everything is smooth
- Domain $U \subseteq \mathbb{R}^3$, bounded/unbounded, topologically trivial
- Media is anisotropic, non-homogeneous, no time dependence
- ε, μ real, positive definite, simultaneously diagonalizable:

$$\varepsilon = R^{-1} \cdot \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \cdot R \qquad \mu = R^{-1} \cdot \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} \cdot R$$

Gaussian beam assumption

$$\text{Trial:} \qquad E(x,t) \quad = \quad \mathrm{Re}\{e^{iP\theta(x,t)} E_0(x,t)\}$$

$P > 0$ big constant, E_0 complex vector field, θ complex phase function

Let $c\colon (0,1) \to U$ be a curve, and assume that

$$\begin{aligned} z &= z(x,t) = x - c(t) \\ \theta(x,t) &= \phi(t) \quad + \quad p(t) \cdot z \quad + \quad \frac{1}{2} z^T \cdot S(t) \cdot z \end{aligned}$$

$$\phi, p \text{ real, } \ \mathrm{Im}\, S \text{ positive definite}$$

Then

$$\begin{aligned} |\exp(iP\theta)| &= \exp\left(-\frac{P}{2}\ z^T \cdot \mathrm{Im}\, S \cdot z\right) \\ &= \text{Gaussian bell curve} \end{aligned}$$

Plugging all this into Maxwell's equations gives condition on c, ϕ, p, S:

- ϕ is constant
- c and p is a solution to Hamilton's equations with Hamiltonian h_+ or h_- [$h_\pm$ depend only on media]
- S is a solution to a matrix Riccati equation (depending on $h_\pm$)

Conclusion:

- Media $\longrightarrow$ Hamiltonians h_+ and h_- $\longrightarrow$ two types of Gaussian beams
- Gaussian beam determined by c, ϕ, p, S and these are determined by "simple" equations

Definition of $h_\pm$

- Let

$$M(x,z) \quad = \quad \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \mu_3 \end{pmatrix} \cdot \begin{pmatrix} & z \times I \\ -z \times I & \end{pmatrix}$$

- Spectrum of $M(x,z) = \pm\{0, h_+(x,z), h_-(x,z)\}$
- $h_\pm \geq 0$ on $U \times \mathbb{R}^3$

Example: In isotropic media,

$$h_\pm(x,z) = \frac{1}{\sqrt{\varepsilon(x)\mu(x)}}$$

Geometrization of Gaussian beams

For some $i \neq j$, $\varepsilon_i \mu_j = \varepsilon_j \mu_i$

$\Rightarrow$ Gaussian beams propagate using Riemannian geometry

Examples

- c is a geodesic in this geometry
- S can be solved from a curvature equation
- In isotropic media: $g_{ij} = \sqrt{\varepsilon\mu}\ \delta_{ij}$
- If $\varepsilon_2\mu_3 = \varepsilon_3\mu_2$, then

$$
\begin{aligned}
g_{+,ij}(x) &= \left(R^{-1} \cdot \operatorname{diag}\left(\sqrt{\varepsilon_2\mu_3}, \sqrt{\varepsilon_1\mu_3}, \sqrt{\varepsilon_1\mu_2}\right) \cdot R\right)_{ij} \\
g_{-,ij}(x) &= \left(R^{-1} \cdot \operatorname{diag}\left(\sqrt{\varepsilon_3\mu_2}, \sqrt{\varepsilon_3\mu_1}, \sqrt{\varepsilon_2\mu_1}\right) \cdot R\right)_{ij}
\end{aligned}
$$

Case: There are no $i \neq j$ such that $\varepsilon_i \mu_j = \varepsilon_j \mu_i$