A geometric interpretation of the complex tensor Riccati equation for Gaussian beams

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Abstract

We study the complex Riccati tensor equation

$$D_{\dot{c}}G + GCG - R = 0$$

on a geodesic c on a Riemannian 3-manifold. This non-linear equation appears in the study of Gaussian beams. Gaussian beams are asymptotic solutions to hyperbolic equations that at each time instant are concentrated around one point in space. When time moves forward, Gaussian beams move along geodesics, and the Riccati equation determines the Hessian of the phase function for the Gaussian beam. The imaginary part of a solution G describes how a Gaussian beam decays in different directions of space. The main result of the present work is that the real part of G is the shape operator of the phase front for the Gaussian beam. This result generalizes a known result for the Riccati equation in $\mathbb{R}^3$. The idea of the proof is to express the Riccati equation in Fermi coordinates adapted to the underlying geodesic. In Euclidean geometry we also study when the phase front is contained in the area of influence, or light cone.

1 Introduction

We shall study the following equation:

> Suppose c is a geodesic parametrized with respect to pathlength in a Riemannian manifold. The *complex tensor Riccati equation* for tensor $G = G_{ij}(t)dx^i \otimes dx^j|_{c(t)}$ on c is the equation
>
> $$\begin{aligned} D_{\dot{c}}G + GCG - R &= 0, \\ G(0) &= G_0, \end{aligned} \tag{1}$$
>
> where tensors $C = C^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ and $R = R_{ij}dx^i \otimes dx^j$ on c are defined as
>
> $$\begin{aligned} C^{ij}(t) &= (g^{ij} - \dot{c}^i\dot{c}^j)(t), \\ R_{ij}(t) &= (R^m_{ijk}\dot{c}^k g_{ml}\dot{c}^l)(t), \end{aligned}$$
>
> and G_0 is a symmetric complex initial value with positive definite imaginary part. (Precise terminology is given below.)

This non-linear equation arises in the study of *Gaussian beams*. These are asymptotic solutions to Maxwell's equations [3, 8, 9, 10] (or wave equation [11, 12, 17], or elastic equation [13]) that have a very characteristic feature. Namely, at each time instant the whole energy of the solution is concentrated around one point in space. When time moves forward, the energy concentration traverses a curve in space, but its envelope is always a Gaussian bell curve. Gaussian beams are named after this property. Electromagnetic Gaussian beams are also known as quasi-photons [9]. A historical account on the development of Gaussian beams can be found in [16].

The motivation for studying Gaussian beams is that their propagation is completely determined by a set of ordinary differential equations. This means that it is much more easier to propagate a Gaussian beam than to solve the original equations (say, Maxwell's equations in the time domain). In view of this property they have been used to study the *traveltime problem*: how long does it take for a signal to travel between two points in a possibly inhomogeneous anisotropic media [11, 12]. In this setting, the physical media determines the Riemannian metric, and Gaussian beams propagate along geodesics of this metric. What is more, the complex tensor Riccati equation determines the Hessian of the phase function for a Gaussian beam (see Section 2.1).

It is well known that the imaginary part of a solution to the complex Riccati equation describes how the Gaussian beam decays in different directions of space (see Section 2.1). The main result of the present work is Theorem 4.5. It shows that the real part of a solution is essentially the shape operator of the phase front for the Gaussian beam (see Section 2.2). This provides a geometric interpretation of solutions to the complex tensor Riccati equation. Even if the equation is non-linear and the real and complex parts are coupled, both parts have a geometric interpretation. A similar result holds for the real Riccati equation in differential geometry. The shape operator for a family of surfaces determined by a distance function satisfy the real Riccati equation [6, 18].

In Euclidean space, the geometric character of the real part of a solution to equation (1) is known [2]. However, in this case, the Riccati equation is not a tensor equation, and the shape operator is calculated with respect to the Euclidean geometry and not with respect to the geometry induced by the physical media. The main tool for proving Theorem 4.5 are explicit expressions for Fermi coordinates around a geodesic. These are presented in Section 3, and in Section 4 we study the Riccati equation in these coordinates.

In this work we also study properties of the shape operator for solutions in the Euclidean geometry corresponding to homogeneous media (Section 5). In particular, we study when the phase front is contained in the area of influence (or the light cone) and give a sufficient condition on the initial values for this to hold for large t.

Basic definitions

We shall work in a differential geometric setting and assume that all objects are smooth. By a manifold M we mean a Hausdorff, second countable, topological manifold with smooth transition maps. Its tangent and cotangent bundles (over $\mathbb{R}$) are denoted by TM and T^*M, respectively. Let us also assume that I is an open interval containing 0 (but also sometimes the identity matrix). Furthermore, let us assume that M is equipped with a Riemannian metric tensor g and locally $g = g_{ij}\ dx^i \otimes dx^j$. The *Einstein's summation convention* is employed throughout the paper. By g^{ij} we mean the ij:th entry of the inverse of (g_{ij}). *Christoffel symbols* Γ_{ijk}, Γ^i_{jk} are defined as

$$\begin{aligned} \Gamma_{ijk} &= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right), \\ \Gamma^i_{jk} &= g^{ia}\Gamma_{ajk}. \end{aligned}$$

A curve $c\colon I \to M$ is a geodesic parametrized with respect to pathlength if

$$\begin{aligned} \frac{d^2c^i}{dt^2} + \Gamma^i_{jk}(t)\dot{c}^j\dot{c}^k &= 0, \\ g_{ij}(t)\dot{c}^i\dot{c}^j &= 1. \end{aligned}$$

As above, we will frequently use notation $g_{ij}(t)$, $\Gamma^i_{jk}(t)$, etc., which simply means that g_{ij} and Γ^i_{jk} are evaluated at $c(t)$. The components of the *Riemann curvature tensor* $R = R^m_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^m}$ are

$$R^m_{ijk} = \frac{\partial \Gamma^m_{ik}}{\partial x^j} - \frac{\partial \Gamma^m_{ij}}{\partial x^k} + \Gamma^s_{ik}\Gamma^m_{js} - \Gamma^s_{ij}\Gamma^m_{ks}. \tag{2}$$

If (x^i, y^i) are local coordinates for TM induced by x^i-coordinates for M, we define

$$N^i_j(x,y) = \Gamma^i_{jk}(x)y^k.$$

Let us also introduce notation

$$p_i(t) = g_{ij}(t)\dot{c}^j(t). \tag{3}$$

A *complex tensor on a curve* c is a mapping that map a point $t \in I$ into a tensor on M at $c(t)$ with possibly complex coefficients. Locally such a tensor α is determined by suitable functions $\alpha^{k_1\cdots k_m}_{i_1\cdots i_n}\colon I \to \mathbb{C}$, such that

$$\alpha(t) = \alpha^{k_1\cdots k_m}_{i_1\cdots i_n}(t)\ dx^{i_1} \otimes \cdots \otimes dx^{i_n} \otimes \frac{\partial}{\partial x^{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{k_m}}\Big|_{c(t)}.$$

