ELECTROMAGNETIC FIELDS FROM CONTACT- AND SYMPLECTIC GEOMETRY

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Abstract. We give two constructions that give a solution to the sourceless Maxwell's equations from a contact form on a 3-manifold. In both constructions the solutions are standing waves. Here, contact geometry can be seen as a differential geometry where the fundamental quantity (that is, the *contact form*) shows a constantly rotational behaviour due to a non-integrability condition. Using these constructions we obtain two main results. With the first construction we obtain a solution to Maxwell's equations on $\mathbb{R}^3$ with an arbitrary prescribed and time independent energy profile. With the second construction we obtain solutions in a medium with a skewon part for which the energy density is time independent. The latter result is unexpected since usually the skewon part of a medium is associated with dissipative effects, that is, energy losses. Lastly, we describe two ways to construct a solution from symplectic structures on a 4-manifold.

One difference between acoustics and electromagnetics is that in acoustics the wave is described by a scalar quantity whereas an electromagnetics, the wave is described by vectorial quantities. In electromagnetics, this vectorial nature gives rise to *polarisation*, that is, in anisotropic medium differently polarised waves can have different propagation and scattering properties. To understand propagation in electromagnetism one therefore needs to take into account the role of polarisation. Classically, polarisation is defined as a property of plane waves, but generalising this concept to an *arbitrary* electromagnetic wave seems to be difficult. To understand propagation in inhomogeneous medium, there are various mathematical approaches: using Gaussian beams (see [Dah06, Kac04, Kac05, Pop02, Ral82]), using Hadamard's method of a propagating discontinuity (see [HO03]) and using microlocal analysis (see [Den92]). In all of these approaches polarisation is modelled in different ways, and in all approaches one needs to fix the polarisation of a wave to describe how the wave propagates. Polarisation can also be studied using the *helicity decomposition* for vector fields on $\mathbb{R}^3$. This does not yield specific information about propagation. However, it yields a decomposition of Maxwell's equations, which can be seen as a refinement of Helmholtz' decomposition [Bla93], or a generalisation of the Bohren decomposition [Dah04, Mos71]. This decomposition has also been studied in fluid mechanics [Mos71, Tur00]. For a similar decomposition for electromagnetic fields in homogeneous spacetime, see [Kai04]. The advantage of all the above helicity decompositions is that they relate polarisation to handed behaviour. This is a phenomenon that has not only proven useful in electromagnetism [Lak94], but also seems to be an important phenomena more generally in both physics and nature [HK90].

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In [Dah04] a helicity decomposition in $\mathbb{R}^3$ was used to study relations from electromagnetic fields to *contact geometry* (see Section 3), which is a branch of differential geometry that describes objects that contain a certain constantly rotating phenomena. This rotational behaviour is due to a non-integrability condition. For results that relate contact geometry and hydrodynamics, see [EG00]. In [Dah04] we used the helicity decomposition (or the Bohren decomposition) to construct examples of contact forms from particular solutions to Maxwell's equations. The motivation for studying this relation is that contact geometry can be seen as a differential geometric framework for studying handed behaviour, and this could be an approach to understand polarisation in electromagnetism. The present work can be seen as a continuation of [Dah04], and here we study the converse relation. In Section 3 we give two ways to construct solutions to the sourceless Maxwell's equations from contact forms on an arbitrary 3-manifold (see Theorems 3.7 and 3.13). These solutions are standing wave solutions, that is, the time-average of the Poynting vector is always zero, so there is no net transfer of energy. Using the first construction we describe an electromagnetic field on $\mathbb{R}^3$ with a prescribed and time independent energy density $\mathscr{E} = \frac{1}{2}(E \wedge D + H \wedge B)$. See Theorem 3.8. In Section 3.5 we use the second construction to obtain a solution to Maxwell's equations such that *(a)* the electromagnetic medium has a skewon component, and *(b)* Poynting's theorem holds, that is, the Poynting vector $\mathscr{S} = E \wedge H$ describes the flow of energy density $\mathscr{E}$. Since the time-average of $\mathscr{S}$ is zero, it is not completely clear how to interpret Poynting's theorem. In Example 3.16 we also show that medium with a skewon part can support solutions with time independent energy density $\mathscr{E}$. The last two results are somewhat unexpected as usually the skewon part of a medium is described as a component related to dissipative effects, that is, energy losses. (In this work we assume that the medium and the fields are real valued. In the complex case, skewon medium can also be lossless [LSTV94, Section 2.6].)

Essentially, one can view contact geometry as an odd dimensional analogue to symplectic geometry, which is the geometry of phase space in Hamiltonian mechanics. It is well known that contact- and symplectic geometry are closely related theories with many common results. In Section 4 we show how to construct solutions to Maxwell's equations on a 4-dimensional manifold starting from symplectic forms (see Theorems 4.2 and 4.3).

In Section 1 formulate Maxwell's equations on 3- and 4-manifolds and define various derived quantities. In Section 2 we review the decomposition of electromagnetic medium into its irreducible components [HO03, Section D.1.2]. This paper contains a number computations best done by computer algebra. Mathematica notebooks for these computations can be found on the author's homepage.

1. Maxwell's equations

By a manifold M we mean a second countable topological Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ with C^∞-smooth transition maps. All objects are real valued and smooth where defined. By TM and T^*M we denote the tangent and cotangent bundles, respectively. Let $\Omega^k_l(M)$ be $\binom{k}{l}$-tensors that are antisymmetric in their k upper indices and their l lower indices. In particular, let $\Omega^k(M)$ be the set of k-forms. Let also $\mathfrak{X}(M)$ be the set of vector fields, and let $C^\infty(M)$ be the set of functions. By $\Omega^k(M)\times\mathbb{R}$ we denote the set of k-forms that depend smoothly on a parameter $t\in\mathbb{R}$. The Einstein summing convention is used throughout.

We will use differential forms to write Maxwell's equations. On a 3-manifold M, *Maxwell equations* read [BH96, HO03]

$$dE = -\frac{\partial B}{\partial t}, \tag{1}$$

$$dH = \frac{\partial D}{\partial t} + J, \tag{2}$$

$$dD = \rho, \tag{3}$$

$$dB = 0, \tag{4}$$

for field quantities $E, H \in \Omega^1(M) \times \mathbb{R}$, $D, B \in \Omega^2(M) \times \mathbb{R}$ and sources $J \in \Omega^2(M) \times \mathbb{R}$ and $\rho \in \Omega^3(M) \times \mathbb{R}$. Let us emphasise that equations (1)–(4) are completely differential-topological, and depend only on the smooth structure of M.

1.1. **Maxwell's equations on a 4-manifold.** Suppose E, D, B, H are time dependent forms $E, H \in \Omega^1(M) \times \mathbb{R}$ and $D, B \in \Omega^2(M) \times \mathbb{R}$ and N is the 4-manifold $N = M \times \mathbb{R}$. Then we can define forms $F, G \in \Omega^2(N)$ and $j \in \Omega^3(N)$,

$$F = B + E \wedge dt, \tag{5}$$

$$G = D - H \wedge dt, \tag{6}$$

$$j = \rho - J \wedge dt. \tag{7}$$

Now fields E, D, B, H solve Maxwell's equations (1)–(4) if and only if

$$dF = 0, \tag{8}$$

$$dG = j, \tag{9}$$

where d is the exterior derivative on N. More generally, if N is a 4-manifold and F, G, j are forms $F, G \in \Omega^2(N)$ and $j \in \Omega^3(N)$ we say that F, G solve *Maxwell's equations* (for a source j) when equations (8)–(9) hold.

1.2. **Derived quantities and energy.** From a solution F, G to Maxwell's equations on a 4-manifold N we can define 4-forms $I_1, I_2, I_3 \in \Omega^4(N)$ as

$$I_1 = F \wedge F, \tag{10}$$

$$I_2 = F \wedge G, \tag{11}$$

$$I_3 = G \wedge G. \tag{12}$$

What is more, the *energy density* is the $\binom{3}{1}$-tensor $\Sigma \in \Omega^3_1(N)$ defined as [HO03]

$$\Sigma(y) = \frac{1}{2}\left(F \wedge \iota_y(G) - G \wedge \iota_y(F)\right), \quad y \in TM, \tag{13}$$

where ι_y is tensor contraction. Then Σ is anti-symmetric in F and G, so that $\Sigma_{F,G} = -\Sigma_{G,F}$. If $\theta \in \mathbb{R}$, then

$$\Sigma_{F,G} = \Sigma_{F_\theta, G_\theta},$$

where $F_\theta = \cos\theta\, F - \sin\theta\, G$ and $G_\theta = \sin\theta\, F + \cos\theta\, G$. The transformation $F, G \mapsto F_\theta, G_\theta$ is called a *duality transformation.* If there are no sources, then this transformation also preserves solutions to Maxwell's equations [BH96, Kai04].

