Non-dissipative electromagnetic medium with two Lorentz null cones

Matias F. Dahl[1]

Aalto University, Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland

Abstract

We study Maxwell's equations on a 4-manifold where the electromagnetic medium is modelled by an antisymmetric $\binom{2}{2}$-tensor with real coefficients. In this setting the *Fresnel surface* is a fourth order polynomial surface that describes the dynamical response of the medium in the geometric optics limit. For example, in isotropic medium the Fresnel surface is a Lorentz null cone. The contribution of this paper is the pointwise description of all electromagnetic medium tensors κ with real coefficients that satisfy the following three conditions:

(i) medium κ is invertible,
(ii) medium κ is *skewon-free*, or non-dissipative,
(iii) the Fresnel surface of κ is the union of two distinct Lorentz null cones.

We show that there are only three classes of media with these properties and give explicit expressions in local coordinates for each class.

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We will study Maxwell's equations on an orientable 4-manifold N where the electromagnetic medium is represented by a suitable tensor κ on N. We also assume that the medium is non-dissipative (or *skewon-free*). An advantage of this formulation is that at each point in N, the medium may depend on up to 21 real parameters. This freedom allows the modelling of a wide array of different anisotropic behaviour including magneto-electric

Email address: `matias.dahl@tkk.fi` (Matias F. Dahl)
URL: `http://www.math.tkk.fi/~fdahl` (Matias F. Dahl)

effects. In this setting, the *Fresnel surface* is a fourth order polynomial surface that describes the dynamical response of the medium in the geometric optics limit. Classically, the Fresnel surface can be seen as the analogue of the dispersion equation, which parameterises signal speeds as a function of direction [1, 2, 3]. For example, in isotropic medium the Fresnel surface is a Lorentz null cone. In general the Fresnel surface is a fourth order surface and can have multiple sheets and singular points.

From the above we see that there are two ways to describe an electromagnetic medium: One description is the tensor κ that prescribes the coefficients in Maxwell's equations. This is the information required to solve the initial value problem for Maxwell's equations. The second description is the Fresnel surface of κ that gives the dynamical response of the medium for high-frequency waves. If κ is known we can compute the Fresnel surface by an explicit equation (see equation (11) below). A less well understood problem is the converse dependence, or inverse problem [4, 5]: If the Fresnel surface of a medium is known at a point $p \in N$, what can we say about the coefficients in $\kappa|_p$? In other words, if the behaviour of signal speed for an electromagnetic medium is known, what can we say about the anisotropic structure of the medium? In general, this relation is not well understood, and therefore it is motivated to consider the problem under additional assumptions. The purpose of this paper we study the special case when the Fresnel surface is two distinct Lorentz null cones. For example, this is the characteristic behaviour for signal speed in *uniaxial crystals* like calcite ($CaCO_3$). In such media, propagation does not only depend on direction, but can also depend on the polarisation of the wave. That is, the medium can be *birefringent*, and each of the two Lorentz null cones describes the signal speed behaviour for different modes of polarisation.

The main result of this paper is Theorem 2.1. It gives the complete pointwise description of all medium tensors κ with real coefficients for which

(i) medium κ is invertible,
(ii) medium κ is *skewon-free*, that is, non-dissipative,
(iii) the Fresnel surface of κ is the union of two distinct Lorentz null cones.

We show that there are only three medium classes with these properties and give explicit expressions in local coordinates for each class. The first class is a generalisation of uniaxial medium. The second and third classes do not seem to have been studied before. The second class has the peculiar property that a wave can propagate with three different phase velocities in the same direction.

Regarding the related problem of characterising media with a single Lorentz null cone, see Section 1.2. For a further discussions and results on the factorisability of the Fresnel surface, see [6, 7] and in 3 dimensions [8, 9]. Of these [6], also studies the question in the presence of a skewon-component.

The main idea of the proof is as follows. We will use the normal form theorem by Schuller, Witte and Wohlfarth [10], which pointwise divides area metrics into 23 metaclasses and gives explicit expressions in local coordinates for each metaclass [10]. This result was also used in [11], and by [10] we only need to consider the first 7 metaclasses. For each of these metaclasses, the Fresnel surface can be written as $F_p(\kappa) = \{\xi \in \mathbb{R}^4 : f(\xi) = 0\}$ for a homogeneous 4th order polynomial $f\colon \mathbb{R}^4 \to \mathbb{R}$ with coefficients determined by $\kappa|_p$. By the assumption f factors as

$$f(\xi) \;\;=\;\; f_+(\xi) f_-(\xi), \quad \xi \in \mathbb{R}^4,$$

where $f_\pm$ are quadratic forms $f_\pm\colon \mathbb{R}^4 \to \mathbb{R}$ with Lorentz signatures. By identifying coefficients we obtain a system of polynomial equations in coefficients of f and $f_\pm$. In the last step we eliminate the coefficients in $f_\pm$ from these equations whence we obtain constraints on f (and hence on $\kappa|_p$). To eliminate variables we use the technique of *Gröbner bases*, which is a computer algebra technique for manipulating polynomial equations.

A limitation of Theorem 2.1 is that the explicit expression is only valid at a point. The reason for this is that the normal form theorem in [10] essentially relies on the Jordan normal form theorem for matrices, which is unstable under perturbations. Another limitation is that we do not allow for complex coefficients in κ which is common when working with time harmonic fields.

This paper relies on computations by computer algebra. Mathematica notebooks for these computations can be downloaded from the author's homepage.

1. Maxwell's equations

By N we denote a 4 dimensional smooth manifold. Moreover, we will assume that N is orientable and oriented so an oriented atlas has been fixed for N. Unless otherwise specified, all local coordinates for N will be assumed to be compatible with this chosen atlas. All objects are assumed to be smooth and real where defined. Let TN and T^*N be the tangent and cotangent bundles, respectively. For $k \geq 1$, let $\Lambda^k(N)$ be the set of antisymmetric k-covectors, so that $\Lambda^1(N) = T^*N$. Also, let $\Omega^k_l(N)$ be $\binom{k}{l}$-tensors fields that are antisymmetric in their k upper indices and l lower indices. In particular, let $\Omega^k(N)$ be the set of k-forms. Let $C^\infty(N)$ be the set of functions. The Einstein summing convention is used throughout. When writing tensors in local coordinates we assume that the components satisfy the same symmetries as the tensor.

1.1. Maxwell's equations on a 4-manifold

By an *electromagnetic medium* on N we mean an antisymmetric $\binom{2}{2}$-tensor $\kappa \in \Omega^2_2(N)$. Then we can treat κ as a map

$$\kappa\colon \Omega^2(N) \rightarrow \Omega^2(N) \tag{1}$$

and we say that fields $F, G \in \Omega^2(N)$ solve the *sourceless Maxwell's equations* if

$$dF = 0, \tag{2}$$
$$dG = 0, \tag{3}$$
$$G = \kappa(F), \tag{4}$$

where d is the exterior derivative on N. Equation (4) is known as the *constitutive equation* and acts as the model for the electromagnetic medium. In this model, the medium is assumed to be linear, smoothly varying and with a pointwise response.

Let us make two comments regarding this formulation. First, since we assume N to be oriented, we may assume that F, G and κ are usual tensors and not twisted tensors [1, 2]. Second, a key motivation for the above formulation is that it allows us to write Maxwell's equations with as little mathematical structure as possible. For example, the above formulation is independent of any geometric structure like a Lorentz metric. Therefore the formulation is also known as the *premetric formulation* for electrodynamics. In special cases, N may have a Lorentz metric that describes a gravitational field. However, in the above formulation any such gravitational effects on electrodynamics is assumed to be contained in κ. For a systematic presentation, see [1, 2].

Let $\{x^i\}_{i=0}^3$ be coordinates for N. We can then write

$$\kappa = \frac{1}{2}\kappa^{ij}_{lm} dx^l \otimes dx^m \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \tag{5}$$

$F = F_{ij} dx^i \otimes dx^j$ and $G = G_{ij} dx^i \otimes dx^j$, whence equation (4) reads

$$G_{ij} = \frac{1}{2}\kappa^{rs}_{ij} F_{rs}. \tag{6}$$

At each point on N, a general antisymmetric $\binom{2}{2}$-tensor κ depends on 36 free real components. In coordinates, it will be convenient to represent these components by a smoothly varying 6×6 matrix. To do this, let O be the ordered set of index pairs $\{01, 02, 03,\ 23, 31, 12\}$. If $I \in O$, we denote the individual indices by I_1 and I_2. Say, if $I = 31$ then $I_2 = 1$. Let also $dx^I = dx^{I_1} \wedge dx^{I_2}$. Then a local basis for $\Omega^2(N)$ is given by $\{dx^J : J \in O\}$, that is, by

$$\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2\}, \tag{7}$$

and equation (6) can be rewritten as

$$\kappa(dx^J) \;=\; \sum_{I \in O} \kappa_I^J dx^I, \quad J \in O, \tag{8}$$

where $\kappa_I^J = \kappa_{I_1 I_2}^{J_1 J_2}$. It follows that κ is locally determined by components $\{\kappa_I^J : I, J \in O\}$. By identifying O with indices $\{1, 2, \ldots, 5, 6\}$ and treating I as a row index and J as a column index, these components define a 6×6 matrix that we denote by $(\kappa_I^J)_{IJ}$.

