Descending Maps Between Slashed Tangent Bundles

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joint work with

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Suppose $F$ is a diffeomorphism

$$F : TM \setminus \{0\} \to TM \setminus \{0\}.$$  

Characterize those $F$ that can be written as

$$F = D\phi|_{TM\setminus\{0\}}.$$  

for a diffeomorphism $\phi : M \to M$.

When $\phi$ exists, one say that $F$ descends.
Definition: Let $M$ be a manifold. Then the **canonical involution** is the diffeomorphism

$$\kappa : TTM \rightarrow TTM$$

that is locally given by

$$\kappa(x, y, X, Y) = (x, X, y, Y).$$

Note:

- $\kappa^2 = \text{identity}$. 

Descending maps between slashed tangent bundles
First main theorem

Suppose $F$ is a diffeomorphism

\[ F : TM \setminus \{0\} \rightarrow TM \setminus \{0\} \]

and suppose that $M$ is connected, simply connected, compact, $\dim M \geq 2$.

Then the following are equivalent:

(i) There exists a diffeomorphism $\phi : M \rightarrow M$ such that

\[ F = D\phi \mid_{TM \setminus \{0\}}. \]

(ii) $DF = \kappa \circ DF \circ \kappa$
A related result

Theorem [Robbin-Weinstein-Lie]:

Let $F$ be a diffeomorphism

$$F : T^* M \rightarrow T^* M.$$

Then the following are equivalent:

(i) $F = \phi^*$ for a diffeomorphism $\phi : M \rightarrow M$.

(ii) $F^* \theta = \theta$.

Here:

- $\phi^*$ = pullback of $\phi$, $\phi(x, \xi) = \left( (\phi^{-1})^i(x), \frac{\partial (\phi^{-1})^i}{\partial x^a} \xi_i \right)$

- $\theta =$ canonical 1-form $\theta \in \Omega^1(T^* M)$, $\theta = \xi_i dx^i$
Suppose: \( DF = \kappa \circ DF \circ \kappa \).

Claim: There exists a map \( \phi: M \to M \) such that \( F = D\phi|_{TM \setminus \{0\}} \).
Suppose: $DF = \kappa \circ DF \circ \kappa$.

Claim: There exists a map $\phi: M \to M$ such that $F = D\phi\big|_{TM\setminus\{0\}}$.

Proof: Let locally $F(x, y) = (F_1(x, y), F_2(x, y))$. Then $DF(x, y, X, Y)$

$$DF(x, y, X, Y) = \left( F_1(x, y), F_2(x, y), \frac{\partial F_1}{\partial x^a}(x, y)X^a + \frac{\partial F_1}{\partial y^a}(x, y)Y^a, \right.$$

$$\left. \frac{\partial F_2}{\partial x^a}(x, y)X^a + \frac{\partial F_2}{\partial y^a}(x, y)Y^a \right),$$
Suppose: $DF = \kappa \circ DF \circ \kappa$.

Claim: There exists a map $\phi: M \to M$ such that $F = D\phi|_{TM\setminus\{0\}}$.

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$$\left. \quad \frac{\partial F_2}{\partial x^a}(x, y)X^a + \frac{\partial F_2}{\partial y^a}(x, y)Y^a \right),$$

$$\kappa \circ DF \circ \kappa(x, y, X, Y) = \left( F_1(x, X), \frac{\partial F_1}{\partial x^a}(x, X)y^a + \frac{\partial F_1}{\partial y^a}(x, X)Y^a, F_2(x, X), \right.$$ 

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Suppose: $DF = \kappa \circ DF \circ \kappa$.

Claim: There exists a map $\phi: M \to M$ such that $F = D\phi|_{TM\setminus\{0\}}$.