Let us next study these derived quantities in the special case that $N = M \times \mathbb{R}$ where M is a 3-manifold, and F and G are given by equations (5)–(6). Then

$$\begin{aligned} I_1 &= 2E \wedge B \wedge dt, \\ I_2 &= (E \wedge D - H \wedge B) \wedge dt, \\ I_3 &= -2H \wedge D \wedge dt. \end{aligned}$$

Since $N = M \times \mathbb{R}$, there is a global vector field $T \in \mathfrak{X}(N)$ given by $T = \frac{\partial}{\partial t}$ and

$$\Sigma(T) = \mathscr{E} - \mathscr{S} \wedge dt, \tag{14}$$

where $\mathscr{E} \in \Omega^3(M) \times \mathbb{R}$ and $\mathscr{S} \in \Omega^2(M) \times \mathbb{R}$ are defined as

$$\begin{aligned} \mathscr{E} &= \frac{1}{2}\left(E \wedge D + H \wedge B\right), \\ \mathscr{S} &= E \wedge H. \end{aligned}$$

Here, $\mathscr{S}$ corresponds to the *Poynting vector* in classical electromagnetism, and $\mathscr{E}$ is the *energy density* of the solution. To prove equation (14) we have used that tensor contraction ι is an *anti-derivation* [AMR88, p. 429]. That is, if $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$ then for $X \in \mathfrak{X}(M)$ we have

$$\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_X \beta). \tag{15}$$

In consequence, $\iota_T(F) = -E$ and $\iota_T(G) = H$.

The next proposition shows that if a suitable condition is met, then quantity $\mathscr{S}$ describes the flow of $\mathscr{E}$ over time (see equation (16)). However, let us emphasise that all conditions in the below proposition are conditions on fields F and G (or fields E, D, B, H) and not in the electromagnetic medium. At this point we have not introduced any electromagnetic medium.

Theorem 1.1 (Poynting theorem)**.** *Let F, G is a solution to the sourceless Maxwell's equations on $N = M \times \mathbb{R}$, let E, D, B, H be as in equations* (5)–(6)*, and let T be as above. Then the following conditions are equivalent:*

(i) $\Sigma(T) \in \Omega^3(N)$ *is closed,*
(ii) $F \wedge \mathscr{L}_T(G) = \mathscr{L}_T(F) \wedge G$*, where $\mathscr{L}$ is the Lie derivative,*
(iii) $\frac{\partial}{\partial t}\mathscr{E} = -d\mathscr{S}$*, where d is the exterior derivative on M,*
(iv) for every open set $U \subset M$ with smooth boundary ∂U and compact closure, we have

$$\frac{\partial}{\partial t}\int_U \mathscr{E} = -\int_{\partial U} \mathscr{S}. \tag{16}$$

Proof. The equivalence of *(i)* and *(ii)* follows from equation (13) using Poincaré's formula $\mathscr{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$. The equivalence of *(i)* and *(iii)* follows by taking the exterior derivative of equation (14) and noting that $d_N\mathscr{E} = -\frac{\partial}{\partial t}\mathscr{E} \wedge dt$ where d_N is the exterior derivative on N. Equivalence of *(iii)* and *(iv)* follows by Stokes theorem and Lemma 14.22 in [Lee03]. ☐

2. The constitutive equation

Suppose N is a 4-manifold. By an *electromagnetic medium* on N we mean a map

$$\kappa \colon \Omega^2(N) \to \Omega^2(N).$$

We then say that 2-forms $F, G \in \Omega^2(N)$ *solve Maxwell's equations in medium* κ if F and G satisfy equations (8)–(9), and

$$G = \kappa(F). \tag{17}$$

Equation (17) is known as the *constitutive equation*. If κ is also invertible, it implies that one can eliminate half of the free variables in Maxwell's equations (8)–(9). In general the constitutive equation κ can be very complicated. For example, magnetic medium usually possess hysteresis. When studying constitutive equations it is therefore customary to introduce certain additional conditions that ensure that

κ is physically relevant. For example, medium can have an internal memory, so fields at one time instant can depend on past values of the fields, but, by causality, they can not depend on future values. We will here only consider electromagnetic medium that is linear and local. Then we can represent κ by an antisymmetric $\binom{2}{2}$-tensor $\kappa \in \Omega^2_2(N)$, and if locally

$$\kappa \quad = \quad \kappa^{ij}_{lm} dx^l \otimes dx^m \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \tag{18}$$

then constitutive equation (17) reads

$$G_{ij} \quad = \quad \kappa^{rs}_{ij} F_{rs}, \tag{19}$$

where $F = F_{ij} dx^i \otimes dx^j$ and $G = G_{ij} dx^i \otimes dx^j$.

Suppose (x^0, x^1, x^2, x^3) are local coordinates for $N = M \times \mathbb{R}$ such that x^0 is the coordinate for $\mathbb{R}$ and (x^1, x^2, x^3) are coordinates for M. If forms F, G are given by equations (5)–(6), then

$$F_{i0} = E_i, \quad F_{ij} = B_{ij}, \quad G_{i0} = -H_i, \quad G_{ij} = D_{ij}$$

for all $i, j = 1, 2, 3$ and equation (19) then reads

$$H_i \quad = \quad -2\kappa^{r0}_{i0} E_r - \kappa^{rs}_{i0} B_{rs}, \tag{20}$$

$$D_{ij} \quad = \quad 2\kappa^{r0}_{ij} E_r + \kappa^{rs}_{ij} B_{rs}, \tag{21}$$

where $i, j = 1, 2, 3$ and r, s are summed over $1, 2, 3$.

2.1. **Decomposition of electromagnetic medium.** At each point of a 4-manifold N a general electromagnetic medium κ is described by a antisymmetric $\binom{2}{2}$-tensor which depends on 36 free parameters. Next we review the decomposition of such a tensor into its three irreducible pieces; the principle part ${}^{(1)}\kappa$, the skewon part ${}^{(2)}\kappa$ and the axion part ${}^{(3)}\kappa$ [HO03]. The motivation for this decomposition is that different components of the medium enter in different parts of electromagnetics. For example, when $G = \kappa(F)$, the energy momentum tensor is independent of the axion part ${}^{(3)}\kappa$, whereas the Lagrangian $L = F \wedge G$ is independent of the skewon part ${}^{(2)}\kappa$. For a further discussion, see [HO03, Section D.1.3].

Let N be a 4-manifold. If $\kappa \in \Omega^2_2(N)$ we define the *trace* of κ as the smooth function $N \to \mathbb{R}$ given by

$$\operatorname{trace} \kappa \quad = \quad \kappa^{ij}_{ij}$$

when κ is locally given by equation (18). Let

$$\langle \cdot, \cdot \rangle \colon \Omega^2_2(N) \times \Omega^2_2(N) \quad \to \quad C^\infty(N)$$

be the canonical map defined as

$$\langle \chi, \kappa \rangle \quad = \quad \frac{1}{6} \operatorname{trace}(\chi \circ \kappa), \qquad \chi, \kappa \in \Omega^2_2(N).$$

That is, if χ and κ are written as in equation (18) then $\langle \chi, \kappa \rangle = \frac{1}{6} \chi^{ij}_{lm} \kappa^{lm}_{ij}$.

The factor $\frac{1}{6}$ is chosen such that $\langle \mathrm{Id}, \mathrm{Id} \rangle = 1$ when $\dim N = 4$. One way to see this is to use the local expression for Id given by $\mathrm{Id}^{ij}_{lm} = \delta^i_{[l} \delta^j_{m]} = \frac{1}{2}\left(\delta^i_l \delta^j_m - \delta^i_m \delta^j_l\right)$, where $[\cdot]$ is tensor anti-symmetrisation. Then $\langle \cdot, \cdot \rangle$ is symmetric, bilinear (over C^∞ functions), and *non-degenerate*, that is, if $\langle \chi, \kappa \rangle = 0$ for all $\kappa \in \Omega^2_2(N)$ then $\chi = 0$. (Take $\kappa^{ij}_{lm} = \delta^i_{[p} \delta^j_{q]} \delta^{[a}_l \delta^{b]}_m$.) Hence $\langle \cdot, \cdot \rangle$ defines a *scalar product* on $\Omega^2_2(N)$ in the sense of [O'N83, p. 47]. The dimension of $\Omega^2_2(N)$ is 36 and using computer algebra one can show that $\langle \cdot, \cdot \rangle$ has signature $(21, 15)$.