To understand the physical interpretation of the different components in $(\kappa_I^J)_{IJ}$, let us treat coordinate x^0 as the time variable on N. Then we may locally decompose F and G into temporal and spatial components and write $F = B + E \wedge dx^0$ and $G = D - H \wedge dx^0$ for suitable forms B, E, D, H. If we further define

$$\begin{aligned} \mathbf{E} &= (F_{10}, F_{20}, F_{30}), \quad \mathbf{B} = (F_{23}, F_{31}, F_{12}), \\ \mathbf{H} &= (G_{01}, G_{02}, G_{03}), \quad \mathbf{D} = (G_{23}, G_{31}, G_{12}), \end{aligned}$$

then equations (2)–(3) are the usual Maxwell's equations in $\mathbb{R}^3$ for vector fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$. Moreover, equation (6) reads

$$\begin{aligned} \begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} &= \left(\begin{array}{ccc|ccc} \kappa_{01}^{01} & \kappa_{01}^{02} & \kappa_{01}^{03} & \kappa_{01}^{23} & \kappa_{01}^{31} & \kappa_{01}^{12} \\ \kappa_{02}^{01} & \kappa_{02}^{02} & \kappa_{02}^{03} & \kappa_{02}^{23} & \kappa_{02}^{31} & \kappa_{02}^{12} \\ \kappa_{03}^{01} & \kappa_{03}^{02} & \kappa_{03}^{03} & \kappa_{03}^{23} & \kappa_{03}^{31} & \kappa_{03}^{12} \\ \hline \kappa_{23}^{01} & \kappa_{23}^{02} & \kappa_{23}^{03} & \kappa_{23}^{23} & \kappa_{23}^{31} & \kappa_{23}^{12} \\ \kappa_{31}^{01} & \kappa_{31}^{02} & \kappa_{31}^{03} & \kappa_{31}^{23} & \kappa_{31}^{31} & \kappa_{31}^{12} \\ \kappa_{12}^{01} & \kappa_{12}^{02} & \kappa_{12}^{03} & \kappa_{12}^{23} & \kappa_{12}^{31} & \kappa_{12}^{12} \end{array}\right) \cdot \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix} \\ &= \left(\begin{array}{c|c} \mathscr{C}^t & \mathscr{B}^t \\ \hline \mathscr{A}^t & \mathscr{D}^t \end{array}\right) \cdot \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix}, \end{aligned} \tag{9}$$

where $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ are suitably defined 3×3 matrices [2, Section D.1.7], and ^{t} denotes the usual matrix transpose. In other words, the 6×6 matrices in equation (9) represent the map (1) with respect to the basis (7). From equation (9) we also see that $-\mathscr{A}^t$ correspond to the permittivity matrix, $\mathscr{B}^{-t}$ (when invertible) corresponds to the permeability matrix and matrices $\mathscr{C}$ and $\mathscr{D}$ correspond the the magneto-electric effect [1, 2, 10, 12, 13].

The medium is called *axion-free* if trace $\kappa = 0$ [2, 14]. In the main result of this paper (Theorem 2.1) we will assume that κ is *skewon-free*, that is,

$$\kappa(u) \wedge v \quad = \quad u \wedge \kappa(v), \qquad \text{for all} \quad u, v \in \Omega^2_2(N),$$

or in coordinates, $\mathscr{A} = \mathscr{A}^t$, $\mathscr{B} = \mathscr{B}^t$ and $\mathscr{C} = \mathscr{D}^t$ for the matrices in equation (9). This symmetry reduces κ to only 21 free components at each point [2]. Physically, such medium describe non-dissipative medium. For example, in skewon-free medium Poynting's theorem holds under suitable assumptions.

Let us note that invertible skewon-free media are essentially in a one-to-one correspondence with *area metrics*. See [10] and [15, Proposition 2.4]. Area metrics also appear when studying the propagation of a photon in a vacuum with a first order correction from quantum electrodynamics [10, 16]. The Einstein field equations have also been generalised into equations where the unknown field is an area metric [17]. For further examples, see [3, 10] and for the differential geometry of area metrics, see [17, 18].

1.2. The Fresnel surface

Let $\kappa \in \Omega^2_2(N)$. In this section we define the Fresnel surface of κ, which describes the dynamical response of κ in the geometric optics limit. If κ is locally given by equation (5) in coordinates $\{x^i\}_{i=0}^3$, let

$$\mathscr{G}_0^{ijkl} \quad = \quad \frac{1}{48} \kappa^{a_1 a_2}_{b_1 b_2} \kappa^{a_3 i}_{b_3 b_4} \kappa^{a_4 j}_{b_5 b_6} \varepsilon^{b_1 b_2 b_5 k} \varepsilon^{b_3 b_4 b_6 l} \varepsilon_{a_1 a_2 a_3 a_4},$$

where $\varepsilon_{l_1 \cdots l_n}$ and $\varepsilon^{l_1 \cdots l_n}$ are the *Levi-Civita permutation symbols*. In overlapping coordinates $\{\widetilde{x}^i\}_{i=0}^3$, these coefficients transform as

$$\widetilde{\mathscr{G}}_0^{ijkl} \quad = \quad \det\left(\frac{\partial x^r}{\partial \widetilde{x}^s}\right) \mathscr{G}_0^{abcd} \frac{\partial \widetilde{x}^i}{\partial x^a} \frac{\partial \widetilde{x}^j}{\partial x^b} \frac{\partial \widetilde{x}^k}{\partial x^c} \frac{\partial \widetilde{x}^l}{\partial x^d}. \tag{10}$$

Components $\mathscr{G}_0^{ijkl}$ thus define a tensor density $\mathscr{G}_0$ on N of weight 1. The *Tamm-Rubilar tensor density* is the symmetric part of $\mathscr{G}_0$ and we denote this tensor density by $\mathscr{G}$ [1, 2, 5]. In coordinates, $\mathscr{G}^{ijkl} = \mathscr{G}_0^{(ijkl)}$, where parenthesis indicate that indices $ijkl$ are symmetrised with scaling $1/4!$. Using tensor density $\mathscr{G}$, the *Fresnel surface* at a point $p \in N$ is defined as

$$F_p(\kappa) \quad = \quad \{\xi \in T^*_p N : \mathscr{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = 0\}. \tag{11}$$

In each cotangent space, the Fresnel surface $F_p(\kappa)$ is a homogeneous fourth order polynomial surface, so it can have multiple sheets and singular points. There are various ways to derive the Fresnel surface: by studying a propagating weak discontinuity [1, 2, 19], using geometric optics [5, 20], or as the characteristic polynomial of the full Maxwell's equations [10]. Classically, the Fresnel surface can be seen as the dispersion equation for a medium, so that it constrains possible wave speed(s) as a function of direction.

By a *pseudo-Riemann metric* on N we mean a symmetric $\binom{0}{2}$-tensor g that is non-degenerate. If N is not connected we also assume that g has constant signature. By a *Lorentz metric* we mean a pseudo-Riemann with signature $(-+++)$ or $(+---)$. At $p \in N$ we define the *null cone* for a pseudo-Riemann metric g as the set

$$N_p(g) \quad = \quad \{\xi \in T_p^*N : g(\xi,\xi) = 0\}.$$

If g is a pseudo-Riemann metric on N, then the *Hodge star operator* is the $\binom{2}{2}$-tensor $\kappa = *_g \in \Omega_2^2(N)$ with components

$$\kappa^{ij}_{rs} \quad = \quad \sqrt{|\det g|}\; g^{ia} g^{jb} \varepsilon_{abrs} \tag{12}$$

when κ is written as in equation (5), $g = g_{ij} dx^i \otimes dx^j$, $\det g = \det g_{ij}$ and g^{ij} is the ijth entry of $(g_{ij})^{-1}$. One can show that $*_g$ is both skewon-free and axion-free.