A more detailed discussion is given in [3]. For a tensor $\alpha = \alpha_i(t)dx^i$, its *covariant derivatives* along the curve c is defined as

$$D_{\dot{c}}\alpha = \left(\frac{d\alpha_i}{dt} - \alpha_k N^k_i \circ \dot{c}\right) dx^i\Big|_{c(t)},$$

and for higher tensors $D(u\otimes v) = D(u)\otimes v + u \otimes D(v)$. By $\mathfrak{X}(M)$ we denote the set of smooth vector fields on M. Suppose $X = X^i\frac{\partial}{\partial x^i} \in \mathfrak{X}(M)$, and $\alpha = \alpha_i dx^i$

is a 1-form on M. Then for a vector $v = v^i \frac{\partial}{\partial x^i} \in TM$ we define *covariant derivatives*

$$\begin{aligned} D_v X &= v^k \left(\frac{\partial X^i}{\partial x^k} + X^j \Gamma^i_{jk} \right) \frac{\partial}{\partial x^i}, \\ D_v \alpha &= v^k \left(\frac{\partial \alpha_i}{\partial x^k} - \alpha_j \Gamma^j_{ik} \right) dx^i. \end{aligned}$$

By $\operatorname{Re} L$, $\operatorname{Im} L$ we mean the real and imaginary parts of a tensor or matrix L, and δ_{ij}, δ_i^j are *Kronecker delta symbols.* By $\overline{A}$ we mean the *closure* of a set A.

Shape operator

Suppose S is a orientable hypersurface of a manifold M (that is, an $n-1$ dimensional submanifold), and suppose that $n\colon S \to TM$ is a unit normal to S. Let $p \in S$, and let $U \subset M$ be a neighbourhood of p. By shrinking U, we can extend n into a vector field $\hat{n}\colon U \to TU$. We may also assume that $g(\hat{n}, \hat{n}) = 1$ on U. For $v \in T_p S$, let

$$L(v) = (D_v \hat{n})(p).$$

The identity $v(g(X, Y)) = g(D_v X, Y) + g(X, D_v Y)$ for $v \in T_p M$, $X, Y \in \mathfrak{X}(M)$ implies that $L(v) \in T_p S$. What is more, if $(x^1, \ldots, x^n)$ are submanifold coordinates for S such that $(0, x^2, \ldots, x^n)$ parametrize S around p, then writing out $(D_v \hat{n})(p)$ shows that $L(v)$ does not depend on the extension $\hat{n}$. The mapping

$$L\colon T_p S \to T_p S$$

is the *shape operator* (or *Weingarten map*) of hypersurface S. It is determined up to a sign with respect to n. Let us also point out that sometimes the shape operator is defined as $-D_v(\hat{n})_p$ [20]. One can show that the eigenvalues of L are real. On a 3-manifold, L has 2 eigenvalues κ_1, κ_2 called *principal curvatures.* The *Gaussian curvature* is defined as $K = \kappa_1 \kappa_2$, and the *mean curvature* is defined as $S = \frac{1}{2}(\kappa_1 + \kappa_2)$.

2 Riccati equation for Gaussian beams

2.1 Physical background

Let us briefly describe Gaussian beams and their relation to the complex tensor Riccati equation. Detailed expositions on this topic are [3, 9, 10, 11, 17]. Let us start with the source-less Maxwell's equations for differential forms

$$\begin{aligned} dE &= -\frac{\partial B}{\partial t}, \\ dH &= \frac{\partial D}{\partial t}, \end{aligned}$$

and let us consider an electric field of the form

$$E(x,t) \;=\; \operatorname{Re}\left\{E_0(x,t)\,\exp\left(iP\theta(x,t)\right)\right\}, \quad (x,t)\in M\times I.$$

Here $P>0$ is a large constant, I is an open interval representing time, E_0 is a complex 1-form, and the function $\theta\colon M\times I\to\mathbb{C}$ is the *phase function* for E. The advantage of representing the electric field as above is that qualitatively E_0 and θ contain different type of information. The 1-form E_0 completely determines how E is polarized. To understand the role of θ, let us write

$$\exp\left(iP\theta(x,t)\right) = \exp(iP\operatorname{Re}\theta)\exp(-P\operatorname{Im}\theta).$$

Thus, $\operatorname{Re}\theta$ describes high frequency oscillations of E, that is, information about propagation, while $\operatorname{Im}\theta$ influences the amplitude of E. In order for E to be stable in the limit $P\to\infty$, let us assume that $\operatorname{Im}\theta\geq 0$.

Suppose $c\colon I\to M$ is a smooth curve. Furthermore, suppose c is covered with local coordinates x^i, and suppose that

$$\phi\colon I\to\mathbb{C},\quad p\colon I\to\mathbb{C}^3,\quad H\colon I\to\mathbb{C}^{3\times 3}$$

are the first three coefficients in the Taylor expansion of θ evaluated on $c(t)$. That is,

$$\begin{aligned}\phi(t) &= \theta(c(t),t),\\ p_j(t) &= \frac{\partial\theta}{\partial x^j}(c(t),t),\\ H_{jk}(t) &= \frac{\partial^2\theta}{\partial x^j\,\partial x^k}(c(t),t).\end{aligned}$$

Then E is a *Gaussian beam* on c provided that for all $t\in I$,

1. $p(t)=\left(p_i(t)\right)_i$ is non-zero,
2. $\phi(t)$ and $p(t)$ are real,
3. the imaginary part of $H(t)=(H_{ij}(t))_{ij}$ is positive definite.

One can also show that these conditions do not depend on local coordinates [3, 11]. Using the chain rule, it follows that [3]

$$\theta(x,t) \;=\; \phi_0(t)+p_i(t)z^i+\frac{1}{2}H_{ij}(t)z^iz^j+o(|z|^3),$$

where $z^i=z^i(x,t)=x^i-c^i(t)$. In consequence, $|\exp\left(iP\theta(x,t)\right)|\approx\exp\left(-\frac{P}{2}z^i\operatorname{Im}H_{ij}z^j\right)$. In other words, at time t, the energy of E is completely concentrated around $c(t)$, and $\operatorname{Im}H$ describes the shape of the field.

Plugging $E(x,t)$ into Maxwell's equation (see [3, 8, 9, 10] for details) yields the following conditions on θ: ϕ_0 is constant, c is a geodesic with respect to a suitable Riemannian geometry depending on the media (see below), and $p_i(t)$ are given by equation (3). Furthermore, if $\Lambda(t) = (\Gamma^m_{ij} p_m)_{ij}$, then

$$G(t) = G_{ij}(t) dx^i \otimes dx^j \big|_{c(t)}, \quad G_{ij} = (H - \Lambda)_{ij}$$

is a solution to the complex tensor Riccati equation [3, 11]. Thus $\operatorname{Im} G = \operatorname{Im} H$ determines how a Gaussian beam decays in different directions from $c(t)$. If we are in isotropic media with permittivity ε and permeability μ, then Gaussian beams propagate along geodesics of the scaled Euclidean metric [3]

$$g = \frac{1}{\varepsilon\mu} \delta_{ij} dx^i \otimes dx^j .$$

One can also derive equations for the vector part E_0 in E, and these are studied in [9].