Notation Z, W and U in the next propositionis the same as in the Weyl decomposition of curvature [Küh06, Section 8D].

Proposition 2.1 (Decomposition of a $\binom{2}{2}$-tensors)**.** *Let N be a 4-manifold, and let*

$$\begin{aligned} Z &= \{\kappa \in \Omega^2_2(N) : u\wedge \kappa(v) = \kappa(u)\wedge v \textit{ for all } u,v\in\Omega^2(N), \\ &\qquad \operatorname{trace}\kappa = 0\} \\ &= \{\kappa\in\Omega^2_2(N) : \kappa^{ij}_{lj}(x) = 0\}, \\ W &= \{\kappa\in\Omega^2_2(N) : u\wedge\kappa(v) = -\kappa(u)\wedge v \textit{ for all } u,v\in\Omega^2(N)\} \\ &= \{\kappa\in\Omega^2_2(N) : u\wedge\kappa(v) = -\kappa(u)\wedge v \textit{ for all } u,v\in\Omega^2(N), \\ &\qquad \operatorname{trace}\kappa = 0\}, \\ U &= \{f\,\mathrm{Id}\in\Omega^2_2(N) : f\in C^\infty(N)\}. \end{aligned}$$

Then

$$\Omega^2_2(N) \quad = \quad Z\oplus W\oplus U. \tag{22}$$

Moreover

(i) U, W, Z are mutually orthogonal with respect to $\langle\cdot,\cdot\rangle$.
(ii) $\langle\cdot,\cdot\rangle$ is non-degenerate on Z, W and U.
(iii) Pointwise,

$$\dim Z = 20, \quad \dim W = 15, \quad \dim U = 1.$$

Here $\langle\cdot,\cdot\rangle$ is *non-degenerate* on $A_0\subset\Omega^2_2(N)$ if $a\in A_0$ and $\langle a,z\rangle = 0$ for all $z\in A_0$ implies that $a=0$.

Proof. By computer algebra the two expressions for Z coincide. We will use the first expression for W as the definition of W and the proof of the second expression for W is postponed to the end of the proof. Our first goal is to show that vector subspaces Z, W and U satisfy equation (22). If $z\in Z$ and $u\in U$, then $u = f\,\mathrm{Id}$ for some $f\in C^\infty(N)$ and $\langle u,z\rangle = f\operatorname{trace} z = 0$. Hence U and Z are orthogonal. Using the second expression for Z we see that $U\cap Z = \{0\}$ so $U+Z = U\oplus Z$. It is clear that $\langle\cdot,\cdot\rangle$ is non-degenerate on U, and an analysis using computer algebra shows that $\langle\cdot,\cdot\rangle$ is also non-degenerate on Z. Hence $\langle\cdot,\cdot\rangle$ is non-degenerate on $U+Z$ and by Lemma 23 in [O'N83, p. 49] we have

$$\Omega^2_2(N) \quad = \quad (U\oplus Z)\oplus(U\oplus Z)^\perp \tag{23}$$

where $B^\perp = \{w\in\Omega^2_2(N) : \langle w,b\rangle = 0 \,\text{for all}\, b\in B\}$ for $B\subset\Omega^2_2(N)$. Let us next show that $W = (U+Z)^\perp$. For inclusion $\subset$, let $w\in W$ and let us show that $w\in U^\perp$ and $w\in Z^\perp$. If w is written as in equation (18) with coefficients w^{ij}_{lm}, then $w\in W$ if and only if

$$w^{ij}_{lm}\varepsilon^{lmpq} \quad = \quad -w^{pq}_{lm}\varepsilon^{lmij}. \tag{24}$$

Here ε^{ijkl} is the *Levi-Civita permutation symbol.* If $u\in U$, then $u = f\,\mathrm{Id}$ for some $f\in C^\infty(N)$, and thus

$$\langle w,u\rangle \quad = \quad \frac{1}{6} f w^{ij}_{ij}.$$

On the other hand, using $\mathrm{Id}^{ij}_{lm} = \delta^i_{[l}\delta^j_{m]}$, $\varepsilon^{lmpq}\varepsilon_{lmij} = 4\delta^p_{[i}\delta^q_{j]}$ and equation (24) we have

$$\langle w,u\rangle \quad = \quad -\frac{1}{6} f w^{ij}_{ij},$$

so $w \in U^\perp$. Using computer algebra we find that Z and W are orthogonal. Hence $w \in Z^\perp$, and inclusion $W \subset (U+Z)^\perp$ follows. Inclusion $W \supset (U+Z)^\perp$ follows by computer algebra. Now equation (23) implies equation (22). We have also proven *(i)*. We have shown that $\langle \cdot, \cdot \rangle$ is non-degenerate on U, Z and $U+Z$. Hence (see [O'N83, p. 50]) it is also non-degenerate on $(U+Z)^\perp = W$, and *(ii)* follows. For *(iii)*, we note that equation (24) imposes 21 linearly independent constraints on coefficients w^{ij}_{lm}. Hence W has dimension $36 - 21 = 15$. It is clear that $\dim U = 1$, whence $\dim Z = 36 - 15 - 1 = 20$. The proof of the second expression for W follows by contracting equation (24) by the Levi-Civita permutation symbol ε_{ijpq}. □

When $\kappa \in \Omega^2_2(N)$ is written as in equation (25) in the next proposition we say that ${}^{(1)}\kappa \in Z$ is the *principal part*, ${}^{(2)}\kappa \in W$ is the *skewon part*, and ${}^{(3)}\kappa \in U$ is the *axion part* [HO03].

Proposition 2.2. *Any $\kappa \in \Omega^2_2(N)$ can be written as*

$$\kappa \;=\; {}^{(1)}\kappa \;+\; {}^{(2)}\kappa \;+\; {}^{(3)}\kappa, \tag{25}$$

where

$$\begin{aligned} {}^{(1)}\kappa &= \kappa - {}^{(2)}\kappa - {}^{(3)}\kappa, && \textit{(principal part)} \\ {}^{(2)}\kappa^{ij}_{lm} &= 2\, \not{\kappa}^{[i}_{[l}\delta^{j]}_{m]}, && \textit{(skewon part)} \\ {}^{(3)}\kappa &= \frac{1}{6}\operatorname{trace}(\kappa)\,\mathrm{Id}\,. && \textit{(axion part)} \end{aligned}$$

and $\not{\kappa} \in \Omega^1_1(N)$ is the trace-free tensor defined as $\not{\kappa}^i_j = \kappa^{im}_{jm} - \frac{1}{4}\operatorname{trace}(\kappa)\delta^i_j$. Then ${}^{(1)}\kappa \in Z$, ${}^{(2)}\kappa \in W$ and ${}^{(3)}\kappa \in U$. Moreover, since the sum in equation (22) *is direct, this decomposition is unique.*

Proof. It is clear that ${}^{(3)}\kappa \in U$. From ${}^{(2)}\kappa^{ij}_{lj} = \not{\kappa}^i_l$ and ${}^{(3)}\kappa^{ij}_{lj} = \frac{1}{4}\operatorname{trace}(\kappa)\delta^i_l$ we obtain ${}^{(1)}\kappa^{ij}_{lj} = 0$, so ${}^{(1)}\kappa \in Z$. To see that ${}^{(2)}\kappa \in W$, it suffices to show that ${}^{(2)}\kappa \in Z^\perp$ and ${}^{(2)}\kappa \in U^\perp$. For ${}^{(2)}\kappa \in Z^\perp$ we have

$$12\langle {}^{(2)}\kappa, z\rangle \;=\; \not{\kappa}^i_l z^{lj}_{ij} - \not{\kappa}^i_m z^{jm}_{ij} - \not{\kappa}^j_l z^{li}_{ij} + \not{\kappa}^j_m z^{im}_{ij} = 0, \quad z \in Z$$

since $z^{ij}_{lj} = 0$ and z^{ij}_{lm} is anti-symmetric in both ij and lm. That ${}^{(2)}\kappa \in U^\perp$ follows similarly. □

Lastly, suppose N has a volume form $\omega \in \Omega^4(N)$. Then $\Omega^2(N)$ has a scalar product $\langle \cdot, \cdot \rangle_\omega$ given by

$$\langle u, v\rangle_\omega \; \omega = u \wedge v, \quad u, v \in \Omega^2(N),$$

and $\langle \cdot, \cdot \rangle_\omega$ has signature $(3,3)$. See [Har91] and [HO03, Section A.1.10]. Vector spaces Z and W can then be characterized as the trace-free elements in $\Omega^2_2(N)$ that are symmetric and anti-symmetric with respect to $\langle \cdot, \cdot \rangle_\omega$, respectively.