The importance of the Hodge star operator is that it gives a model for isotropic electromagnetic medium. For example, if $\varepsilon, \mu > 0$ and g is the Lorentz metric $g = \operatorname{diag}(\frac{-1}{\varepsilon\mu}, 1, 1, 1)$ on $\mathbb{R}^4$ then $\kappa = \sqrt{\frac{\varepsilon}{\mu}}\, *_g$ is the medium tensor for isotropic medium with permittivity ε and permeability μ. More generally, we will say that $\kappa \in \Omega_2^2(N)$ is *isotropic* if $\kappa = f *_g$ for a non-zero function $f \in C^\infty(N)$ and a Lorentz metric g. For such an isotropic medium $\kappa = f *_g$, we know that electromagnetic waves propagate along null-geodesics of g. That is, the dynamical response of κ is modelled by Lorentz geometry and

$$F_p(\kappa) \quad = \quad N_p(g), \quad p \in N. \tag{13}$$

The converse claim is that isotropic medium is the only skewon-free and axion-free medium tensor κ for which equation (13) holds for some Lorentz metric g. This is a conjecture that has been formulated and studied in a number of papers [1, 19, 21, 22, 23]. See also the book [2] by Hehl and Obukhov. The conjecture has been proven in a number of cases: when $\mathscr{C} = 0$ (see equation (9)) by Obukhov, Fukui and Rubilar [19] and for a special class of nonlinear media by Obukhov and Rubilar [23]. Also, on the level of the Fresnel polynomial, Favaro and Bergamin [11] have shown that isotropic medium (as defined above) is the only class of skewon-free and axion-free medium tensors with a *bi-quadratic* Fresnel polynomial of Lorentzian signature. For additional results and discussions, see also [2, 4, 5, 24, 25, 26]. In particular, we refer to [5] for a description on how the argument below changes to characterising medium with one Lorentz null cone. Let us here note that when the Fresnel surface decomposes into two distinct Lorentz null cones, then Propositions 1.3 and 1.4 below show that the Fresnel polynomial $\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$ factors into two Lorentzian quadratic forms that essentially are uniquely determined. However, if $F_p(\kappa)$ is only one Lorentz null cone, then only one Lorentzian quadratic factor is uniquely determined up to scaling. Since the Fresnel surface only contains the real roots, one can not determine the second factor from $F_p(\kappa)$ alone since it could (without a further analysis [5]) be a positive definite factor like $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2$. See [5, Example 4.2].

1.3. Fresnel surfaces that decompose into two Lorentz null cones

For the remainder of this paper, we will study medium tensors for which the Fresnel surface can be written as the union of two Lorentz null cones. Let us therefore make the following definition.

Definition 1.1. Suppose $\kappa \in \Omega^2_2(N)$. At $p \in N$ we say that the Fresnel surface of κ *decomposes into two Lorentz null cones* if there exists Lorentz metrics g_+ and g_- defined in a neighbourhood of p such that

$$F_p(\kappa) \;=\; N_p(g_+) \,\cup\, N_p(g_-) \tag{14}$$

and $N_p(g_+) \neq N_p(g_-)$.

Let us make two comments regarding this definition. First, we know that two Lorentz metrics are conformally related if and only if their null cones coincide [27]. The condition $N_p(g_+) \neq N_p(g_-)$ thus excludes the case $F_p(\kappa) = N_p(g_\pm)$ which corresponds to a single Lorentz null cone. Second, the typical example of a medium where the Fresnel surface decomposes into two Lorentz null cones is a *uniaxial crystal* like calcite. The characteristic difference between isotropic media and uniaxial crystals is that the latter show *birefringence*, that is, due to the multiple sheets in the Fresnel surface, propagation speed can not only depend on direction of propagation, but also on the polarisation of the wave.

In the next section we will prove Theorem 2.1, which shows that in addition to uniaxial-type media there are two (and only two) additional medium classes where the Fresnel surface decomposes into two Lorentz null cones. We next collect three propositions that we will need in the proof of Theorem 2.1. To prove of these propositions we will need some terminology from algebraic geometry. If $k = \mathbb{R}$ or $k = \mathbb{C}$, we denote by $k[x_1, \ldots, x_n]$ the ring of polynomials $k^n \to k$ in variables $x_1, \ldots, x_n$. Moreover, a non-constant polynomial $f \in k[x_1, \ldots, x_n]$ is *irreducible* if $f = uv$ for $u, v \in k[x_1, \ldots, x_n]$ implies that u or v is a constant. For a polynomial $r \in k[x_1, \ldots, x_n]$, let $V(r) = \{x \in k^n : r(x) = 0\}$ be the *variety* induced by r, and let $\langle r \rangle = \{fr : f \in k[x_1, \ldots, x_n]\}$ be the the *ideal generated by* r. For what follows the necessary theory for manipulating these objects can, for example, be found in [28].

By a result of Montaldi [29], a Lorentz null cone can not contain a vector subspace of dimension ≥ 2. The next proposition generalises this result to Fresnel surfaces that decompose into two Lorentz null cones.

Proposition 1.2. *Suppose $g_\pm$ are Lorentz metrics on a 4-manifold N. If $\Gamma \subset T_p^*N$ is a non-empty vector subspace such that $\Gamma \subset N(g_+) \cup N(g_-)$, then* $\dim \Gamma \leq 1$.

Proof. By [29, Proposition 2]) we may assume that g_+ and g_- are not conformally related. If $\dim \Gamma \geq 2$, we can find linearly independent $u, v \in \Gamma$ such that $\operatorname{span}\{u, v\} \subset N(g_+) \cup N(g_-)$. We may further assume that $u \in N(g_+)$. Let

$$U = \{\theta \in \mathbb{R} : \cos\theta u + \sin\theta v \notin N(g_-)\}.$$

For $w \in T^*N$ let us write $\|w\|^2 = g_+(w, w)$. If U is empty, then $\operatorname{span}\{u, v\} \subset N(g_-)$ and the result follows from the special case. Otherwise there exists a $\theta_0 \in U$ so that $\|\cos\theta u + \sin\theta v\|^2 = 0$ for all θ in some neighbourhood $I_0 \ni \theta_0$. Differentiating gives

$$\frac{1}{2}\left(\|v\|^2 - \|u\|^2\right) \cdot \sin 2\theta + g_+(u, v) \cdot \cos 2\theta = 0, \quad \theta \in I_0.$$

By computing the Wronskian, it follows that $0 = \|u\|^2 = \|v\|^2$ and $g_+(u, v) = 0$. Thus $\operatorname{span}\{u, v\} \subset N(g_+)$, but this contradicts [29, Proposition 2]. □

For two Lorentz metrics g and h we know that their null cones $N(g)$ and $N(h)$ coincide if and only if g and h are conformally related [27, Theorem 3]. The next two propositions give an analogous uniqueness result for Fresnel surfaces that decompose into two Lorentz null cones. Let us emphasize that these results are pointwise. For example, in equation (15) the two sides have different transformation rules.

Proposition 1.3. *Suppose $\kappa \in \Omega_2^2(N)$, and the Fresnel surface of κ decomposes into two Lorentz null cones at $p \in N$. If $\{x^i\}_{i=0}^3$ are coordinates around p, $\mathscr{G}^{ijkl}$ are components for the Tamm-Rubilar tensor density for κ, and $g_\pm = g_{\pm\, ij} dx^i \otimes dx^j$ are as in Definition 1.1, then*

$$\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l \quad = \quad C\ (g_+^{ij}\xi_i\xi_j)\ (g_-^{kl}\xi_k\xi_l) \quad \text{at } p, \quad \{\xi_i\}_{i=0}^3 \in \mathbb{R}^4, \qquad (15)$$

for some $C \in \mathbb{R}\backslash\{0\}$.

Proof. Let $\gamma\colon \mathbb{R}^4 \to \mathbb{R}$ be the polynomial $\gamma(\xi) = \mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$ for $\xi = (\xi_i)_{i=0}^3 \in \mathbb{R}^4$. Then the following holds:

($*$) If $G = (G^{ij})_{i,j=0}^3 \in \mathbb{R}^{4\times 4}$ has Lorentz signature and if

$$\{\xi \in \mathbb{R}^4 : \gamma(\xi) = 0\} \quad \supset \quad N \text{ for } N = \{\xi \in \mathbb{R}^4 : G^{ij}\xi_i\xi_j = 0\} \quad (16)$$

then there exists a quadratic form $H = (H^{ij})_{i,j=0}^3 \in \mathbb{R}^{4\times 4}$ such that $\gamma(\xi) = (G^{ij}\xi_i\xi_j)(H^{kl}\xi_k\xi_l)$ for all $\xi \in \mathbb{R}^4$.

To see this, we may by changing coordinates and scaling assume that inclusion (16) holds for $G = \operatorname{diag}(-1, 1, 1, 1)$. By insisting that inclusion (16) holds for all $\xi \in \{0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}\}^4 \cap N$ we obtain linear constraints on the coefficients in γ. By eliminating variables in γ using these constraints, computer algebra shows that γ has $G^{ij}\xi_i\xi_j$ as a factor and ($*$) follows. For a similar argument in Lorentzian geometry, see [30, Section II.4].