2.2 Phase front surface

For a fixed $t \in I$, let us define the *phase front surface*,

$$F_t = \{x \in M : \operatorname{Re}\theta(x,t) = \operatorname{Re}\theta(c(t),t)\}.$$

(In the proof of Proposition 2.2 we show that F_t is a surface.) On F_t, $\operatorname{Re}\theta$ is constant, that is, $\exp(iP\theta)$ does not oscillate. Thus all oscillations happen through F_t. For example, for a plane wave, F_t would be sheets orthogonal to the direction of propagation. The next two propositions give some geometric information about F_t. Proposition 2.1 is proved in [1] for real phase functions.

Proposition 2.1. *Phase front surface F_t is orthogonal to $\dot c(t)$ with respect to g.*

Proof. Let $u \in T_{c(t)} F_t$. It follows that $u = \frac{d\gamma^i}{ds}(0) \frac{\partial}{\partial x^i}\Big|_{c(t)}$ for some curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = c(t)$ and $s \mapsto \operatorname{Re}\theta(\gamma(s),t)$ is constant. Since $p_i(t) = \frac{\partial\theta}{\partial x^i}(c(t),t)$, equation (3) implies that

$$g(u, \dot c) \quad = \quad \frac{d\theta(\gamma(s),t)}{ds}\Big|_{s=0},$$

and the proposition follows since $g(u,\dot c)$ is real. $\square$

Proposition 2.2. *Let $\hat F_t$ be the representation of the phase front in local coordinates equipped with the Euclidean inner product. Then the Gaussian curvature K and the mean curvature S of $\hat F_t$ at $c(t)$ are*

$$\begin{aligned} K &= \frac{p \cdot \operatorname{adj}(\operatorname{Re} H) \cdot p^T}{\|p\|_0^4}, \\ S &= -\frac{p \cdot \operatorname{Re}(H - \operatorname{trace}(H) I) \cdot p^T}{2\|p\|_0^3}, \end{aligned}$$

where p is the Euclidean vector $p = (p_1, p_2, p_3)$, $\|\cdot\|_0$ is the Euclidean norm, and adj *is the adjugate of a matrix. Furthermore, p is a unit normal to $\hat{F}_t$ at $c(t)$, and K and S are calculated with respect to this orientation for F_t.*

Proof. Let us first calculate the principal curvatures κ_1, κ_2 of $\hat{F}_t$ at $c(t)$. Let $f(x) = \operatorname{Re}\theta(x,t) - \operatorname{Re}\theta(c(t),t)$. Then $(\nabla f)(c(t)) = p(t)$, and $(\operatorname{Hess} f)(c(t)) = \operatorname{Re} H(t)$. Since $p(t) \neq 0$, F_t is a surface. Next, we employ a result from [20, p. 204]: If s_1, s_2 are the two roots to the 4×4 determinant equation

$$\det\left(\begin{array}{c|c} \operatorname{Re} H - sI & p^T \\ \hline p & 0 \end{array}\right) = 0,$$

then $\kappa_i = \|p\|_0^{-1} s_i$. (Here we have taken into account that in [20], the shape operator is defined with opposite sign.) Using computer algebra, we find that s_i are solutions to $As^2 + Bs + C = 0$, where

$$\begin{aligned} A &= -\|p\|_0^2, \\ B &= -p \cdot \operatorname{Re}(H - \operatorname{trace}(H)I) \cdot p^T, \\ C &= -p \cdot \operatorname{adj}(\operatorname{Re} H) \cdot p^T. \end{aligned}$$

Then, by *Viete's formulas*, $s_1 s_2 = C/A$ and $s_1 + s_2 = -B/A$, and the result follows since $K = \kappa_1\kappa_2$ and $S = \frac{1}{2}(\kappa_1 + \kappa_2)$. □

2.3 Solving the Riccati equation

Assuming that G_0 is symmetric and has positive definite imaginary part, one can show that equation (1) is well posed. A solution $G(t)$ exists for all t where c is defined, and such a solution is unique. In addition, for all t the solution $G(t)$ is symmetric, and its imaginary part is positive definite. In other words, if G represents a Gaussian beam at one t, it will be a Gaussian beam for all t [3, 11].

Let us next outline the standard way to solve equation (1). Namely, how to reduce it into a linear system. For simplicity, let us assume that we can cover geodesic c with some local coordinates, whence we can define matrices $C = (C^{ij})_{ij}$, $G = (G_{ij})_{ij}$, $R = (R_{ij})_{ij}$, $N = (N_i^j \circ \dot{c})_{ij}$ all depending on t. Let us point out that matrices C, G, R are symmetric (see Section 3.1 or [3]). Equation (1) is then equivalent to the matrix equation [3, 11]

$$G' - NG - GN^T + GCG - R = 0. \tag{4}$$

Let Y, Z be the solution to the linear system

$$\begin{aligned} \begin{pmatrix} Y \\ Z \end{pmatrix}' &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} -R & -N \\ -N^T & C \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}, \\ Y(0) &= I, \quad Z(0) = G_0. \end{aligned} \tag{5}$$

Using the assumptions on G_0, one can show that Y is invertible for all t. Furthermore, $G = ZY^{-1}$ is the unique solution to equation (4), and so G, interpreted as a 2-tensor is a solution to equation (1).

The next lemma shows that G and Z are invertible for all t.

Lemma 2.3. *A symmetric complex matrix with positive definite imaginary part is invertible.*

Proof. Let A, B be real symmetric matrices, such that B is positive definite. If $\det(A + iB) = 0$, then $-i \in \sigma(B^{-1}A)$ (the spectrum of $B^{-1}A$). Since B^{-1} is positive definite, it has a symmetric square root, whence $-i \in \sigma(B^{-1}A) = \sigma(\sqrt{B^{-1}}A\sqrt{B^{-1}})$. This is a contradiction since $\sqrt{B^{-1}}A\sqrt{B^{-1}}$ is symmetric. $\square$

In the lemma below, the first equation states that $\operatorname{Im} G$ is determined by Y alone.

Proposition 2.4. *Suppose Y, Z are solutions to equation* (5)*. Then*

$$\begin{aligned} \operatorname{Im} G &= (Y^{-1})^* \operatorname{Im}(G_0)\, Y^{-1}, \\ \operatorname{Im}(G^{-1}) &= -(Z^{-1})^* \operatorname{Im}(G_0)\, Z^{-1}, \end{aligned}$$

where A^* is the conjugate transpose of a matrix A.

Proof. As R and C are symmetric, equation (4) implies that

$$\frac{d}{dt}\left(\begin{pmatrix} Y \\ Z \end{pmatrix}^* \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}\right) = 0,$$

so $Y^*Z - Z^*Y = G_0 - G_0^*$. Since G is symmetric,

$$\begin{aligned} \operatorname{Im} G &= \frac{1}{2i}(G - G^*) \\ &= \frac{1}{2i}(Y^{-1})^*(Y^*Z - Z^*Y)Y^{-1} \\ &= \frac{1}{2i}(Y^{-1})^*(G_0 - G_0^*)Y^{-1}. \end{aligned}$$

In particular, $\operatorname{Im} G_0 = \frac{1}{2i}(G_0 - G_0^*)$, and the first claim follows. The second claim follows similarly since G^{-1} is symmetric. $\square$

3 Fermi coordinates

Next we construct coordinates around a geodesic such that on the geodesic, the metric tensor is Euclidean and all Christoffel symbols vanish. More generally, one can find local coordinates around any curve such that the Christoffel symbols vanish. Original references to this result by Fermi (1922) and Levi-Civita (1925–26) are given in [5, p. 92].