2.2. **Time independent medium.** Let M and $N = M \times \mathbb{R}$ and vector field $T = \frac{\partial}{\partial t}$ be as in Section 1.2. We say that a electromagnetic medium $\kappa \in \Omega^2_2(N)$ is time independent, if $\mathscr{L}_T(\kappa) = 0$. If κ is locally given by equation (18), this condition states that components κ^{ij}_{lm} do not depend on time t.

The next proposition shows that the skewon part of the medium is related to behaviour of energy. See [HO03, Section D.1.5].

Proposition 2.3. *Let $N = M \times \mathbb{R}$ be as above, and let κ be a time independent medium with no skewon part. If F and G is a solution to the sourceless Maxwell's, then all conditions in Proposition 1.1 hold.*

Proof. For an arbitrary 2-form $F \in \Omega^2(N)$ and a vector field $X \in \mathfrak{X}(N)$ we have

$$\mathscr{L}_X(F) = \left(X(F_{ij}) + F_{ik}\frac{\partial X^k}{\partial x^j} - F_{jk}\frac{\partial X^k}{\partial x^i} \right) dx^i \otimes dx^j.$$

Thus $\mathscr{L}_T(\kappa F) = \kappa \mathscr{L}_T(F)$ and condition *(ii)* holds in Proposition 1.1. $\square$

2.3. **Anisotropic medium on a 3-manifold.** Suppose g is a (positive definite) Riemann metric on an orientable 3-manifold M. Then we denote by $\sharp$ and $\flat$ the musical isomorphisms $\sharp\colon T^*M \to TM$ and $\flat\colon TM \to T^*M$ induced by g. If $y \in TM$, then $y^\flat$ is the unique covector $y^\flat \in T^*M$ such that $y^\flat(w) = g(y,w)$ for all $w \in TM$, and $\sharp$ is the inverse of $\flat$. The *Hodge star operator* $*$ is the map $*\colon \Omega^p(M) \to \Omega^{3-p}(M)$ $(p = 0, \dots, 3)$ that acts on basis elements of $\Omega^p(M)$ as

$$*(dx^{i_1} \wedge \dots \wedge dx^{i_p}) = \frac{\sqrt{|g|}}{(3-p)!} g^{i_1 l_1} \cdots g^{i_p l_p} \varepsilon_{l_1 \cdots l_p\, l_{p+1} \cdots l_n} dx^{l_{p+1}} \wedge \dots \wedge dx^{l_n}.$$

Here $g = g_{ij} dx^i \otimes dx^j$, $|g| = \det g_{ij}$, g^{ij} is the ijth entry of $(g_{ij})^{-1}$, and $\varepsilon_{l_1 \cdots l_n}$ is the *Levi-Civita permutation symbol.* We also have the following relation (Proposition 6.2.12 in [AMR88]),

$$\alpha \wedge *\beta = g(\alpha, \beta) dV, \quad \alpha, \beta \in \Omega^p(M),\ p = 1, 2, 3, \tag{26}$$

where $dV = \sqrt{|g|} dx^1 \wedge dx^2 \wedge dx^3$ is the volume form induced by g. Here

$$g(\alpha, \beta) = \frac{1}{p!} \alpha_{i_1 \cdots i_p} g^{i_1 k_1} \cdots g^{i_1 k_p} \beta_{k_1 \cdots k_p}, \tag{27}$$

when $\alpha, \beta \in \Omega^p(M)$ are locally written as $\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}$.

Properties of the Hodge operator depend on the dimension. On a 4-manifold N, the Hodge operator $*\colon \Omega^2(N) \to \Omega^2(N)$ is conformally invariant, so that a scaling of the Riemann metric does not change its Hodge operator. In 4-dimensions, the Hodge operator also determines the metric up to a rescaling [DKS89]. These results do not hold in 3 dimensions (for example, see proof of Proposition 3.5). On the other hand, on any three manifold M the Hodge operator always satisfies $*^2 = \mathrm{Id}\,|_{\Omega^p(M)}$ for all $p = 0, 1, 2, 3$.

Let us consider the constitutive equations

$$D = *_\varepsilon E, \tag{28}$$

$$B = *_\mu H, \tag{29}$$

where $*_\varepsilon$ and $*_\mu$ are Hodge star operators corresponding to two Riemann metrics g_ε and g_μ, respectively. Here, metric g_ε models permittivity and g_μ models permeability. To make the connection clearer to equation (17) one could also use $*_\mu^2 = \mathrm{Id}$ and write equation (29) as $H = *_\mu B$.

On a 3-manifold equations (28)–(29) form a common model for electromagnetic medium [BH96, Kac04, KLS06] even if it does not describe the most general medium. Mathematically, one can show that any invertible linear map from 1-forms to 2-forms (or from vector fields to vector fields) can be realised as a Hodge operator of a Riemann metric. However, one needs to assume a positive-definite condition, so that the Riemann metric will be positive definite. Moreover, one needs to impose a symmetry condition, so that the degrees of freedom match. See [KLS06].

Proposition 2.4. *Equations* (28)–(29) *define an electromagnetic medium that has only a principal part.*

Proof. Any 2-form $F \in \Omega(N)$ can be written as

$$F = \alpha + \alpha' \wedge dt$$

where $\alpha \in \Omega^2(M) \times \mathbb{R}$ and $\alpha' \in \Omega^1(M) \times \mathbb{R}$. With this decomposition, equations (28)–(29) define a tensor $\kappa \in \Omega^2_2(N)$ by

$$\kappa(\alpha + \alpha' \wedge dt) = *_\varepsilon \alpha' - (*_\mu \alpha) \wedge dt.$$

Then $G = \kappa(F)$ when F and G are defined as in (5)–(6) and D and B are defined by equations (28)–(29). Using $*^2 = \mathrm{Id}$, it follows that κ is invertible. Comparing equations (28)–(29) and equations (20)–(21) we see that $\operatorname{trace} \kappa = 0$. By Theorem 2.1 it thus suffices to show that $\kappa(u) \wedge v = u \wedge \kappa(v)$ for all $u, v \in \Omega^2(N)$. If we write $u = \alpha + \alpha' \wedge dt$ and $v = \beta + \beta' \wedge dt$ this is equivalent to

$$(*_\varepsilon \alpha') \wedge \beta' - (*_\mu \alpha) \wedge \beta = \alpha' \wedge (*_\varepsilon \beta') - \alpha \wedge (*_\mu \beta),$$

which holds since $*(\omega) \wedge \nu = *(\nu) \wedge \omega$ when $\omega, \nu \in \Omega^k(M) \times \mathbb{R}$ for $k = 0, \ldots, \dim M$ and $*$ is the Hodge operator for an arbitrary Riemann metric on M. See equation (26). □

3. Contact geometry and electromagnetism

3.1. **Contact geometry.** A *contact form* on a $(2n+1)$-manifold (with $n \geq 1$) M is a 1-form $\alpha \in \Omega^1(M)$ such that the $(2n+1)$-form $\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha$ is never zero [Gei08]. By Frobenius theorem [Boo86], a form $\alpha \in \Omega^1(M)$ is a contact form if and only if the hyperplane field

$$\ker\alpha = \{v \in TM : \alpha(v) = 0\}$$

is nowhere integrable. That is, there is no hypersurface in M that is tangential to $\ker \alpha$. For a contact form α, the pair $(M, \ker \alpha)$ is called a *contact structure*. Thus a contact structure is invariant under a rescaling of the contact form by non-zero function.

Hereafter we specialise to study contact structures on 3-manifolds.

Example 3.1. The *standard tight contact structures* on $\mathbb{R}^3$ are the contact structures induced by contact forms $\alpha_\pm \in \Omega^1(\mathbb{R}^3)$ given by

$$\alpha_\pm = dz \pm x dy.$$

Since $\alpha_\pm = \pm dx \wedge dy \wedge dz$ both are contact forms. When $z = 0$, the plane fields $\ker \alpha_\pm$ are pictured in Figure 1. □

By definition every contact form α on a manifold M induces an orientation on M, and since $\dim M = 3$, the volume form $\alpha \wedge d\alpha$ is invariant under rescalings of α by non-zero functions. As an example, contact forms $\alpha_\pm$ induce opposite orientations $\pm dx \wedge dx \wedge dz$ on $\mathbb{R}^3$. By Darboux theorem [Gei08] every contact form on a 3-manifold is (up to a rescaling) locally diffeomorphic in an orientation preserving way to either α_+ or α_-. This implies that (up to a deformation) all contact structures locally show the same rotational behaviour as the contact structures in Figure 1. We can therefore think of contact structures as a mathematical model for rotational behaviour. We can also interpret the orientation induced by a contact form as the handedness of the rotation.