From the above it follows that there are quadratic forms $H_\pm = (H_\pm^{ij})_{i,j=0}^3$ such that

$$\begin{aligned} \gamma(\xi) &= (g_+^{ij}\xi_i\xi_j)(H_+^{kl}\xi_k\xi_l) \\ &= (g_-^{ij}\xi_i\xi_j)(H_-^{kl}\xi_k\xi_l), \quad \xi \in \mathbb{R}^4. \end{aligned}$$

By Proposition A.1 in Appendix A, $g_\pm^{ij}\xi_i\xi_j$ are irreducible over $\mathbb{R}$, and Proposition 1.2 implies that γ is not constant. Hence unique factorisation into irreducible factors [28, Theorem 5 in Section 3.5] implies that $H_+^{ij}\xi_i\xi_j$ is irreducible if and only if $H_-^{ij}\xi_i\xi_j$ is irreducible. If $H_\pm^{ij}\xi_i\xi_j$ are reducible, then $F_p(\kappa)$ has a two dimensional subspace which contradicts Proposition 1.2. Hence $H_\pm^{ij}\xi_i\xi_j$ are irreducible and since $N_p(g_+) \neq N_p(g_-)$, unique factorisation implies that $H_\pm$ are proportional to $g_\mp$ and equation (15) follows. □

Proposition 1.4. *Suppose $\kappa \in \Omega^2_2(N)$ and the Fresnel surface of κ decomposes into two Lorentz null cones at $p \in N$ as in equation* (14)*. Then $g_+|_p$ and $g_-|_p$ are uniquely determined up to scalings and permutations.*

Proof. The result follows by Proposition 1.3, Proposition A.1 in Appendix A, and since any polynomial has a unique decomposition into irreducible factors [28, Theorem 5 in Section 3.5]. □

On $N = \mathbb{R}^4$ let κ be the $\binom{2}{2}$-tensor determined by the 6×6 matrix $(\kappa_I^J)_{IJ} = \operatorname{diag}(-1, 1, 0, -1, 1, 0)$. Then κ is skewon-free and κ has the Fresnel surface

$$F_p(\kappa) \quad = \quad \{\xi \in T^*_p\mathbb{R}^4 : \xi_0\xi_1\xi_2\xi_3 = 0\}, \quad p \in \mathbb{R}^4. \tag{17}$$

It is clear that $F_p(\kappa)$ has multiple decompositions into second order surfaces and for κ there is no unique decomposition as in Proposition 1.4.

1.4. The normal form theorem for skewon-free medium

In arbitrary coordinates, a general skewon-free medium depends on 21 free components. Next we state the normal form theorem by Schuller, Witte and Wohlfarth [10] in a form sufficient for the present setting. This normal form theorem shows that by choosing the coordinates suitably, the number of free components at a point $p \in N$ can be reduced considerably. For our application in Theorem 2.1, we may assume that κ is invertible and the Fresnel surface contains no 2-dimensional subspace. This will imply that there are only 7 possible normal forms and each normal form depends only on 2, 4 or 6 free parameters. This reduction of parameters will make the computer algebra feasible in the proof of Theorem 2.1.

To formulate the theorem let us first introduce some terminology. Suppose $L\colon V \to V$ is a linear map where V is an n-dimensional real vector space. If the matrix representation of L in some basis is $A \in \mathbb{R}^{n\times n}$ and A is written using the Jordan normal form we say that L has *Segre type* $\left[m_1 \cdots m_r \, k_1\overline{k_1} \cdots k_s\overline{k_s}\right]$ when the blocks corresponding to real eigenvalues have dimensions $m_1 \le \cdots \le m_r$ and the blocks corresponding to complex eigenvalues have dimensions $2k_1 \le \cdots \le 2k_s$. Moreover, by uniqueness of the Jordan normal form, the Segre type depends only on L and not on the basis. For $\kappa \in \Omega^2_2(N)$ and $p \in N$ we define the Segre type of $\kappa|_p$ as the Segre type of the linear map $\Omega^2(N)|_p \to \Omega^2(N)|_p$. By counting how many ways a 6×6 matrix can be decomposed into Jordan normal forms, it follows that there are only 23 Segre types for a $\binom{2}{2}$-tensor. The main result of the normal form theorem in [10] is to establish simple coordinate expressions for each of these Segre types, or *metaclasses*, under the assumption that $\kappa|_p$ is skewon-free and invertible.

The below formulation of the normal form theorem in [10] is based on the restatement in [15] but specialised to the setting of an orientable manifold. The argument for excluding Metaclasses VIII–XXIII is Lemma 5.1 in [10].

Theorem 1.5. *Suppose $\kappa \in \Omega^2_2(N)$ and $p \in N$. If*

(a) *κ has no skewon part at p,*
(b) *κ is invertible at p,*
(c) *the Fresnel surface $F_p(\kappa)$ does not contain a two dimensional vector subspace,*

then there exists coordinates $\{x^i\}_{i=0}^3$ around p such that the 6×6 matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is one of the below matrices:

- *Metaclass I: (Segre type $[1\overline{1}\,1\overline{1}\,1\overline{1}]$)*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -\beta_3 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{18}$$

- *Metaclass II: (Segre type $[2\overline{2}\,1\overline{1}]$)*

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & -\beta_2 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & \alpha_2 \end{pmatrix} \tag{19}$$

- *Metaclass III: (Segre type $[3\overline{3}]$)*

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha_1 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & 1 \\ 0 & 0 & 1 & -\beta_1 & \alpha_1 & 0 \\ 0 & 1 & \beta_1 & 0 & 0 & \alpha_1 \end{pmatrix} \tag{20}$$

- *Metaclass IV:* *(Segre type* $[11\,1\bar{1}\,1\bar{1}]$*)*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{21}$$

- *Metaclass V:* *(Segre type* $[11\,2\bar{2}]$*)*

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_2 \end{pmatrix} \tag{22}$$

- *Metaclass VI:* *(Segre type* $[11\,11\,1\bar{1}]$*)*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_5 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_5 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{23}$$

- *Metaclass VII:* *(Segre type* $[11\,11\,11]$*)*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_6 \\ \alpha_4 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_6 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{24}$$

In each matrix the parameters satisfy $\alpha_1, \alpha_2, \ldots \in \mathbb{R}$*,* $\beta_1, \beta_2, \ldots \in \mathbb{R}\backslash\{0\}$ *and* $\operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2 = \cdots$.

Proof. Let us first note that $\kappa|_p$ can not have a Jordan block of dimension $d \in \{2, \ldots, 6\}$ that corresponds to a real eigenvalue $\lambda \in \mathbb{R}\backslash\{0\}$. In the setting of area metrics, this is Lemma 5.1 in [10]. However, let us also outline a direct proof. For a contradiction, suppose $\kappa|_p$ has such a block. By considering unit vectors in the normal basis, we can find non-zero $e_1, e_2 \in \Lambda^2(N)|_p$ so that $\kappa(e_1) = \lambda e_1$ and $\kappa(e_2) = \lambda e_2 + e_1$. Writing out $\kappa(e_1) \wedge e_2 = e_1 \wedge \kappa(e_2)$ implies that $e_1 \wedge e_1 = 0$, so $e_1 = \eta_1 \wedge \eta_2$ for some linearly independent $\eta_1, \eta_2 \in \Lambda^1(N)|_p$. If $W = \operatorname{span}\{\eta_1, \eta_2\}$, then

$$W \quad \subset \quad \{\alpha \in \Lambda^1(N)|_p : \xi \wedge \kappa(\xi \wedge \alpha) = 0\}$$

for all $\xi \in W$ whence Theorem 3.3 in [5] gives the contradiction $W \subset F_p(\kappa)$. In the terminology of [10], it follows that $\kappa|_p$ must be in one of Metaclasses I–VII. By [15] there are coordinates $\{x^i\}_{i=0}^3$ around p (that are not necessarily compatible with the orientation for N) such that the 6×6-matrix $(\kappa_I^J)_{IJ}$ that represents κ at p is given by one of matrices (18)–(24) for some $\alpha_1, \alpha_2, \ldots \in \mathbb{R}$ and $\beta_1, \beta_2, \ldots > 0$. If coordinates x^i are compatible with the orientation of N the claim follows. Let us therefore assume that coordinates x^i are not compatible with the orientation of N. For each metaclass, we then need to find a coordinate transformation $x^i \mapsto \widetilde{x}^i$ such that *(i)* the transformation is orientation reversing and *(ii)* the transformation preserves the structure of the matrix representation for the metaclass (while possibly exchanging the signs of all β_i). If $\{\widetilde{x}^i\}_{i=0}^3$ are coordinates defined as $\widetilde{x}^i = J^i_j x^j$ for a 4×4 matrix $J = (J^i_j)_{ij}$, then suitable transformation matrices are

Metaclass	I	II	III	IV	V	VI	VII
Transformation matrix J	J_1	J_2	J_2	J_1	J_2	J_1	J_1

where $J_1 = \operatorname{diag}(-1, 1, 1, 1)$ and $J_2 = \operatorname{diag}(1, 1, -1, 1)$. □

2. Non-dissipative medium tensors with two Lorentz null cones

The next theorem is the main result of this paper.