In Section 3.1 we show that in Fermi coordinates, $N = 0$, and C is constant. This will considerably simplify equation (4).

Theorem 3.1 (Fermi coordinates). *Suppose M is a Riemannian n-manifold, $n \geq 2$, and $c\colon J \to M$ is a geodesic parametrized with respect to pathlength that does not intersect itself. Furthermore, suppose $c(J)$ is contained in one* *coordinate chart, and I is an open bounded interval satisfying $\overline{I} \subset J$. Then there exist local coordinates $(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = (t, \mu^1, \ldots, \mu^{n-1})$ that cover $c(I)$, and for $t \in I$,*

$$\begin{aligned} (t, \mu = 0) \quad & \textit{represents } c(t), && (6)\\ \dot{c}(t) &= \frac{\partial}{\partial t}\Big|_{c(t)}, && (7)\\ \tilde{g}_{ij}(t) &= \delta_{ij}, && (8)\\ \frac{\partial \tilde{g}_{ij}}{\partial \tilde{x}^k}(t) &= 0, && (9)\\ \tilde{\Gamma}^i_{jk}(t) &= 0. && (10) \end{aligned}$$

Coordinates t, μ^i as in Proposition 3.1 are called *Fermi coordinates*. Since the first Fermi coordinate is in a special position, we shall use an adapted Einstein summation convention: for Roman letters the summation range is $1, \ldots, n$, and for Greek letters the range is $2, \ldots, n$.

One way to prove Theorem 3.1 is to construct coordinates (t, μ) such that for constant (t, μ), curves $s \mapsto (t, s\mu)$ represent geodesics [15]. The disadvantage, however, with this construction is that it gives the coordinates in a rather implicit form. The proof below is less geometric, but the coordinates are given explicitly by equation (11). This expression will be needed in Section 4 and in the proof of Theorem 4.5.

Proof of Theorem 3.1. Let x^i be coordinates that cover $c(J)$. For some $t_0 \in I$, let vectors $P_2(t_0), \ldots, P_n(t_0)$ be such that $\{\dot{c}(t_0), P_2(t_0), \ldots, P_n(t_0)\}$ form an orthonormal basis. For a general $t \in J$, let $P_\alpha(t)$ be the the parallel transport of $P_\alpha(t_0)$ along c to $c(t)$. In other words,

$$P_\alpha(t) = P^i_\alpha(t) \frac{\partial}{\partial x^i}\Big|_{c(t)}, \quad \alpha = 2, \ldots, n,$$

where coefficients $P^i_\alpha(t)$ are solutions to ordinary differential equations

$$\dot{P}^i_\alpha + N^i_s(t) P^s_\alpha \quad = \quad 0.$$

For $\tilde{x}^i = (t, \mu^2, \ldots, \mu^n) \in J \times \mathbb{R}^{n-1}$, let us define a map $\tilde{x} \mapsto x(\tilde{x})$ by

$$\begin{aligned} x^i(\tilde{x}) &= c^i(t) - P^i_\alpha(t)\mu^\alpha - \frac{1}{2} Q^i_{\alpha\beta}(t)\mu^\alpha \mu^\beta, && (11)\\ Q^i_{\alpha\beta} &= \Gamma^i_{ab} P^a_\alpha P^b_\beta. \end{aligned}$$

Let B_r be the Euclidean open ball in $\mathbb{R}^{n-1}$ with center 0 and radius $r > 0$. For some $r > 0$, $x(\overline{I} \times B_r)$ is completely contained in x^i-coordinates. (Proof: Let U be the chart for the x^i coordinates. Suppose that for each $l = 1, 2, \ldots$, there exists an $y_l \in x(\overline{I} \times \overline{B}_{1/l})$ such that $y_l \notin U$. On a subsequence, $y_l \to c(a)$ for some $a \in \overline{I}$. This contradicts $y_l \notin U$ since $c(a) \in c(\overline{I}) \subset c(J) \subset U$, and U is open.)

The rest of the proof is divided into two steps. First we show that $\tilde{x}^i$-coordinates are coordinates. Thereafter we show that the coordinates satisfy the sought conditions. *Step 1.* The parallel transport preserves the inner product between two vectors. Thus $\{\dot{c}(t), P_2(t), \dots, P_n(t)\}$ are orthonormal for all $t \in J$ and, in particular, linearly independent. Since

$$\begin{aligned} \frac{\partial x^i}{\partial t}(t,\mu) &= \dot{c}^i + N^i_s(t) P^s_\alpha \mu^\alpha - \frac{1}{2}\dot{Q}^i_{\alpha\beta}\mu^\alpha\mu^\beta, \\ \frac{\partial x^i}{\partial \mu^\alpha}(t,\mu) &= -P^i_\alpha - Q^i_{\alpha\beta}\mu^\beta, \end{aligned} \tag{12}$$

the Jacobian of $\tilde{x} \mapsto x(\tilde{x})$ is pointwise invertible on $\overline{I} \times \{0\}$. It follows that there exists an open set $X \subset J \times B_r$ such that $x \colon X \to x(X)$ is locally a diffeomorphism. Lemma 3.2 implies that for some $r > 0$, $x \colon I \times B_r \to x(I \times B_r)$ is a diffeomorphism.

Step 2. Equations (6) and (7) follow from equation (11). As $\{\dot{c}(t), P_2(t), \dots, P_n(t)\}$ is orthonormal, we have

$$\begin{aligned} g_{ij}(t) P^i_\alpha P^j_\beta &= \delta_{\alpha\beta}, \\ g_{ij}(t) P^i_\alpha \dot{c}^j &= 0, \\ g_{ij}(t) \dot{c}^i \dot{c}^j &= 1. \end{aligned}$$

Equation (8) now follows from transformation rules $\tilde{g}_{ij} = \frac{\partial x^a}{\partial \tilde{x}^i}\frac{\partial x^b}{\partial \tilde{x}^j} g_{ab}$, and equation (10) follows from transformation rules

$$\tilde{\Gamma}_{ijk} = \frac{\partial x^a}{\partial \tilde{x}^i}\left(\frac{\partial^2 x^b}{\partial \tilde{x}^j \partial \tilde{x}^k} g_{ab} + \frac{\partial x^b}{\partial \tilde{x}^j}\frac{\partial x^c}{\partial \tilde{x}^k}\Gamma_{abc} \right).$$

Here it suffices to show that the parenthesis vanishes for all j, k. Equation (9) follows from the identity

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}. \tag{13}$$

$\square$

Lemma 3.2. *Suppose X is open in $\mathbb{R}^n$, $\overline{X}$ is compact, $f \colon \overline{X} \to f(\overline{X})$ is continuous, and $f \colon X \to f(X)$ is smooth and locally a diffeomorphism. Furthermore, suppose $S \subset X$ is closed and $f \colon S \to f(S)$ is a bijection. Then there exists an open set W in X such that $S \subset W \subset X$, and $f \colon W \to f(W)$ is a diffeomorphism.*