A contact form α induces a unique vector field R called the *Reeb vector field*. It is the unique vector field $R \in \mathfrak{X}(M)$ determined by

$$d\alpha(R, \cdot) = 0, \quad \alpha(R) = 1, \tag{30}$$

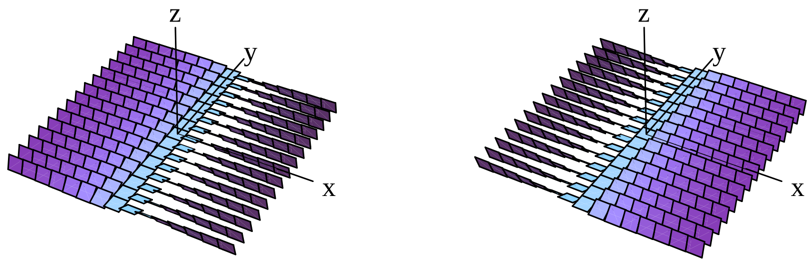

FIGURE 1. Contact structures α_+ (left) and α_- (right).

and if $p \in M$, then [dS01]

$$T_pM = \operatorname{span} R_p \oplus \ker \alpha_p = \ker d\alpha|_p \oplus \ker \alpha_p. \tag{31}$$

Also, in each $2n$-vector space $\ker \alpha_p$, the 2-form $d\alpha|_p$ is a symplectic quadratic form. (That is, $d\alpha|_p$ is a bilinear, and non-degenerate). For contact forms $\alpha_\pm$ in Example 3.1, we have $R_\pm = \frac{\partial}{\partial z}$.

3.2. **Adapted Riemann metrics and Beltrami fields.** In this section we describe how any contact form on a 3-manifold induces a non-unique Riemann metric that is compatible with the contact structure [CH85]. In [EG00] this result was used to study relations between hydrodynamics and contact geometry.

Definition 3.2 (Adapted Riemann metric)**.** A contact form $\alpha \in \Omega^1(M)$ on a 3-manifold M and a Riemann metric g on M are *adapted* if

$$d\alpha = *\alpha, \quad g(\alpha, \alpha) = 1, \tag{32}$$

where $*$ is the Hodge star operator induced by g.

When α and g are as in Definition 3.2 local computations show that $\alpha(\alpha^\sharp) = 1$ and $d\alpha(\alpha^\sharp, \cdot) = *\alpha(\alpha^\sharp, \cdot) = 0$ whence the Reeb vector field of α is given by $R = \alpha^\sharp$.

To understand the relevance of adapted Riemann metrics, let us define the curl of a vector field $X \in \mathfrak{X}(M)$ on a Riemann manifold as the unique vector field $\nabla \times X \in \mathfrak{X}(M)$ determined by

$$(\nabla \times X)^\flat = *d(X^\flat).$$

For the Reeb vector field $R = \alpha^\sharp$, conditions (32) read

$$\nabla \times R = R, \quad g(R, R) = 1.$$

That is, an adapted metric turns the contact form into a non-vanishing Beltrami vector field. A *Beltrami vector field* is a vector field $F \in \mathfrak{X}(M)$ such that $\nabla \times F = fF$ for some function $f \in C^\infty(M)$ [EG00]. In this note we only work with forms; we study contact forms and electromagnetic fields, and both of these are most naturally represented using forms, and not vector fields. We will therefore work with Beltrami 1-forms instead of Beltrami vector fields.

Definition 3.3 (Beltrami 1-form)**.** Let $*$ be the Hodge star operator induced by a Riemann metric on a 3-manifold M. A 1-form $\alpha \in \Omega^1(M)$ is a *Beltrami* 1*-form* if

$$d\alpha = f * \alpha \tag{33}$$

for some $f \in C^\infty(M)$. Moreover, α is a *rotational Beltrami* 1*-form* if f is nowhere zero.

The next proposition is due to Chern and Hamilton [CH85] (who also studied conditions on curvature for g). Direct proofs can be found in [EG00, Kom06].

Proposition 3.4 (Chern, Hamilton – 1984)**.** *Every contact form on a* 3*-manifold has an (non-unique) adapted Riemann metric.*

Proposition 3.5 (Etnyre, Ghrist – 2000)**.** *Let α be a* 1*-form on a* 3*-manifold M.*

(i) If α is a rotational Beltrami 1*-form that in nowhere zero, then α is a contact form.*
(ii) If α is a contact form, and f is a strictly positive function $f \in C^\infty(M)$, then there exists a (non-unique) Riemann metric on M such that equation (33) *holds. In this case, α is a rotational Beltrami* 1*-form.*

Proof. For *(i)* we have $\alpha \wedge d\alpha = f\alpha \wedge *\alpha$, and by equation (26), $\alpha \wedge d\alpha$ vanishes only if α or f vanishes. For *(ii)*, let us first note that if g and $\widetilde{g}$ are metrics such that $\widetilde{g} = \mu\, g$ for some strictly positive function $\mu \in C^\infty(M)$, then corresponding Hodge star operators $*, \widetilde{*} \colon \Omega^1(M) \to \Omega^2(M)$ satisfy $\widetilde{*} = (\mu)^{1/2} *$. For the proof, let g be a Riemann metric such that (32) holds. A suitable Riemann metric is then $\widetilde{g} = 1/f^2 g$. □

Example 3.6. Let $k > 0$. With coordinates x, y, z for $\mathbb{R}^3$, let

$$\beta_\pm \quad = \quad \cos(kx)\, dz \pm \sin(kx)\, dy.$$

With the Euclidean Riemann metric on $\mathbb{R}^3$ we have $*d\beta_\pm = \pm k\beta_\pm$, so $\beta_\pm$ are rotational Beltrami 1-forms and since $\beta_\pm \wedge d\beta_\pm = \pm k dx \wedge dy \wedge dz$ forms $\beta_\pm$ are contact structures with opposite induced orientations. Figure 2 shows how the contact structures $\ker \beta_\pm$ rotate with opposite handedness.

In [Dah04] the above contact forms where erroneously described as the "standard overtwisted contact structures". However, contact structures $\beta_\pm$ are tight and not overtwisted. For the terminology, see [Gei08]. In fact, for diffeomorphisms $f_\pm \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f_\pm(x,y,z) \quad = \quad \begin{pmatrix} 0 & -\cos kx & \pm\sin kx \\ k & 0 & 0 \\ 0 & \pm\sin kx & \cos kx \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we have $\beta_\pm = f_\pm^* \alpha_\pm$, where $\alpha_\pm$ are the contact forms in Example 3.1. I would like to thank professor Hansjörg Geiges for pointing this out. □

Theorem 3.7. *Suppose $\alpha \in \Omega^1(M)$ is a contact form on a* 3*-manifold M and $\omega > 0$. Then there exists a (non-unique) Riemann metric with Hodge operator $*$ such that time harmonic forms*

$$\begin{aligned} E(x,t) &= \alpha \cdot \cos \omega t, \\ H(x,t) &= -\alpha \cdot \sin \omega t, \\ D(x,t) &= *E(x,t), \\ B(x,t) &= *H(x,t), \qquad (x,t) \in M \times \mathbb{R} \end{aligned}$$

solve the source-less Maxwell's equations on M.

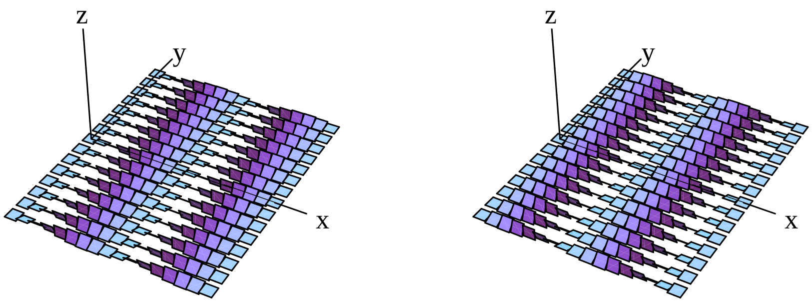

Figure 2. Contact structures β_+ (left) and β_- (right) in Example 3.6.