Theorem 2.1. *Suppose $\kappa \in \Omega^2_2(N)$ and suppose that at some $p \in N$*

(i) $\kappa|_p$ *has no skewon component,*

(ii) $\kappa|_p$ *is invertible as a linear map* $\Lambda^2(N)|_p \to \Lambda^2(N)|_p$,

(iii) the Fresnel surface $F_p(\kappa)$ *decomposes into two Lorentz null cones (see Definition 1.1).*

Then $\kappa|_p$ *must have Segre type* $[1\overline{1}\, 1\overline{1}\, 1\overline{1}]$, $[2\overline{2}\, 1\overline{1}]$ *or* $[1 1\, 1\overline{1}\, 1\overline{1}]$.

I. ***Metaclass I:*** *If $\kappa|_p$ has Segre type $[1\bar{1}\,1\bar{1}\,1\bar{1}]$, there are coordinates $\{x^i\}_{i=0}^3$ around p such that the matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is given by equation (18) for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{R}\backslash\{0\}$ with*

$$\alpha_2 = \alpha_3, \quad \beta_2 = \beta_3, \quad \operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2 = \operatorname{sgn}\beta_3$$

and either $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ or both inequalities hold. Moreover, equation (14) holds for Lorentz metrics

$$g_\pm = \operatorname{diag}\left(1, -1, \frac{1}{2}\left(-D_3 \pm \sqrt{D_3^2 - 4}\right), \frac{1}{2}\left(-D_3 \pm \sqrt{D_3^2 - 4}\right)\right)^{-1},$$

where $D_3 > 2$ is defined by

$$D_3 = \frac{(\alpha_1 - \alpha_2)^2 + \beta_1^2 + \beta_2^2}{\beta_1 \beta_2}. \tag{25}$$

II. ***Metaclass II:*** *If $\kappa|_p$ has Segre type $[2\bar{2}\,1\bar{1}]$, there are coordinates $\{x^i\}_{i=0}^3$ around p such that the matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is given by equation (19) for some $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}\backslash\{0\}$ with*

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2.$$

Moreover, equation (14) holds for Lorentz metrics

$$g_\pm = \begin{pmatrix} \pm 1 & 0 & 0 & \beta_1 \\ 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}^{-1}.$$

IV. ***Metaclass IV:*** *If $\kappa|_p$ has Segre type $[11\,1\bar{1}\,1\bar{1}]$, there are coordinates $\{x^i\}_{i=0}^3$ around p such that the matrix $(\kappa_I^J)_{IJ}$ that represents $\kappa|_p$ in these coordinates is given by equation (21) for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}\backslash\{0\}$ with*

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \alpha_4 \neq 0, \quad \alpha_3^2 \neq \alpha_4^2.$$

Moreover, equation (14) holds for Lorentz metrics

$$g_\pm = \operatorname{diag}\left(1, \frac{1}{2}\left(-D_1 \pm \sqrt{D_1^2 + 4}\right), \frac{1}{2}\left(-D_1 \pm \sqrt{D_1^2 + 4}\right), -1\right)^{-1},$$

where $D_1 \in \mathbb{R}$ is defined by

$$D_1 = \frac{(\alpha_2 - \alpha_3)^2 + \beta_2^2 - \alpha_4^2}{\beta_2 \alpha_4}. \tag{26}$$

Conversely, if $\kappa|_p$ is defined by equation (18), (19) *or* (21) *and α_i and β_i satisfy conditions listed in cases I., II. or IV., respectively, then $\kappa|_p$ satisfy properties (i)–(iii).*

For an outline of the proof, see the introduction. The main mathematical technique in the proof is to use the computer algebra technique of *Gröbner basis* to eliminate variables from systems of polynomial equations. In more detail, suppose $V \subset \mathbb{C}^n$ is the solution set to polynomial equations $f_1 = 0, \dots, f_N = 0$ where $f_i \in \mathbb{C}[x_1, \dots, x_n]$. If I is the ideal generated by $f_1, \dots, f_N$, the *elimination ideals* are the polynomial ideals defined as

$$I_k \quad = \quad I \cap \mathbb{C}[x_{k+1}, \dots, x_n], \quad k \in \{0, \dots, n-1\}.$$

Thus, if $(x_1, \dots, x_n) \in V$ then $p(x_{k+1}, \dots, x_n) = 0$ for any $p \in I_k$, and I_k contain polynomial consequences of the original equations that only depend on variables $x_{k+1}, \dots, x_n$. Using Gröbner basis, one can explicitly compute I_k [28, Theorem 2 in Section 3.1]. In the below proof, the computation of Gröbner bases has done using the built-in routine GroebnerBasis in Mathematica.

Proof. In the below proof we will use the following index notation. If $i \in \{1, 2, 3\}$, then i', i'' are indices in $\{1, 2, 3\}$ such that $\{i, i', i''\} = \{1, 2, 3\}$ and $i' < i''$. For example, if $i = 1$, then $i'' = 3$ and if $i = 3$, then $i'' = 2$. Suppose *(i)–(iii)* holds. Then Theorem 1.5 implies that $\kappa|_p$ is of Metaclass I–VII, and the proof divides into seven cases. In each case, there are coordinates $\{x^i\}_{i=0}^3$ such that $(\kappa_I^J)_{IJ}$ at p is given by equation (18)–(24) for constants α_i, β_i as in Theorem 1.5.

Metaclass I. We may exclude the possibility $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2 = \beta_3$ since that corresponds to a single Lorentz null cone. Let D_1, D_2 be defined as

$$D_1 \quad = \quad \frac{(\alpha_2 - \alpha_3)^2 + \beta_2^2 + \beta_3^2}{\beta_2 \beta_3}, \tag{27}$$

$$D_2 \quad = \quad \frac{(\alpha_3 - \alpha_1)^2 + \beta_3^2 + \beta_1^2}{\beta_3 \beta_1}, \tag{28}$$

and let D_3 be defined as in equation (25). Then $D_1, D_2, D_3 \geq 2$, and by applying an orientation preserving coordinate change, we may assume that $2 \leq D_1 \leq D_2 \leq D_3$. Since $D_1 = D_2 = D_3 = 2$ implies that $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2 = \beta_3$, it follows that $D_3 > 2$. Furthermore, since $D_1 = D_2 = 2$ implies that $D_3 = 2$ we also have $D_2 > 2$.

The Tamm-Rubilar tensor density for $\kappa|_p$ satisfies

$$C^{-1}\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \xi_0^4+\xi_1^4+\xi_2^4+\xi_3^4-D_0\xi_0\xi_1\xi_2\xi_3 + \sum_{i=1}^{3} D_i\left(\xi_{i'}^2\xi_{i''}^2-\xi_0^2\xi_i^2\right), \qquad (29)$$

where $C=\beta_1\beta_2\beta_3$ and D_0 is given explicitly in terms of $\alpha_1,\ldots,\beta_3$, and implicitly D_0 satisfies

$$D_0^2 = 4\left(4+D_1D_2D_3-D_1^2-D_2^2-D_3^2\right). \qquad (30)$$

By Proposition 1.3, there are real symmetric matrices $A=(A^{ij})_{i,j=0}^3$ and $B=(B^{ij})_{i,j=0}^3$ such that

$$C^{-1}\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \left(\xi^t\cdot A\cdot\xi\right)\ \left(\xi^t\cdot B\cdot\xi\right), \quad \xi\in\mathbb{R}^4. \qquad (31)$$

Writing out these equations shows that $A^{00}B^{00}=1$. Hence A^{00} is non-zero, and by rescaling A and B, we may assume that $A^{00}=1$. This substitution simplifies the equations so that by polynomial substitutions we may eliminate all variables in B and variable D_0. This results in a system of polynomial equations that only involve D_1,D_2,D_3 and the variables in A. By eliminating the variables in A using a Gröbner basis, we obtain constraints on D_1,D_2,D_3 and these constraints imply that $D_1=2$ and $D_2=D_3$, whence equation (30) implies that $D_0=0$. The result follows since equation (31) holds with $A=g_+^{-1}$ and $B=g_-^{-1}$, where $g_\pm$ are the matrices in the theorem formulation.

Metaclass II. As in Metaclass I, there are matrices A and B such that the Fresnel polynomial satisfies equation (31) (with $C=1$). As in Metaclass I we can eliminate variables in B. Further eliminating all variables in A using a Gröbner basis implies that $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$. Then a direct computation shows that equation (31) holds with $A=g_+^{-1}$, $B=g_-^{-1}$ and $C=\beta_1$. Metrics $g_\pm$ both have Lorentz signatures since $\det g_\pm<0$.

Metaclass III. Decomposing the Fresnel polynomial as in equation (31) (with $C=1$) gives a system of polynomial equations for the variables in A, B and $\kappa|_p$. Computing the Gröbner basis for these equations implies that $\beta_1=0$. Thus $\kappa|_p$ can not be in Metaclass III.