Proof. (Following [19].) Let

$$C = \{(x,y) \in X \times X : f(x) = f(y), x \neq y\}.$$

Then C is closed: If (x_n, y_n) is a sequence in C converging to $(x,y) \in \mathbb{R}^{2n}$, then $f(x) = \lim f(x_n) = \lim f(y_n) = f(y)$. Also, if $x = y$, then there exists a neighbourhood $B \ni x$ where f is a bijection. For some n, $x_n, y_n \in B$ whence $f(x_n) = f(y_n)$ and $x_n = y_n$. Let $d(x, S)$ be the distance from x to the set S (see [14]), and let $\delta\colon C \to \mathbb{R}$ be the function

$$\delta(x, y) = d(x, S) + d(y, S), \quad (x, y) \in C.$$

As $\overline{X}$ is compact, C is compact, and as δ is continuous, δ has a minimum $m > 0$, so that $\delta \geq m$ on C. A suitable set is $W = \{x \in X : d(x, S) < m/2\}$. □

3.1 Curvature

In Fermi coordinates, the Riemann curvature tensor takes the form

$$\tilde{R}^m_{ijk}(t, \mu) \quad = \quad \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{mk}}{\partial \tilde{x}^i\, \partial \tilde{x}^j} - \frac{\partial^2 \tilde{g}_{ik}}{\partial \tilde{x}^m\, \partial \tilde{x}^j} + \frac{\partial^2 \tilde{g}_{ij}}{\partial \tilde{x}^m\, \partial \tilde{x}^k} - \frac{\partial^2 \tilde{g}_{mj}}{\partial \tilde{x}^i\, \partial \tilde{x}^k}\right). \tag{14}$$

This follows by inserting $\tilde{\Gamma}^m_{ik} = \tilde{g}^{ms}\tilde{\Gamma}_{sik}$ into equation (2), differentiating, using $\tilde{g}_{ij} = \delta_{ij}$, $\tilde{\Gamma}^i_{jk} = 0$, and the definition of Γ_{ijk}. (At the expansion point in normal coordinates, the Riemann curvature tensor satisfies the same formula. See [7, p. 130])

In Fermi coordinates, tensors C and R have matrix representations

$$\tilde{C} \quad = \quad \operatorname{diag}(0, 1, 1), \tag{15}$$

$$\tilde{R} \quad = \quad \operatorname{diag}(0, \mathscr{R}), \tag{16}$$

where $\mathscr{R}\colon I \to \mathbb{R}^{2\times 2}$ is a 2×2 symmetric matrix. The latter claim follows since $\tilde{R}_{1i} = \tilde{R}_{i1} = 0$ and $\tilde{R}_{ij} = \tilde{R}^1_{ij1}$, as can be seen from equation (14). Furthermore, as $\tilde{\Gamma}^i_{jk} = 0$ we have $\tilde{N}^i_j = 0$.

4 Gaussian beams in Fermi coordinates

Definition 1 (Surface S_t). For a fixed t, let S_t be the Fermi coordinate surface $t =$ constant passing through $c(t)$.

On S_t, a second order approximation of θ is

$$\begin{aligned} \theta(x(t, \mu), t) \quad &\approx \quad \phi_0 + \frac{1}{2} G_{jk} P^j_\alpha \mu^\alpha P^k_\beta \mu^\beta \\ &= \quad \phi_0 + \frac{1}{2} \tilde{G}_{\alpha\beta}(t) \mu^\alpha \mu^\beta. \end{aligned}$$

This gives a physical interpretation of tensor G and surface S_t. Namely, on S_t a Gaussian beam decays approximately as $\exp(-\frac{P}{2} \operatorname{Im} \tilde{G}_{\alpha\beta}\mu^\alpha\mu^\beta)$.

4.1 Decomposition of the Riccati equation

For matrices $\tilde{G}, \tilde{C}, \tilde{R}$ representing tensors G, C, R in Fermi coordinates, equation (4) reads

$$\tilde{G}' + \tilde{G}\tilde{C}\tilde{G} - \tilde{R} = 0. \tag{17}$$

In other words, when we formulate the Riccati equation in Fermi coordinates, two terms vanish, and $\tilde{C}$ becomes a constant matrix. In addition, since matrices $\tilde{C}, \tilde{R}$ are non-zero only in their lower 2×2 blocks, it turns out that equation (17) is only a 2×2 Riccati equation (equation (20)). Let us next make this precise.

In view of equations (15)-(16), let us also partition $\tilde{G}$ and $\tilde{G}_0$ as

$$\tilde{G} = \left(\begin{array}{c|c} \lambda & u^T \\ \hline u & \mathscr{G} \end{array} \right), \qquad \tilde{G}_0 = \left(\begin{array}{c|c} \lambda_0 & u_0^T \\ \hline u_0 & \mathscr{G}_0, \end{array} \right),$$

where $\lambda \colon I \to \mathbb{C}$, $u \colon I \to \mathbb{C}^{2\times 1}$, and $\mathscr{G} \colon I \to \mathbb{C}^{2\times 2}$ are functions, and $\lambda_0 \in \mathbb{C}, u_0 \in \mathbb{C}^{2\times 1}, \mathscr{G}_0 \in \mathbb{C}^{2\times 2}$ are initial values. That $\operatorname{Im} \tilde{G}_0$ is positive definite means in terms of $\lambda_0, u_0, \mathscr{G}$ that

$$\operatorname{Im} \lambda_0 > \operatorname{Im} u_0^T \, (\operatorname{Im} \mathscr{G}_0)^{-1} \operatorname{Im} u_0, \qquad \operatorname{Im} \mathscr{G}_0 \text{ positive definite.}$$

This follows using a Shur complement on $\operatorname{Im} \tilde{G}$ and the fact: if S is invertible, then A is positive definite if and only if $S^T A S$ is positive definite.

With above notation, equation (17) decomposes into three differential equations

$$\begin{aligned} \lambda' &= -u^T u, && (18) \\ u' &= -\mathscr{G} u, && (19) \\ \mathscr{G}' + \mathscr{G}^2 - \mathscr{R} &= 0, && (20) \end{aligned}$$

with initial values $\lambda(0) = \lambda_0$, $u(0) = u_0$, $\mathscr{G}(0) = \mathscr{G}_0$. If we write $\mathscr{G} = \mathscr{G}_r + i\mathscr{G}_i$, where $\mathscr{G}_r, \mathscr{G}_i$ are real, then equation (20) reads

$$\begin{aligned} \mathscr{G}_r' + \mathscr{G}_r^2 - \mathscr{R} &= \mathscr{G}_i^2, \\ \mathscr{G}_i' + \mathscr{G}_r \mathscr{G}_i + \mathscr{G}_i \mathscr{G}_r &= 0. \end{aligned}$$

Physically, $\operatorname{Im} \lambda$ describes how rapidly the Gaussian beam decays in the direction of propagation, and $\operatorname{Im} \mathscr{G}$ describes its decay in the transversal plane. If $\operatorname{Im} u = 0$, the decay of the Gaussian beam is determined by an ellipsoid aligned with one axis along $\dot{c}$. Equation (20) shows that we can solve $\mathscr{G}$, that is $\tilde{G}_{\alpha\beta}$, without solving u or λ.