Proof: Let locally $F(x, y) = (F_1(x, y), F_2(x, y))$. Then

$$DF(x, y, X, Y) = \left(F_1(x, y), F_2(x, y), \frac{\partial F_1}{\partial x^a}(x, y)X^a + \frac{\partial F_1}{\partial y^a}(x, y)Y^a, \frac{\partial F_2}{\partial x^a}(x, y)X^a + \frac{\partial F_2}{\partial y^a}(x, y)Y^a \right),$$

$$\kappa \circ DF \circ \kappa(x, y, X, Y) = \left(F_1(x, X), \frac{\partial F_1}{\partial x^a}(x, X)y^a + \frac{\partial F_1}{\partial y^a}(x, X)Y^a, F_2(x, X), \frac{\partial F_2}{\partial x^a}(x, X)y^a + \frac{\partial F_2}{\partial y^a}(x, X)Y^a \right).$$

First components: $F_1(x, y) = F_1(x, X)$. Let $\phi$ be the unique map $\phi: M \to M$ determined by $\phi \circ \pi = \pi \circ F$. Locally $\phi(x) = F_1(x, y)$. 
Suppose: \( DF = \kappa \circ DF \circ \kappa \).

**Claim:** There exists a map \( \phi: M \rightarrow M \) such that \( F = D\phi|_{TM\setminus\{0\}} \).

**Proof:** Let locally \( F(x, y) = (F_1(x, y), F_2(x, y)) \). Then

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DF(x, y, X, Y) = \left( F_1(x, y), F_2(x, y), \frac{\partial F_1}{\partial x^a}(x, y)X^a + \frac{\partial F_1}{\partial y^a}(x, y)Y^a, \right.
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- **First components:** \( F_1(x, y) = F_1(x, X) \). Let \( \phi \) be the unique map \( \phi: M \rightarrow M \) determined by \( \phi \circ \pi = \pi \circ F \). Locally \( \phi(x) = F_1(x, y) \).
- **Second components:** \( F_2(x, y) = \frac{\partial \phi}{\partial x^a}(x)y^a \). Thus \( F = D\phi|_{TM\setminus\{0\}} \).
Second main theorem: Suppose $F$ is a diffeomorphism

$$F: TM \setminus \{0\} \rightarrow TM \setminus \{0\},$$

and $M$ is connected, simply connected, compact, and $\dim M \geq 2$. If $M$ has two Riemann metrics $g$ and $\tilde{g}$ such that

(i) $g$ has a trapping hypersurface $\Sigma \subset M$;

$$\forall p \in M \quad \forall y \in T_pM \setminus \{0\} \quad \exists T \in \mathbb{R} \text{ s.t. } \exp(Ty) \in \Sigma.$$ (ii) for all $p \in \Sigma$,

$$g(y, y) = \tilde{g}(y, y) \quad y \in T_pM \setminus \{0\}$$

$$S(y) = \tilde{S}(y), \quad y \in T_pM \setminus \{0\}$$

$$DF(\xi) = \xi, \quad \xi \in T(T_pM \setminus \{0\})$$

(iii) If $J: I \rightarrow TM \setminus \{0\}$ is a Jacobi field for $g$ then

$F \circ J: I \rightarrow TM \setminus \{0\}$ is a Jacobi field for $\tilde{g}$.

Then $F = D\phi|_{TM\setminus\{0\}}$ for a diffeomorphism $\phi: M \rightarrow M$ and $\phi$ is an isometry.
Outline of proof:

1. $F$ preserves integral curves since:
   ▶ every integral curve is a Jacobi field
   ▶ $F$ preserves Jacobi fields
   ▶ $S = \tilde{S}$ and $DF = \text{Id}$ on $\Sigma$

2. $DF = \kappa \circ DF \circ \kappa$ since:
   ▶ $F$ preserves Jacobi fields
   ▶ $F$ preserves integral curves

3. Thus there exists a diffeomorphism $\phi: M \to M$ such that $F = D\phi |_{TM \setminus \{0\}}$.

4. $\phi$ is totally geodesic since:
   ▶ $F = D\phi |_{TM \setminus \{0\}}$ preserves integral curves

5. Proposition: Let $M$ be a connected manifold with two Riemann metrics. If $\phi: M \to M$ is totally geodesic and $\phi$ is an isometry at one point, then $\phi$ is an isometry.
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