Metaclass IV. Let us first note that $\alpha_4\neq 0$ since otherwise we have $\operatorname{span}\{dx^1|_p, dx^2|_p\}\subset F_p(\kappa)$ which is not possible by Proposition 1.2. Then the Tamm-Rubilar tensor density satisfies

$$C^{-1}\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \xi_0^4 - \xi_1^4 - \xi_2^4 + \xi_3^4 + D_0\xi_0\xi_1\xi_2\xi_3 - \sum_{i=1}^{3} D_i\xi_0^2\xi_i^2 + D_1\xi_2^2\xi_3^2 + D_2\xi_1^2\xi_3^2 - D_3\xi_1^2\xi_2^2,$$

where $C = \beta_1\beta_2\alpha_4$, D_0 is determined explicitly in terms of $\alpha_1, \ldots, \beta_2$, D_1 is defined in equation (26), $D_3 \geq 2$ is defined in equation (25) and

$$D_2 = \frac{(\alpha_1 - \alpha_3)^2 + \beta_1^2 - \alpha_4^2}{\beta_1\alpha_4}.$$

By decomposing and eliminating variables as in Metaclass I, it follows that that $D_0 = 0$ and $D_3 = 2$. Thus we have proven that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ whence $D_1 = D_2$ and equation (31) holds with $A = g_+^{-1}$, $B = g_-^{-1}$ and C as above. Moreover, since $\det g_\pm < 0$, metrics $g_\pm$ both have Lorentz signatures. Condition $\alpha_3^2 \neq \alpha_4^2$ follows since $\det \kappa|_p \neq 0$.

Metaclass V. We may assume that $\alpha_3 \neq 0$, since otherwise we have $\operatorname{span}\{dx^i|_p\}_{i=1}^3 \subset F_p(\kappa)$. Decomposing and eliminating variables as in Metaclass I gives that β_1 is purely complex and $\kappa|_p$ can not be in Metaclass V.

Metaclass VI. We may assume that α_4 and α_5 are non-zero since otherwise $\operatorname{span}\{dx^i, dx^3\} \subset F_p(N)$ for some $i \in \{0, 1\}$ as in Metaclass IV. Then the Tamm-Rubilar tensor density satisfies

$$C^{-1}\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \xi_0^4 + \xi_1^4 - \xi_2^4 - \xi_3^4 + D_0\xi_0\xi_1\xi_2\xi_3 - \sum_{i=1}^{3} D_i\xi_0^2\xi_i^2 + D_1\xi_2^2\xi_3^2 - D_2\xi_1^2\xi_3^2 - D_3\xi_1^2\xi_2^2,$$

where $C = \beta_1\alpha_4\alpha_5$ and $D_0, D_1, D_2, D_3 \in \mathbb{R}$ are defined in terms of α_i and β_1. By decomposing the Fresnel tensor as in equation (31) and eliminating variables using a Gröbner basis, it follows that there exists a $\sigma \in \{\pm 1\}$ such that

$$D_0 = 0, \quad D_1 = \sigma 2, \quad D_2 = -\sigma D_3,$$

and moreover, equation (31) holds for $A = g_+^{-1}$, $B = g_-^{-1}$ and C as above, where

$$g_\pm = \operatorname{diag}\left(1, -\sigma, \frac{1}{2}\left(\sigma D_3 \pm \sqrt{D_3^2 + 4}\right), \frac{1}{2}\left(-D_3 \mp \sigma\sqrt{D_3^2 + 4}\right)\right)^{-1}.$$

These metrics satisfy $\det g_\pm > 0$ for $\sigma = 1$ and $\sigma = -1$, so the metrics are not of Lorentz signature. Since this contradicts Proposition 1.3 and unique factorisation, it follows that $\kappa|_p$ can not be in Metaclass VI.

Metaclass VII. We may assume that $\alpha_4, \alpha_5, \alpha_6 \neq 0$ since otherwise $\operatorname{span}\{dx^i|_p, dx^3|_p\} \subset F_p(\kappa)$ for some $i \in \{0,1,2\}$ as in Metaclass IV. Then the Tamm-Rubilar tensor density satisfies

$$C^{-1}\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \xi_0^4 + \xi_1^4 + \xi_2^4 + \xi_3^4 + D_0\xi_0\xi_1\xi_2\xi_3 - \sum_{i=1}^{3} D_i(\xi_{i'}^2\xi_{i''}^2 + \xi_0^2\xi_i^2),$$

where $C = \alpha_4\alpha_5\alpha_6$, constants $D_1, D_2, D_3 \in \mathbb{R}$ are given by

$$D_1 = \frac{(\alpha_2-\alpha_3)^2 - \alpha_5^2 - \alpha_6^2}{\alpha_5\alpha_6}, \tag{32}$$

$$D_2 = \frac{(\alpha_1-\alpha_3)^2 - \alpha_4^2 - \alpha_6^2}{\alpha_4\alpha_6}, \tag{33}$$

$$D_3 = \frac{(\alpha_1-\alpha_2)^2 - \alpha_4^2 - \alpha_5^2}{\alpha_4\alpha_5}, \tag{34}$$

and $D_0 \in \mathbb{R}$ is given explicitly in terms of $\alpha_1, \ldots, \beta_3$, and implicitly D_0 satisfies

$$D_0^2 = 4\left(-4 + D_1D_2D_3 + D_1^2 + D_2^2 + D_3^2\right). \tag{35}$$

Decomposing the Tamm-Rubilar tensor density as in equation (31) and eliminating variables using a Gröbner basis, gives polynomial equations for D_0, D_1, D_2, D_3. Let us consider the cases $D_0 = 0$ and $D_0 \neq 0$ separately. If $D_0 = 0$, there exists an $i \in \{1,2,3\}$ and a $\sigma \in \{\pm 1\}$ such that

$$D_0 = 0, \quad D_i = -\sigma 2, \quad D_{i'} = \sigma D_{i''}, \tag{36}$$

where the last condition is a consequence of equation (35). Suppose $i = 1$. Then Proposition 1.3 implies that for some invertible symmetric matrices $A, B \in \mathbb{R}^{4\times 4}$ with Lorentz signatures we have

$$\left(\xi^t \cdot A \cdot \xi\right)\left(\xi^t \cdot B \cdot \xi\right) = \left(\xi^t \cdot L_+ \cdot \xi\right)\left(\xi^t \cdot L_- \cdot \xi\right), \quad \xi \in \mathbb{C}^4, \tag{37}$$

where matrices $L_\pm \in \mathbb{C}^{4\times 4}$ are defined as

$$L_\pm = \operatorname{diag}\left(1, \sigma, \frac{1}{2}\left(-D_2 \pm \sqrt{D_2^2-4}\right), \frac{\sigma}{2}\left(-D_2 \pm \sqrt{D_2^2-4}\right)\right) \tag{38}$$

Since $L_\pm$ are invertible, equation (37), Proposition A.1 in Appendix A and unique factorisation imply that $L_\pm$ are real and have Lorentz signatures. Thus $\det L_\pm < 0$, but this contradicts equation (38), which implies that

$$\det L_\pm = \frac{1}{4}\left(-D_2 \pm \sqrt{D_2^2 - 4}\right)^2 \geq 0.$$

A similar analysis for $i = 2, 3$ shows that $D_0 = 0$ is not possible. If $D_0 \neq 0$ it follows that there exists $\sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}$ and distinct $i, j, k \in \{1, 2, 3\}$ such that

$$D_0 \neq 0, \quad D_i = \sigma_1 2, \quad D_j = \sigma_2 2, \quad D_k = \frac{1}{2}\left(-4\sigma_1\sigma_2 + \sigma_3 D_0\right), \tag{39}$$

where the last equation follows from equation (35). If $(i, j) = (1, 2)$ then $k = 3$ and Proposition 1.3 implies that for some invertible symmetric matrices $A, B \in \mathbb{R}^{4\times 4}$ with Lorentz signatures, equation (37) holds for matrices $L_\pm \in \mathbb{C}^{4\times 4}$ defined as

$$L_\pm = \begin{pmatrix} 1 & 0 & 0 & \pm\frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} \\ 0 & -\sigma_1 & \mp\frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & 0 \\ 0 & \mp\frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & -\sigma_2 & 0 \\ \pm\frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & 0 & 0 & \sigma_1\sigma_2 \end{pmatrix}. \tag{40}$$

Since both sides in equation (37) should decompose into the same number of irreducible factors, it follows that $\xi^t \cdot L_\pm \cdot \xi$ are irreducible in $\mathbb{C}[\xi_0, \ldots, \xi_3]$. Thus equation (37) and unique factorisation imply that $L_\pm$ are real and have Lorentz signatures, so $\det L_\pm < 0$. However, this contradicts equation (40) which implies that

$$\det L_\pm = \left(\frac{1}{8}D_0 - \sigma_1\sigma_2\sigma_3\right)^2 \geq 0.$$

The cases $(i, j) = (1, 3)$, $(2, 3)$ are excluded by the same argument by using metrics

$$L_\pm = \begin{pmatrix} 1 & 0 & \pm\frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & 0 \\ 0 & -\sigma_1 & 0 & \mp\frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} \\ \pm\frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & 0 & \sigma_1\sigma_2 & 0 \\ 0 & \mp\frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & 0 & -\sigma_2 \end{pmatrix},$$

$$L_\pm = \begin{pmatrix} 1 & \pm\frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & 0 & 0 \\ \pm\frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & \sigma_1\sigma_2 & 0 & 0 \\ 0 & 0 & -\sigma_1 & \mp\frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} \\ 0 & 0 & \mp\frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & -\sigma_2 \end{pmatrix},$$

respectively. Thus $\kappa|_p$ can not be in Metaclasses VII.