Proposition 4.1. *Let $\tilde{G}$ be a solution to equation* (17) *decomposed as above.*

1. *Pointwise $u = 0$ if and only if $\dot{c} = (1, 0, 0)$ is an eigenvector for $\tilde{G}$. In this case, λ is the eigenvalue corresponding to $\dot{c}$.*

2. If $u_0 = 0$, then u is zero for all t, and $\lambda(t) = \lambda_0$ is constant.

Proof. Claim *(i)* is elementary. Claim *(ii)* follows by solution formula for equations (19). □

Let us next give a coordinate invariant formulation of projection $G \mapsto \mathscr{G}$. For this purpose, let $\Pi = \Pi^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ be the tensor determined by coefficients

$$\begin{aligned} \Pi^j_i &= g_{ia} C^{aj} \\ &= \delta^j_i - p_i \dot{c}^j. \end{aligned}$$

Then generally $\Pi \circ \Pi = \Pi$,

$$v = \Pi(v) + g(\dot{c}, v)\dot{c}, \quad v \in T_{c(t)}M,$$

and $\ker \Pi = \operatorname{span}(\dot{c})$. Thus Π is the projection operator mapping $T_{c(t)}M$ onto the orthogonal complement of $\dot{c}(t)$. For a 2-tensor $B = B_{ij} dx^i \otimes dx^j$, let $\Pi B \Pi^T = \Pi^a_i B_{ab} \Pi^b_j dx^i \otimes dx^j$ whence the matrix representing $\Pi B \Pi^T$ in Fermi coordinates is $\tilde{\Pi}\tilde{B}\tilde{\Pi}^T$, where $\tilde{\Pi} = (\tilde{\Pi}^j_i)_{ij} = (\tilde{C}^{ij})_{ij}$. For example, the matrix representing $\Pi G \Pi^T$ in Fermi coordinates is

$$\tilde{\Pi}\tilde{G}\tilde{\Pi}^T = \operatorname{diag}(0, \mathscr{G}).$$

Proposition 4.2 (Riccati equation in transversal plane). *Let G_0 be a symmetric initial value with positive definite imaginary part. If tensor G is a solution to equation* (1) *with initial value G_0, then*

$$F = \Pi G \Pi^T$$

is also a solution to equation (1)*, but with initial value $\Pi G_0 \Pi^T$. Conversely, if G and F are solutions from initial values G_0 and $\Pi G_0 \Pi^T$, respectively, then $F = \Pi G \Pi^T$.*

Proof. Since equation (1) is a tensor equation, it suffices to prove the claim in Fermi coordinates. Thus, by assumption $\tilde{G}$ solves equation (17) with $\tilde{G}(0) = \tilde{G}_0$. Then $\mathscr{G}$ satisfies equation (20) with $\mathscr{G}(0) = \mathscr{G}_0$. If $F = \Pi G \Pi^T$, then $F_0 = \Pi G_0 \Pi^T$, and $\tilde{F} = \operatorname{diag}(0, \mathscr{G})$ satisfies equation (17). For the second claim, let us first note that in Fermi coordinates, $\tilde{F} = \operatorname{diag}(0, \mathscr{F})$ for some 2×2 matrix $\mathscr{F}$. This follows from equations (18)-(19), and their solution formulas. Let $\tilde{C}\tilde{G}\tilde{C} = \operatorname{diag}(0, \mathscr{G})$. Then $\mathscr{F}$ and $\mathscr{G}$ share the same initial value, which is symmetric and has positive definite imaginary part. Since, in addition, $\mathscr{F}$ and $\mathscr{G}$ both solve equation (20), we have $\mathscr{F} = \mathscr{G}$. This gives $\tilde{F} = \operatorname{diag}(0, \mathscr{F}) = \operatorname{diag}(0, \mathscr{G}) = \tilde{C}\tilde{G}\tilde{C}$. □

4.2 Shape operator of surfaces F_t and S_t

Next we calculate shape operators of surfaces F_t and S_t. This will be done using Lemma 4.3 and Lemma 4.4. To formulate these lemmas, we need the *Legendre transformation* $\sharp\colon T^*M \to TM$ defined as $\sharp\colon \alpha_i dx^i \mapsto g^{ij}\alpha_i \frac{\partial}{\partial x^j}$. Let us also write $\alpha^\sharp = \sharp(\alpha)$.

Lemma 4.3 (Shape operator for an equipotential surface). *Let $p \in M$. Suppose $f\colon M \to \mathbb{R}$ is a smooth function with $df_p \neq 0$, whence*

$$S = \{x \in M : f(x) = f(p)\}$$

is a hypersurface passing trough p. Suppose further that $g((df)^\sharp, (df)^\sharp) = 1$ at p, and S is oriented with a unit normal n such that $n = df^\sharp$ at p. Then the shape operator of S at p equals

$$L(v) = \pi \circ \sharp \circ D_v(df), \quad v \in T_pS,$$

where $\pi\colon T_pM \to T_pS$ is the projection $\pi(v) = v - g(v,n)n$ for $v \in T_pM$.

Proof. Suppose $E \in \mathfrak{X}(M)$ is vector field

$$E(x) = (df)^\sharp(x), \quad x \in M,$$

whence $g(E,E) = 1$ at $p \in M$, and $\hat{n} = E/\sqrt{g(E,E)}$ is an extension of a correctly oriented unit normal on S. Suppose $v \in T_pM$. Then $D_v(fX) = v(f)X + fD_v(X)$ for $X \in \mathfrak{X}(M)$, and $D_v(\alpha^\sharp) = \sharp \circ D_v(\alpha)$ for $\alpha \in \Omega^1(M)$. These identities together with $\pi \circ L = L$ imply the result. $\square$

Lemma 4.4 (Shape operator in Fermi coordinates). *Suppose f and S are as in the previous lemma, c is a geodesic parametrized with respect to pathlength such that $\dot{c}(t) = (df)^\sharp(p)$, and (s,μ) are Fermi coordinates around c. Then the shape operator of S at $c(t)$ is*

$$L(v) = \sum_{\alpha,\beta=2}^{3} \tilde{v}^\alpha \frac{\partial^2 f}{\partial\mu^\alpha\,\partial\mu^\beta}(p)\frac{\partial}{\partial\mu^\beta}, \quad v \in T_p(S).$$

Theorem 4.5 (Shape operator of surface F_t). *Let phase front F_t be oriented with unit normal n such that $n_{c(t)} = \dot{c}(t)$. Then the shape operator of F_t at $c(t)$*

$$L\colon T_{c(t)}F_t \to T_{c(t)}F_t$$

is given by

$$L(v) = v^i(\Pi \operatorname{Re} G \Pi^T)_{ij} g^{jk}\frac{\partial}{\partial x^k}, \quad v \in T_{c(t)}F_t,$$

or in Fermi coordinates,

$$L(v) = \sum_{\alpha,\beta=2}^{3} \tilde{v}^\alpha \operatorname{Re}\tilde{G}_{\alpha\beta}\frac{\partial}{\partial\mu^\beta}, \quad v \in T_{c(t)}F_t.$$

Proof. We only need to prove the claim in Fermi coordinates. Let us fix t, whence $F_t = \{x \in M : f(x) = f(c(t))\}$ for $f(x) = \operatorname{Re}\theta(x,t)$. Since $(df)^\sharp(c(t)) = \dot{c}(t)$, Lemma 4.4 implies that

$$L(v) = \sum_{\alpha,\beta=2}^{3} \tilde{v}^\alpha \frac{\partial^2 f}{\partial\mu^\alpha\,\partial\mu^\beta}\frac{\partial}{\partial\mu^\beta}, \quad v \in T_{c(t)}F_t.$$

By the chain rule we have $\frac{\partial^2 f}{\partial \mu^\alpha \partial \mu^\beta} = \frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial \mu^\alpha}\frac{\partial x^j}{\partial \mu^\beta} + \frac{\partial f}{\partial x^i}\frac{\partial^2 x^i}{\partial \mu^\alpha \partial \mu^\beta}$. Evaluating at $\mu = 0$ and using equation (12) yields $\frac{\partial^2 f}{\partial \mu^\alpha \partial \mu^\beta}(t, 0) = \operatorname{Re} \tilde{G}_{\alpha\beta}(t)$. □

In the Euclidean inner product, the shape operator vanishes on hyperplanes. Intuitively this means that the Euclidean inner product sees no shape on a hyperplane. The next proposition gives an analogous result for g and S_t.