Proof. By Proposition 3.5 *(ii)*, there exists a non-unique Riemann metric such that $d\alpha = \omega * \alpha$. Maxwell's equations (1)-(4) now follow directly from the definitions of E, H, D, B given in the proposition. $\square$

For the solution in Theorem 3.7 we obtain

$$I_1 = -\frac{1}{\omega}\sin 2\omega t\, \alpha \wedge d\alpha \wedge dt, \tag{34}$$

$$I_2 = \frac{1}{\omega}\cos 2\omega t\, \alpha \wedge d\alpha \wedge dt, \tag{35}$$

$$I_3 = \frac{1}{\omega}\sin 2\omega t\, \alpha \wedge d\alpha \wedge dt, \tag{36}$$

$$\mathscr{E} = \frac{1}{2\omega}\alpha \wedge d\alpha, \tag{37}$$

$$\mathscr{S} = 0. \tag{38}$$

Moreover, E and H are Beltrami 1-forms for a Riemann metric that represents an electromagnetic medium with scalar wave impedance [KLS06]. By Proposition 2.4 the medium is also of principal type. Since α is a contact form, energy density $\mathscr{E}$ is independent of t and nowhere zero. Solutions given by Theorem 3.7 are examples where E and H are everywhere proportional. For such solutions, the Poynting vector $\mathscr{S}$ vanishes identically, and there is no net energy transfer. The solution in Theorem 3.7 is then best described as a standing wave solution to Maxwell's equations. For further discussion and examples of such solutions, see [UKS89, SKU90, ZB93].

The next theorem shows that an electromagnetic field can have any prescribed energy profile in $\mathbb{R}^3$. In particular, one can create a standing wave that is highly focused and whose energy profile is time independent. Let us also mention that the energy profile can have two signs, and from the proof we see that different signs are realised using contact forms $\beta_\pm$ that rotate with opposite handedness.

Theorem 3.8. *Suppose $e \in C^\infty(\mathbb{R}^3)$ is a strictly non-vanishing function. Then there exists an electromagnetic field on $\mathbb{R}^3$ in an electromagnetic medium of purely principal type such that the Poynting vector $\mathscr{S}$ and energy density $\mathscr{E}$ are given by*

$$\mathscr{S} = 0, \quad \mathscr{E} = e dV,$$

where $dV = dx \wedge dy \wedge dz$ is the standard volume form on $\mathbb{R}^3$.

Proof. Let $\alpha = \sqrt{|e|}\beta$ where $\beta = \beta_+$ if $e > 0$ and $\beta = \beta_-$ if $e < 0$, and $\beta_\pm$ are the contact forms in Example 3.6 with $k = 1$. Then $\alpha \wedge d\alpha = edV$. Setting $\omega = 1/2$ in Theorem 3.7 gives a time-harmonic solution E, D, B, H on $\mathbb{R}^3$ in a medium, which by Proposition 2.4, is of principal type. The given expressions for for $\mathscr{E}$ and $\mathscr{S}$ follow by equations (37)–(38). □

3.3. **Contact structures from electromagnetic fields.** It is well known that rotational behaviour plays an important role in electromagnetism. See for example, [Lak94, LSTV94]. It is therefore not surprising that there is a relation between electromagnetism and contact geometry, although it is not clear how general this relation might be. For plane waves this relation is easy to understand. Any plane wave in isotropic medium can be decomposed into a sum of one right hand circularly polarised plane wave and one left hand circularly polarised plane wave. Once one fixes time, each of these circularly polarised plane waves (when non-zero) becomes a contact form on $\mathbb{R}^3$. In fact, the contact forms are just rotated versions of $\beta_\pm$ in Example 3.6. Thus every non-zero plane wave induces one or two contact structures [Dah04].

By means of the Bohren decomposition, contact forms can be constructed also from other solutions to Maxwell's equations in isotropic medium. The key idea is to start with a solution to Helmholtz equation and decompose it into a sum of two Beltrami fields. Then, assuming that the Beltrami fields do not vanish, they induce contact forms by Proposition 3.5 *(i)*. Let us here consider two examples. For details, see [Dah04]. Figures 3 and 4 show slices of contact forms constructed from TE_{11} and TM_{11} solutions in a rectangular waveguide, respectively. For these solutions, the induced planefields are π-periodic in z (with z being the direction of the waveguide). Both figures show the characteristic rotational behaviour for contact structures. Figure 5 shows the set in the wave-guide where the contact condition $\alpha \wedge d\alpha \neq 0$ fails. In both cases this set is a union of isolated points and lines. In the TE_{11} case, the condition fails in the corners when $z = k\pi$ for $k \in \mathbb{Z}$ and in the centre of the waveguide for all z. In the TM_{11} case, the condition fails in the corners for all z, and in the centre of the waveguide for $z = \pi/2 + k\pi$ for $k \in \mathbb{Z}$.

If u is an eigenfunction to the Laplace-Beltrami operator on a 2-dimensional Riemann manifold, then the *nodal set* of u is the set where u vanishes. These sets can be used to visualise different oscillation modes. Say, on an oscillating plate, the nodal set indicate parts of the plate that do not oscillate. Physically these can be revealed by placing a thin layer of sand on the vibrating plate. Mathematically, nodal sets have also been studied since they satisfy many properties. See for example [Bär97, Kom06].

Since the forms that induces the contact structures in Figures 3 and 4 are Beltrami fields, the sets in Figure 5 are characterised as the sets where these Beltrami fields vanish. Therefore the sets in Figure 5 can be seen as an analogue of nodal sets for Beltrami fields. It is interesting to note that the lines where the contact condition fails are always parallel to the waveguide (that is, to the direction of propagating energy). For a similar analysis of zero-sets of electromagnetic Beltrami-type fields in spacetime, see [Kai04].

3.4. **Hodge-like operators induced by contact structures.** It is well known that a Riemann metric induces a linear operator that maps p-forms into $(n-p)$-forms. This is the Hodge operator defined in Section 2.3. It is less well known that every contact form also induces a similar linear map between p-forms and

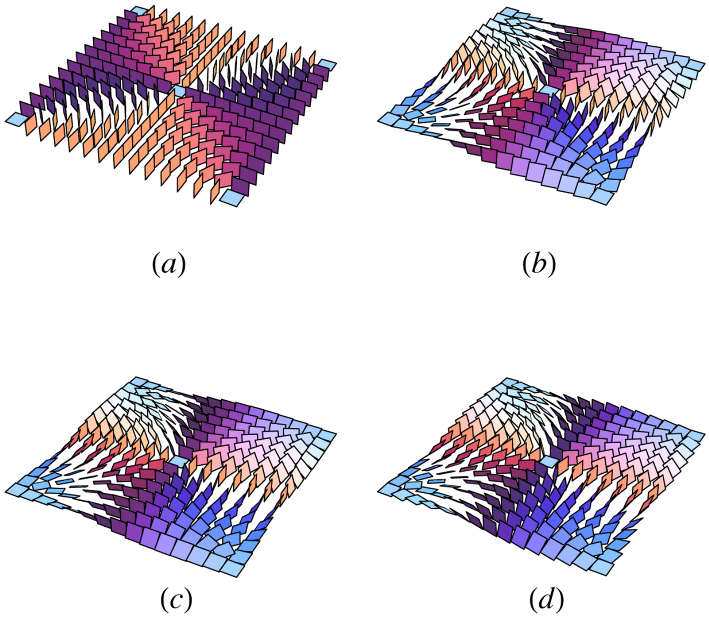

FIGURE 3. Planefield for contact form induced by a TE_{11} wave when $t = 0$ and $z = \pi k/4$ for $k = 0, 1, 2, 3$.

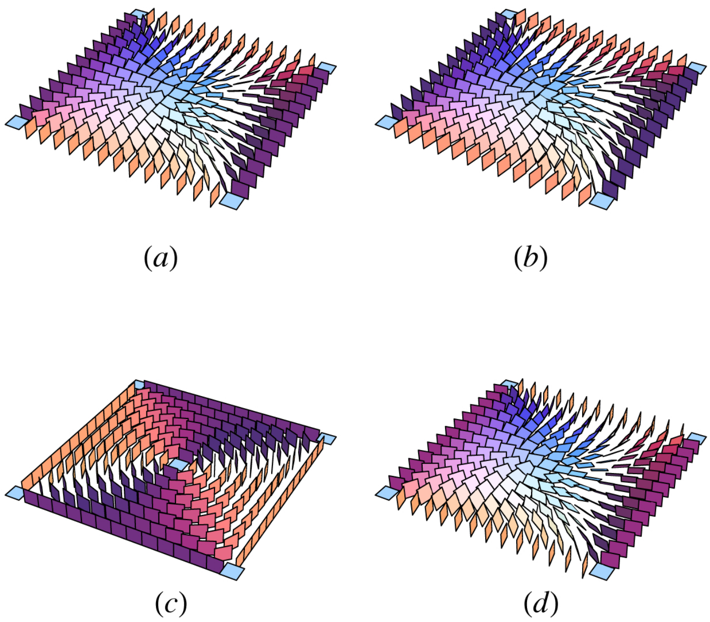

FIGURE 4. Planefield for contact form induced by a TM_{11} wave when $t = 0$ and $z = \pi k/4$ for $k = 0, 1, 2, 3$.