The converse direction follows by computer algebra. For example, if $\kappa|_p$ is defined as in equation (21) we have $\det \kappa|_p = (\alpha_3^2 - \alpha_4^2)(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)$. □

3. Analysis of the three medium classes

In this section we give a more detailed analysis of the three medium classes in Theorem 2.1. Let us first make four comments. First, it should be emphasised that in Theorem 2.1, κ is assumed to be real. For complex coefficients in κ, the setting becomes more involved. A *chiral medium* would be an example of a physical electromagnetic medium, where the medium tensor contains complex coefficients and for time harmonic fields, propagation is determined by two Lorentz null cones. In such a medium, right and left hand circularly polarised plane waves propagate with different wavespeeds [13, 31]. However, (as a complex medium) it is not listed as a possibility in Theorem 2.1. Second, the assumption that κ is invertible is only used to show that $\alpha_3^2 \neq \alpha_4^2$ in Metaclass IV and to exclude Metaclasses VIII—XXIII. Third, from a mathematical point of view it is interesting to note that if we set $\xi_0 = 0$ in equation (29) we obtain the ternary quartic studied in [32] and for this polynomial, D_0^2 in equation (30) is one of the factors in the discriminant.

3.1. Relative configuration of the two Lorentz null cones

In Theorem 2.1 each of the three medium classes is parameterised by 2 or 4 scalar parameters in the normal form coordinates, but the configuration of the Lorentz null cones is parameterised by only one parameter; $D_3 > 2$ in Metaclass I, $\beta_1 \in \mathbb{R}\backslash\{0\}$ in Metaclass II and $D_1 \in \mathbb{R}$ in Metaclass IV. Here D_3 is defined by equation (25) and D_1 is defined by equation (26). Next we consider the relative configuration of the two Lorentz null cones for each metaclass in Theorem 2.1.

Metaclass I.

Let us first note that by equation (25) we have $D_3 \geq 2$. Thus inequalities $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$ exclude the possibility $D_3 = 2$, which would imply that $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2 = \beta_3$. Then $\kappa|_p = -\beta_1 *_g +\alpha_1 \,\mathrm{Id}$ for the locally defined Lorentz metric $g = \mathrm{diag}(-1, 1, 1, 1)$, and $\kappa|_p$ models isotropic medium with a possible axion component. For this medium the Fresnel surface is the single Lorentz null cone $F_p(\kappa) = N_p(g)$.

If we treat x^0 as time and define $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ as in equation (9), then the constitutive equation at p reads

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \left(\begin{array}{ccc|ccc} -\alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & -\alpha_2 & 0 & 0 & -\beta_2 \\ \hline -\beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & -\beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & -\beta_2 & 0 & 0 & \alpha_2 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \qquad (41)$$

where we have incorporated the minus sign in equation (9) into the coefficient matrix. Here the parameters satisfy $\alpha_1, \alpha_2 \in \mathbb{R}$, $\beta_1, \beta_2 \in \mathbb{R}\backslash\{0\}$, $\mathrm{sgn}\,\beta_1 = \mathrm{sgn}\,\beta_2$ and [$\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ or both inequalities hold].

This medium has permittivity matrix $\varepsilon = -\,\mathrm{diag}(\beta_1, \beta_2, \beta_2)$, permeability matrix $\mu = -\,\mathrm{diag}(1/\beta_1, 1/\beta_2, 1/\beta_2)$, and a possible magnetic-electric effect modelled by matrices $\pm\,\mathrm{diag}(\alpha_1, \alpha_2, \alpha_2)$. When $\alpha_1 = \alpha_2 = 0$ and $\beta_1, \beta_2 < 0$ the medium is usual *uniaxial medium*, where wave propagation is well understood [33, Section 15.3]. However, this medium class also allows $\beta_1, \beta_2 > 0$ whence the medium has negative permittivity and permeability. For $\alpha_1 + 2\alpha_2 \neq 0$ the medium has an axion component.

For the medium in equation (41), the Fresnel polynomial $\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$ is a function of $\xi_0, \xi_1, \xi_2^2+\xi_3^2$. It is therefore motivated to set $\xi_2 = 0$ whence we can plot $F_p(\kappa)$ as a surface in $\mathbb{R}^3$. Figure 1 shows the relative configuration of the two Lorentz null cones in $F_p(\kappa)$ for different values of D_3, where D_3 is as in equation (25).

If metrics $g_\pm$ are as in Theorem 2.1, so that equation (14) holds then writing out $g_+(\xi,\xi) \pm g_-(\xi,\xi) = 0$ implies that

$$N_p(g_+) \cap N_p(g_-) = \{\xi_i dx^i|_p : \xi_0 = \pm\xi_1,\ \xi_2 = \xi_3 = 0\}.$$

Thus the intersection of the null cones $N(g_\pm)$ for Metaclass I is two distinct lines through the origin in $T^*_p N$. When ξ_0 is time, it follows that the two sheets of the Fresnel surface coincide on the ξ_1-axis. See Figure 1. Such an axis is also known as an *optical axis* (motivating the name uniaxial medium).

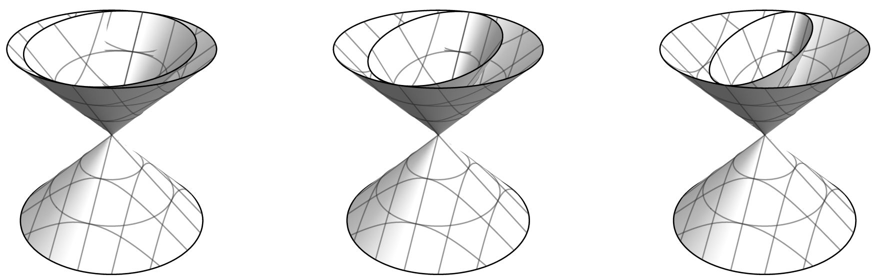

Figure 1: Relative configuration for the two Lorentz null cones in Metaclass I for $D_3 = 2.05$ (left), $D_3 = 2.3$ and $D_3 = 3$ (right). The Lorentz null cones coincide when $D_3 = 2$.

Metaclass II.

Metaclass II in Theorem 2.1 is a subclass of the medium defined by equation (19). To understand the relative configuration of $N_p(g_\pm)$ in $F_p(\kappa)$ let us first note that if we flip the sign of β_1, then $N_p(g_+)$ and $N_p(g_+)$ exchange places. Let us therefore assume that $\beta_1 > 0$. Since the Fresnel surface does not depend on the axion component [2, equation (D.2.39)], let us assume that $\operatorname{trace}\kappa = 0$, whence $\alpha_1 = 0$.

Instead of studying the medium in the coordinates given by the normal form theorem [10], it will be useful to study the medium in coordinates $\{\widetilde{x}^i\}_{i=0}^3$ where $\widetilde{x}^i = \sum_{j=0}^3 L_{ij} x^j$ and $L = (L_{ij})$ is the Jacobian matrix

$$L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \frac{1}{2\beta_1}(1-w) & \frac{1}{2\beta_1}(1+w) \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1}, \quad (42)$$

for

$$w = \sqrt{1+4\beta_1^2}. \quad (43)$$

(The motivation for these coordinates is that the latter matrix diagonalizes g_+ and the former is a matrix that permutates coordinates and diagonalises the permittivity and permeability matrices in a plane where $g_\pm$ are

Euclidean.) Since $\det L > 0$, it follows that x^i and $\widetilde{x}^i$ have the same orientation. If we treat $\widetilde{x}_0$ as time as in equation (41) then in these coordinates $\kappa|_p$ represents the constitutive equation

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \frac{\beta_1}{w} \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -\frac{w^2}{\beta_1} & 0 & 0 \\ 0 & 0 & 2 & 0 & -w-2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -w \\ \hline -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w & 0 & 0 & 0 & 0 \\ 0 & 0 & -w+2 & 0 & -2 & 0 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix},$$

where $\beta_1 > 0$ and w is as in equation (43). In $\widetilde{x}^i$-coordinates, the medium always has a magneto-electric effect, and the permittivity and permeability 3×3 matrices are both diagonal. Moreover, the permeability matrix is always negative definite, and the permittivity matrix is negative definite for $w > 2$, singular for $w = 2$ and indefinite for $w \in (0, 2)$.