Proposition 4.6 (Shape operator of surface S_t). *The curve c is orthogonal to S_t, and the shape operator on S_t (with respect to g) is identically zero at $c(t)$.*

Proof. The first claim follows since $T_{c(t)}S_t = \operatorname{span}\{\frac{\partial}{\partial \mu^2}, \frac{\partial}{\partial \mu^3}\}$. The second claim follows from Lemma 4.4 as $S_t = \{(s, \mu) : s - t = 0\}$. □

Let us recall Gauss' lemma in Riemannian geometry [1, 19]. It states that if B is a geodesic ball around a point p, then geodesics from p will intersect the boundary of B orthogonally. In the setting of Gaussian beams, a geodesic ball around a point represents the *area of influence*. In time t, no Gaussian beam can propagate outside a geodesic ball of radius t. Proposition 2.1 and 4.6 state that for small t, surfaces F_t and S_t are tangential to the geodesic sphere of radius t.

4.3 Linear equations for Y and Z

In Fermi coordinates, equations (5) takes the simple form

$$\begin{aligned} \tilde{Y}' &= \tilde{C}\tilde{Z}, \\ \tilde{Z}' &= \tilde{R}\tilde{Y}, \\ \tilde{Y}(0) &= I, \quad \tilde{Z}(0) = \tilde{G}_0. \end{aligned}$$

Since $\tilde{C}$ is constant, this system can also be written as a second order ordinary differential equation

$$\begin{aligned} \tilde{Y}'' &= \tilde{R}\tilde{Y}, \\ \tilde{Y}'(0) &= \tilde{C}\tilde{G}_0, \\ \tilde{Y}(0) &= I. \end{aligned}$$

Structurally, equation $\tilde{Y}'' = \tilde{R}\tilde{Y}$ has the form of a Jacobi equation [7]. However, this reduction only seems to be possible in Fermi coordinates. A more general result in this direction can be found in [4].

5 Gaussian beams in Euclidean geometry

Let us consider the case when g is Euclidean, that is, $g = \delta_{ij} dx^i \otimes dx^j$, and the geodesic is $t \mapsto (t, 0, 0)$. Then Fermi coordinates (s, μ) coincide with Euclidean coordinates; $s = x^1$, $\mu^2 = x^2$, and $\mu^3 = x^3$.

5.1 Geometry of surface F_t

Let us study the shape of F_t for $t = 0$. For simplicity, let us assume that $\operatorname{Re} G_{1i} = 0$ for all i, $c(t) = 0$, and θ is a second order polynomial in z. The equation for F_t then reads

$$s = -\frac{1}{2} \operatorname{Re} G_{\alpha\beta} \mu^\alpha \mu^\beta.$$

If we assume that $\operatorname{Re} G_{11}$ is non-zero, then one obtains a second order equation for t. Of the two solutions, one must be excluded as it does not pass through $c(t)$, and if one approximates $\sqrt{1+x} = 1 + x/2$ in the other, it will coincide with the above solution. Figure 1 shows how F_t depends on the definiteness of $\operatorname{Re} \mathscr{G}$.

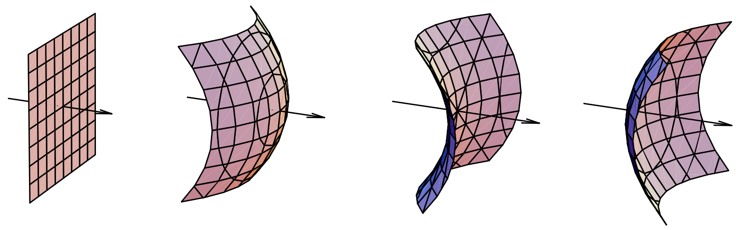

Figure 1: Phase front F_t in cases $\operatorname{Re} \mathscr{G} = 0$, $\operatorname{Re} \mathscr{G}$ positive definite, $\operatorname{Re} \mathscr{G}$ semi-definite, and $\operatorname{Re} \mathscr{G}$ negative definite. The arrow shows direction of propagation.

5.2 Explicit solution to $\mathscr{G}$

In Euclidean geometry, $R = 0$, and the solution to equation (17) is

$$G(t) = G_0 (I + tCG_0)^{-1}, \quad C = \operatorname{diag}(0, 1, 1),$$

or, in partitioned form,

$$\begin{aligned} \lambda(t) &= u_0^T \mathscr{G}_0^{-1} (I + \mathscr{G}_0 t)^{-1} u_0 + k, \\ u(t) &= (I + \mathscr{G}_0 t)^{-1} u_0, \\ \mathscr{G}(t) &= \mathscr{G}_0 (I + \mathscr{G}_0 t)^{-1}, \end{aligned}$$

where

$$k = \lambda_0 - u_0^T \mathscr{G}_0^{-1} u_0 = \det \mathscr{G}_0^{-1} \, \det G_0.$$

The expressions for G and $\mathscr{G}$ can be derived using the standard solution method for a matrix Riccati equation as outlined in Section 2.3. However, it is maybe easiest to just verify these by direct substitution using matrix identity $\frac{dA^{-1}}{dt} =$

$-A^{-1}\frac{dA}{dt}A^{-1}$, and the fact that A and $(I+A)^{-1}$ commute. Inverse $(I+\mathscr{G}_0 t)^{-1}$ exists by Lemma 2.3, and $(I+tCG_0)^{-1}$ exists since $\det(I+tCG_0) = \det(I+\mathscr{G}_0 t)$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If B is an invertible 2×2 symmetric matrix, then $B^{-1} = \frac{-1}{\det B} JBJ$. Thus

$$(I+\mathscr{G}_0 t)^{-1} = \frac{I - J\mathscr{G}_0 J t}{\det(I - J\mathscr{G}_0 J t)}.$$

From this expression it follows that $\lambda(t), u(t)$, and $\mathscr{G}(t)$ decay as $1/t$. Thus, if $I = \mathbb{R}$, then

$$\lim_{t\to\infty} \lambda(t) = \det \mathscr{G}_0^{-1}\ \det G_0, \quad \lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} \mathscr{G}(t) = 0,$$

In particular, at infinity the Gaussian beam is infinitely wide ($\mathscr{G} = 0$)

Example 5.1 (Eigenvalues of $\operatorname{Re}\mathscr{G}$ and $\operatorname{Im}\mathscr{G}$). Let us study solutions from initial values of the form

$$\mathscr{G}_0 = \operatorname{diag}(a+i, c+i), \tag{21}$$

where a, c are real eigenvalues of $\operatorname{Re}\mathscr{G}$. In this case, the Riccati equation decouples into two scalar equations. Figure 2 shows the eigenvalues of $\operatorname{Re}\mathscr{G}$ and $\operatorname{Im}\mathscr{G}$ for six values for a and c. The signum-type function in the plots will be studied in Section 5.3.