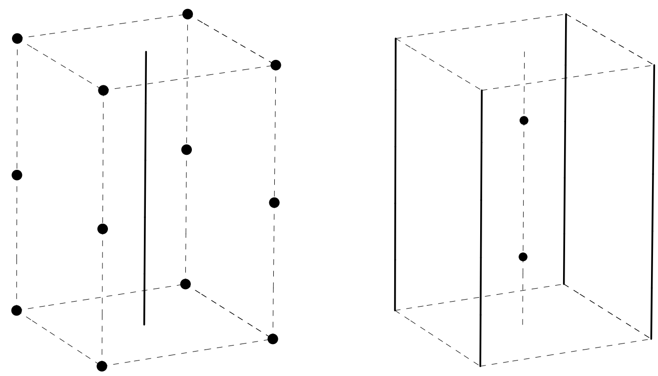

FIGURE 5. Sets where $\alpha \wedge d\alpha \neq 0$ fails for contact forms induced from the TE_{11}-wave (left) and TM_{11}-wave (right). The dashed box represents the waveguide for $z \in [0, 2\pi]$ at time $t = 0$. The dots and the solid lines indicate where the contact condition fails.

$(n-p)$-forms. For a general treatment of these operators, see [Bel02, LM87]. Even if the general theory of these operators is valid in any odd dimensions, let us here assume that the base manifold is 3-dimensional. A similar Hodge-like operator is also induced by a symplectic form on an even dimensional manifold (see Section 4).

The next proposition shows that a contact form identifies TM and T^*M. One can think of this as a contact-geometric analogue to the Legendre transformation in Riemannian geometry.

Proposition 3.9. *Let $\alpha \in \Omega^1(M)$ be a contact form on a 3-manifold M, and let $\flat_\alpha$ be the map $\flat_\alpha \colon TM \to T^*M$ defined as*

$$\flat_\alpha(y) = d\alpha(y, \cdot) + \alpha(y)\alpha, \quad y \in TM. \tag{39}$$

Then

(i) for each $x \in M$, the map $\flat_\alpha \colon T_xM \to T^*_xM$ is a linear isomorphism.
(ii) For any $\xi \in \Omega^1(M)$ we have

$$\alpha(\flat_\alpha^{-1}\xi) = \xi(R), \tag{40}$$

$$d\alpha(\flat_\alpha^{-1}(\xi), \cdot) = \xi - \xi(R)\alpha. \tag{41}$$

(iii) $\flat_\alpha(R) = \alpha$.

In (ii) and (iii), R is the Reeb vector field for α.

Before the proof of Proposition 3.9, let us note that if $\alpha = \alpha_i(x)dx^i$ and $y = y^i \left.\frac{\partial}{\partial x^i}\right|_x$ then locally

$$\flat_\alpha(y) = y^i h_{ik}(x) dx^k|_x,$$

where

$$h_{ik}(x) = \frac{\partial \alpha_k}{\partial x^i}(x) - \frac{\partial \alpha_i}{\partial x^k}(x) + \alpha_i(x)\alpha_k(x).$$

Proof of Proposition 3.9. For *(i)* suppose that $\flat_\alpha(y) = 0$ for some $y \in T_xM$. By decomposition (31) we may write $y = y_\perp + CR$ for some $y_\perp \in \ker\alpha$ and $C \in \mathbb{R}$. From conditions (30) it follows that

$$d\alpha(y_\perp, w) + C\alpha(w) \quad = \quad 0, \quad w \in T_xM.$$

Setting $w = R_x$ gives $C = 0$. Thus $d\alpha(y_\perp, \cdot) = 0$, and $y_\perp = 0$ since $d\alpha$ is non-degenerate in $\ker\alpha$. Thus $y = 0$ and *(i)* follows. Part *(iii)* follows using conditions (30). For part *(ii)*, equation (39) implies that

$$\xi \quad = \quad d\alpha(\flat_\alpha^{-1}(\xi), \cdot) + \alpha(\flat_\alpha^{-1}(\xi))\alpha \tag{42}$$

holds for all $\xi \in T^*M$. Evaluating both sides for R gives equation (40). Equation (41) follows using equations (40) and (42). Part *(iii)* follows by equations (40) and (41) since conditions (30) characterise the Reeb vector field. □

By definition a contact form α on a 3-manifold induces a volume form $\alpha \wedge d\alpha$ and by Proposition 3.9, α also induces an invertible map $\flat_\alpha \colon TM \to T^*M$. We may then define a linear map $*_\alpha \colon \Omega^1(M) \to \Omega^2(M)$ by setting

$$\begin{aligned} *_\alpha(\xi) \quad &= \quad \iota_{\flat_\alpha^{-1}(\xi)}(\alpha \wedge d\alpha) \\ &= \quad \alpha(\flat_\alpha^{-1}(\xi))d\alpha - \alpha \wedge \iota_{\flat_\alpha^{-1}(\xi)}(d\alpha) \\ &= \quad \xi(R)d\alpha - \alpha \wedge \xi, \quad \xi \in \Omega^1(M). \end{aligned} \tag{43}$$

Here the second equality follows by equation (15) and the last equality follows by equations (40)–(41). Equation (43) define the map $*_\alpha$. It should be emphasised that $*_\alpha$ is purely contact-geometrical and does not depend on any Riemann metric; it is determined by the contact form α alone. In this work we slightly generalise the above construction. Instead of starting with one contact form we start with *two* contact forms that are compatible in the following sense.

Definition 3.10 (Compatible contact forms)**.** Two contact forms α, β on a 3-manifold M are *compatible* if the 3-forms

$$\alpha \wedge d\beta, \quad \beta \wedge d\alpha,$$

are both volume forms on M.

The above notion of compatible contact forms does not seem to have been studied before. If α is contact form, then α and $f\alpha$ for any non-vanishing $f \in C^\infty(M)$ are compatible contact forms. The next example shows that compatible contact forms need not be proportional. In particular, part *(ii)* shows that compatible contact forms may or may not induce the same orientations on M.

Example 3.11 (Compatible contact forms)**.** Let x, y, z be coordinates for $\mathbb{R}^3$ and let dV be the volume form $dV = dx \wedge dy \wedge dz$.

(i) Contact forms $\alpha_\pm$ in Example 3.1 are compatible.
(ii) For non-zero constants C, D, let

$$\begin{aligned} \alpha \quad &= \quad dz + Cxdy, \\ \beta \quad &= \quad dz + Dydx. \end{aligned}$$

Then α and β are compatible contact forms with

$$\begin{aligned} \alpha \wedge d\alpha \quad &= \quad \beta \wedge d\alpha = C\,dV, \\ \beta \wedge d\beta \quad &= \quad \alpha \wedge d\beta = -D\,dV. \end{aligned}$$

(iii) For $k, \phi \in \mathbb{R} \setminus \{0\}$, let $\beta_\pm$ be as in Example 3.6 and let $\gamma_\pm$ be 1-forms

$$\gamma_\pm = \cos(kx+\phi)\, dz \pm \sin(kx+\phi)\, dy.$$

That is, 1-forms $\gamma_\pm$ are obtained by rotating the planes in $\beta_\pm$ by ϕ radians around the x-axis. Then

$$\begin{aligned} \beta_\pm \wedge d\beta_\pm &= \gamma_\pm \wedge d\gamma_\pm = \pm k\, dV, \\ \beta_\pm \wedge d\gamma_\pm &= \gamma_\pm \wedge d\beta_\pm = \pm k \cos\phi\, dV, \end{aligned}$$

so $\beta_\pm$ and $\gamma_\pm$ are compatible contact forms provided that $\cos\phi \neq 0$. □