Computing the Fresnel polynomial $\widetilde{\mathscr{G}}^{ijkl}\widetilde{\xi}_i\widetilde{\xi}_j\widetilde{\xi}_k\widetilde{\xi}_l$ shows that it is a function of $\widetilde{\xi}_0, \widetilde{\xi}_1, \widetilde{\xi}_2^2 + \widetilde{\xi}_3^2$. As in Metaclass I, we project $F_p(\kappa)$ onto $\widetilde{\xi}_2 = 0$, and we can plot $F_p(\kappa)$ as a surface in $\mathbb{R}^3$. Figure 2 shows this projection for three different values of β_1. From these illustrations (or from metrics $g_\pm$ directly) we see that the null cones $N(g_\pm)$ coincide in the limit $\beta_1 \to \infty$. A peculiar property of this medium is that it can have one, two, or three phase velocities in the same direction. This is illustrated in Figure 3, which shows the behaviour of the inverse phase velocity in one spatial plane.

If metrics $g_\pm$ are as in Theorem 2.1, so that equation (14) holds then

$$N_p(g_+) \cap N_p(g_-) = \{\widetilde{\xi}_i dx^i|_p : \widetilde{\xi}_0 = \widetilde{\xi}_1,\ \widetilde{\xi}_2 = \widetilde{\xi}_3 = 0\}.$$

Thus the intersection of the null cones $N(g_\pm)$ for Metaclass II is one line through the origin in T_p^*N. When $\widetilde{\xi}_0$ is time, it follows that the two sheets of the Fresnel surface coincide on the $+\widetilde{\xi}_1$-direction, but not in the opposite spatial direction. See Figures 2 and 3.

Metaclass IV

Metaclass IV in Theorem 2.1 is a subclass of the medium defined by equation (21). Treating x^0 as time and writing the constitutive equation as

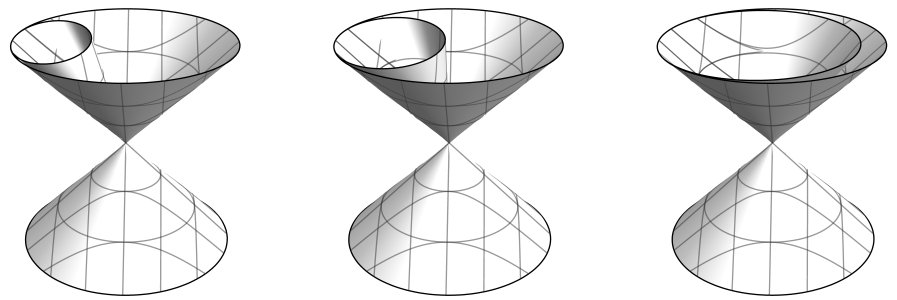

Figure 2: Relative configuration for the two Lorentz null cones in Metaclass II for $\beta_1 = 0.2$ (left), $\beta_1 = 0.8$ and $\beta_1 = 8$ (right).

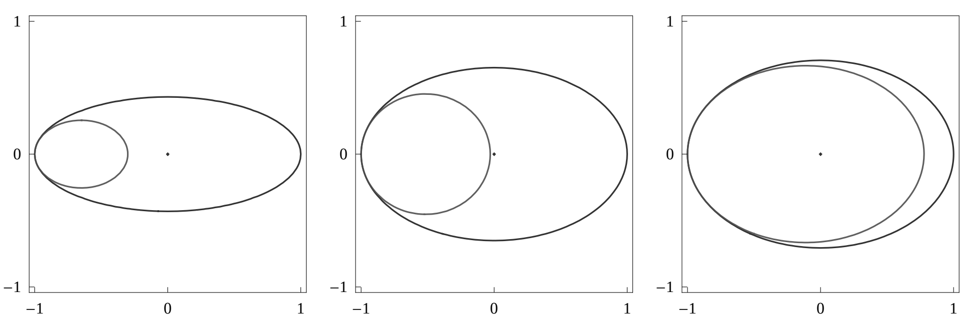

Figure 3: Inverse phase velocity in the $\xi_1\xi_2$-plane for the media in Figure 2.

in equation (41) gives the constitutive equation

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = \left(\begin{array}{ccc|ccc} -\alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & -\alpha_1 & 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & -\alpha_3 & 0 & 0 & \alpha_4 \\ \hline -\beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & -\beta_1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & -\alpha_4 & 0 & 0 & \alpha_3 \end{array}\right) \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}.$$

where $\alpha_1, \alpha_3 \in \mathbb{R}$, $\beta_1, \alpha_4 \in \mathbb{R}\backslash\{0\}$ and $\alpha_3^2 \neq \alpha_4^2$. In these coordinates, the medium has a magneto-electric effect unless $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and an axion component unless $2\alpha_1 + \alpha_3 = 0$. The permittivity and permeability 3×3 matrices are always invertible, and one is always indefinite.

The Fresnel polynomial is a function of $\xi_0, \xi_1^2 + \xi_2^2, \xi_3$. We may therefore plot the Fresnel surface in the same way as in Metaclass I. This is illustrated in Figure 4 for three different values of $D_1 \in \mathbb{R}$, where D_1 is given by equation (26).

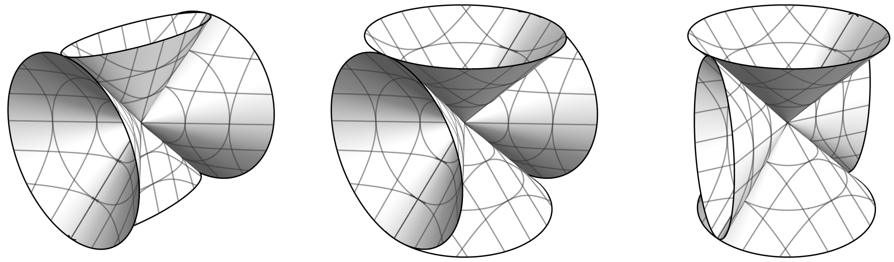

Figure 4: Relative configuration for Lorentz null cones in Metaclass IV for $D_1 = -3$ (left), $D_1 = 0$ and $D_1 = 3$ (right).

If metrics $g_\pm$ are as in Theorem 2.1, so that equation (14) holds then

$$N_p(g_+) \cap N_p(g_-) = \{\xi_i dx^i|_p : \xi_0 = \pm\xi_3,\ \xi_1 = \xi_2 = 0\}.$$

As in Metaclass I, the intersection of the null cones $N(g_\pm)$ for Metaclass IV is two distinct lines through the origin in T_p^*N. When ξ_0 is time, the two sheets of the Fresnel surface coincide on the ξ_3-axis. See Figure 4.

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Appendix A. Irreducible quadratic forms

The next proposition characterises when a quadratic form is irreducible as a complex polynomial. This technical result was used in Section 1.3 and in the proof of Theorem 2.1. In the proposition, adj Q is the *adjugate matrix* of all cofactor expansions of Q, and ξ^t is the matrix transpose.

Proposition A.1. *Suppose $Q \in \mathbb{C}^{4\times 4}$ is a symmetric non-zero matrix and $f \in \mathbb{C}[\xi_0, \ldots, \xi_3]$ is the polynomial $f(\xi) = \xi^t \cdot Q \cdot \xi$ for $\xi = (\xi_0, \ldots, \xi_3) \in \mathbb{C}^4$.*

Then f is irreducible in $\mathbb{C}[\xi_0, \ldots, \xi_3]$ if and only if $\operatorname{adj} Q \neq 0$.

Proof. If f is not irreducible, then $f = uv$ for first order polynomials $u, v \in \mathbb{C}[\xi_0, \ldots, \xi_3]$. Then u and v are linear, that is, $u(0) = v(0) = 0$. (To see this, we know that $f(0) = 0$, so we may assume that $u(0) = 0$. For a contradiction, suppose that $v(0) \neq 0$. Then $df|_0 = 0$ implies that $du|_0 = 0$, but then $u = 0$, whence Q would vanish identically.) It follows that there are nonzero vectors $a, b \in \mathbb{C}^n$ such that

$$\begin{aligned} f(\xi) &= (\xi^t \cdot a)\,(\xi^t \cdot b) \\ &= \xi^t \cdot \frac{1}{2}(ab^t + ba^t) \cdot \xi, \quad \xi \in \mathbb{C}^n. \end{aligned} \tag{A.1}$$

Thus Q has rank 1 or 2, and by [34, Theorem 3.9.4] all 3×3 cofactor matrices for Q vanish whence $\operatorname{adj} Q = 0$. Conversely, if $\operatorname{adj} Q = 0$, then all 3×3 cofactor matrices for Q vanish whence Q has rank 1 or 2, and by [35, Section 48] or [36, Theorem 12.3.6], there are non-zero a and b such that equation (A.1) holds and f is not irreducible. □

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