In the plots we can see three phenomena. First, in all plots the magnitudes of the eigenvalues tend to zero. In other words, for large t, the phase front will resemble more and more a plane. Secondly, even if initial value $\operatorname{Re}\mathscr{G}_0$ is semidefinite or zero, the solution $\operatorname{Re}\mathscr{G}$ always tends to a positive definite matrix. This suggests that the solution is unstable if $\operatorname{Re}\mathscr{G}$ is not positive definite. This is also physically reasonable. In free space, the phase surface should look like a sphere, not as a hyperbola (see Figure 1). Lastly, eigenvalues of $\operatorname{Re}\mathscr{G}$ and $\operatorname{Im}\mathscr{G}$ are coupled.

5.3 When is the phase front within the area of influence?

Let us write the general initial value $\mathscr{G}_0$ as

$$\mathscr{G}_0 = \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} + i \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

where A, B, C are complex numbers, and $a, b, c, \alpha, \beta, \gamma$ are real. Computer algebra gives the following expressions for K and S,

$$\begin{aligned} K(t) &= \frac{C_4 t^2 + \frac{1}{2}C_3 t + C_0}{C_4 t^4 + C_3 t^3 + C_2 t^2 + C_1 t + 1}, \\ S(t) &= \frac{1}{2}\frac{2C_4 t^3 + \frac{3}{2}C_3 t^2 + C_2 t + \frac{1}{2}C_1}{C_4 t^4 + C_3 t^3 + C_2 t^2 + C_1 t + 1}, \end{aligned}$$

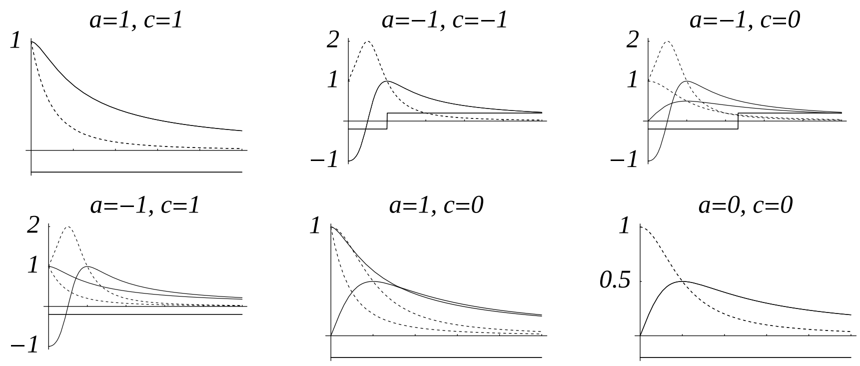

Figure 2: Eigenvalues of $\operatorname{Re}\mathscr{G}$ (solid lines) and $\operatorname{Im}\mathscr{G}$ (dashed lines) for $t = 0, \ldots, 5$ from initial value (21). The signum-type functions is described in Section 5.3.

where

$$\begin{aligned}
C_0 &= \det \operatorname{Re}\mathscr{G}_0, \\
C_1 &= 2 \operatorname{trace} \operatorname{Re}\mathscr{G}_0, \\
C_2 &= \left(\operatorname{trace}\left(\operatorname{Re}\mathscr{G}_0\right)\right)^2 + 2\det\operatorname{Re}\mathscr{G}_0 + \left(\operatorname{trace}\left(\operatorname{Im}\mathscr{G}_0\right)\right)^2 - 2\det\operatorname{Im}\mathscr{G}_0, \\
C_3 &= 2\left(c|A|^2 + a|C|^2 - \operatorname{Re}\left(B^2 \operatorname{trace}(\mathscr{G}_0^*)\right)\right), \\
C_4 &= |\det \mathscr{G}_0|^2,
\end{aligned}$$

and M^* is the complex conjugate of a matrix M. These expressions can be derived either using Proposition 2.2, or from the eigenvalues of $\mathscr{G}$ and using Theorem 4.5. It follows that K decays as $1/t^2$, and S decays as $1/t$. Thus for large t the Gaussian curvature of F_t (at $c(t)$) approaches $1/t^2$ which is the curvature for a sphere of radius t. The sphere with center $c(0)$ and radius t represents the *area of influence*; in time t, a signal can propagate at most a distance of t units. As the phase front describes how the field propagates, it would be natural to assume that near $c(t)$, phase front F_t is contained inside this sphere. In other words, it would be natural to assume that Gaussian beams should satisfy

$$K(t) > \frac{1}{t^2}, \quad S(t) > 0. \tag{22}$$

The first condition is only reasonable for large t. The latter condition assures that $\operatorname{Re}\mathscr{G}$ is positive definite, so that phase front F_t is bending in the right direction (see Figure 1). The signum-type function plotted in Figure 2 indicates when condition (22) holds. When the function is positive the condition holds and when negative the condition does not hold. Let us point out that $G(t)$ in $a = c = 1$ is identical to $G(t+1)$ in $a = c = -1$. However, condition

$K(t) > 1/t^2$ is not translationally invariant. Therefore the signum functions do not coincide. Schematically this condition is illustrated in Figure 3. In the left figure, condition (22) holds. In other words, phase fronts are contained within the area of influence. The right figure shows phase fronts where $\mathrm{Re}\,\mathscr{G}$ are positive definite, but $K(t) < \frac{1}{t^2}$.

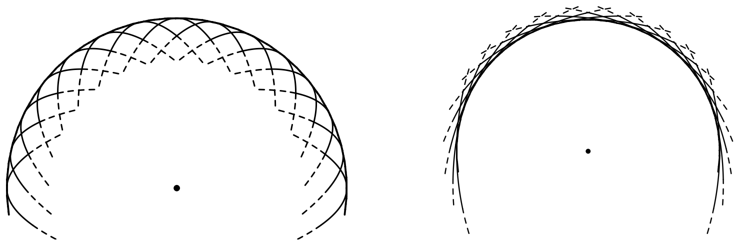

Figure 3: A $2D$ illustration of phase fronts in cases $K(t) > 1/t^2$ (left) and $K(t) < 1/t^2$ (right).

For large t, condition (22) holds if and only if

$$\lim_{t\to\infty} \frac{1}{2} C_3 t^3 + (C_2 - C_0)t^2 + C_1 t + 1 = -\infty.$$

For example, if $C_3 < 0$, the condition holds for large t. Suppose $B = 0$. Then matrix Riccati equation (20) decomposes into two scalar equations, and in this case, condition $C_3 < 0$ simplifies into $c|A|^2 + a|C|^2 < 0$. It follows that in Figure 2, condition (22) will be satisfied only in cases $(a, c) = (-1, 0)$ and $(-1, -1)$. In addition, by slightly perturbing initial values $(-1, 1)$ and $(0, 0)$, these too satisfy the condition.

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