Proposition 3.12. *Suppose α is a contact form on a 3-manifold M and $\beta \in \Omega^1(M)$ is a 1-form such that $\alpha \wedge d\beta$ is a volume form. Then the map*

$$L\colon \Omega^1(M) \to \Omega^2(M)$$

defined as

$$L(\xi) = \xi(R) d\beta - \alpha \wedge \xi, \quad \xi \in \Omega^1(M), \tag{44}$$

where $R \in \mathfrak{X}(M)$ is the Reeb vector field of α satisfies:

(i) *L is an invertible linear map.*
(ii) *$L(\alpha) = d\beta$.*
(iii) *If $\alpha = \beta$ then*

$$L(\xi) = \iota_{\flat_\alpha^{-1}(\xi)}(\alpha \wedge d\alpha), \quad \xi \in \Omega^1(M).$$

Proof. In part *(i)* we only need to prove that L is invertible, so suppose that

$$L(\xi) = \xi(R) d\beta - \alpha \wedge \xi = 0$$

for a $\xi \in \Omega^1(M)$. Taking the wedge product with α gives $\xi(R) = 0$. Hence $\alpha \wedge \xi = 0$, and contracting by R gives $\xi = 0$. Part *(ii)* follows from the definition of L, and part *(iii)* follows by equation (43). □

3.5. Electromagnetic fields form two contact forms.

Theorem 3.13. *Let $\omega > 0$ and let α and β be contact forms on a 3-manifold M such that*

$$\alpha \wedge d\beta, \quad \beta \wedge d\alpha,$$

are volume forms on M. Let L_e and L_m be the invertible linear maps

$$L_e, L_m \colon \Omega^1(M) \to \Omega^2(M)$$

determined by Proposition 3.12 such that

$$L_e(\alpha) = \frac{1}{\omega}\, d\beta, \tag{45}$$

$$L_m(\beta) = \frac{1}{\omega}\, d\alpha. \tag{46}$$

Then time harmonic fields

$$\begin{aligned} E(x,t) &= \alpha \cos\omega t, \\ H(x,t) &= -\beta \sin\omega t, \\ D(x,t) &= L_e(E), \\ B(x,t) &= L_m(H), \qquad (x,t) \in M \times \mathbb{R} \end{aligned}$$

solve the source-less Maxwell's equations on M.

Proof. For $\xi \in \Omega^1(M)$ let

$$L_e(\xi) = \frac{1}{\omega}\xi(R_\alpha)d\beta - \alpha\wedge\xi, \tag{47}$$

$$L_m(\xi) = \frac{1}{\omega}\xi(R_\beta)d\alpha - \beta\wedge\xi, \tag{48}$$

where R_α and R_β are the Reeb vector fields for α and β, respectively. Here, L_e is obtained by setting $\alpha\mapsto\alpha$ and $\beta/\omega\mapsto\beta$ in Proposition 3.12, and L_m is obtained similarly by setting $\beta\mapsto\alpha$ and $\alpha/\omega\mapsto\beta$. It follows that L_e and L_m are invertible linear maps in ξ such that equations (45)–(46) hold. Then

$$D = \frac{1}{\omega}d\beta\,\cos\omega t,$$
$$B = -\frac{1}{\omega}d\alpha\,\sin\omega t,$$

and the result follows using Maxwell's equations (1)–(4). □

For the solution in Proposition 3.13 we have

$$I_1 = -\frac{\sin 2\omega t}{\omega}\,\alpha\wedge d\alpha\wedge dt, \tag{49}$$

$$I_2 = \frac{1}{\omega}\left(\cos^2\omega t\ \alpha\wedge d\beta - \sin^2\omega t\ \beta\wedge d\alpha\right)\wedge dt, \tag{50}$$

$$I_3 = \frac{1}{\omega}\sin 2\omega t\,\beta\wedge d\beta\wedge dt, \tag{51}$$

$$\mathscr{E} = \frac{1}{2\omega}\left(\cos^2\omega t\ \alpha\wedge d\beta + \sin^2\omega t\ \beta\wedge d\alpha\right), \tag{52}$$

$$\mathscr{S} = -\frac{\sin 2\omega t}{2}\,\alpha\wedge\beta. \tag{53}$$

In particular we see that the Poynting vector $\mathscr{S}$ can be non-zero, but its time-average is zero, so there is no net flux of energy. The next proposition shows that nevertheless the conclusion in Poynting's theorem (equation (16)) is still valid for the fields in Theorem 3.13.

Proposition 3.14. *For the electromagnetic fields in Theorem 3.13, the energy density $\mathscr{E}$ and the Poynting vector $\mathscr{S}$ satisfy*

$$\frac{\partial}{\partial t}\int_U \mathscr{E} = -\int_{\partial U}\mathscr{S}$$

for all open sets $U\subset M$ with smooth boundary ∂U and compact closure.

Proof. By equations (52)–(53), we have $\frac{\partial}{\partial t}\mathscr{E} = -d\mathscr{S}$, whence the claim follows by Stokes theorem. □

The next proposition together with Propositions 3.14 imply that the assumptions in Proposition 2.3 are not sharp; even if a medium has a skewon part, equation (16) can still hold. From the present analysis we can not say if equation (16) holds for *all* electromagnetic fields in the medium in Theorem 3.13, or if equation (16) holds for only the particular fields in Theorem 3.13. However, from equations (47)–(48) we see that pointwise the medium in Theorem 3.13 depends on 1-forms α and β and their first order derivatives. Thus the skewon part of the medium is pointwise determined by at most 12 constants (and probably less as some constants would parameterise the principal part.) This is less than 15 which is the number of free parameters in the most general skewon medium.

Proposition 3.15. *Let κ be the medium in Proposition 3.13 determined by two compatible contact forms α and β. Then κ has both a principal part and a skewon part, but no axion part.*

Proof. We know that any 2-form on $N = M \times \mathbb{R}$ can be written as $\xi + \xi' \wedge dt$ for some $\xi \in \Omega^2(M) \times \mathbb{R}$ and $\xi' \in \Omega^1(M) \times \mathbb{R}$. With this decomposition, the medium in Proposition 3.13 is given by

$$\kappa(\xi + \xi' \wedge dt) = L_e(\xi') - L_m^{-1}(\xi) \wedge dt. \tag{54}$$

Indeed, for this κ we have $\kappa(F) = G$. We can also express the medium as

$$D_{ab} = P^r_{ab} E_r, \quad H_a = Q^{rs}_a B_{rs}$$

for some $\binom{1}{2}$- and $\binom{2}{1}$-tensors P and Q, respectively. Comparing with equations (20)–(21) we find that locally medium (54) is given by

$$\kappa^{r0}_{i0} = 0, \quad \kappa^{rs}_{i0} = -Q^{rs}_i, \quad \kappa^{r0}_{ij} = \frac{1}{2} P^r_{ij}, \quad \kappa^{rs}_{ij} = 0$$

for all $i, j, r, s = 1, 2, 3$. Thus $\operatorname{trace} \kappa = 0$, and κ has no axion part by Proposition 2.2. To show the remaining two claims, let us assume that κ is purely of principal type or purely of skewon type, and derive a contradiction. By the counter-assumption, Theorem 2.1 implies that

$$(\xi + \xi' \wedge dt) \wedge \kappa(\eta + \eta' \wedge dt) = \pm \kappa\,(\xi + \xi' \wedge dt) \wedge (\eta + \eta' \wedge dt) \tag{55}$$

for all $\xi, \eta \in \Omega^2(M) \times \mathbb{R}$ and $\xi', \eta' \in \Omega^1(M) \times \mathbb{R}$. Using equation (54) and setting $\eta = \xi = 0$, equation (55) implies that

$$\xi' \wedge L_e(\eta') = \pm L_e(\xi') \wedge \eta'. \tag{56}$$

If we take $\xi' = \eta' = \alpha$ in equation (56), we see that we must choose the $+$-sign in equation (55), that is, medium κ is of principal type. On the other hand, if we take $\xi', \eta' \in T^*_x M$ for some $x \in M$ such that $\xi'(R_\alpha) = \eta'(R_\alpha) = 0$, then equations (47) and (56) imply that

$$\alpha \wedge \eta' \wedge \xi' = \pm \alpha \wedge \xi' \wedge \eta'.$$

Contracting by R_α gives $\eta' \wedge \xi' = \pm \xi \wedge \eta'$, and we need to take the $-$-sign in equation (55), that is, medium κ is of skewon type. In conclusion, we can not choose only one sign in equation (55), so κ has both a skewon part and a principal part. $\square$

The next example shows that a medium with a skewon part can support solutions with time independent energy density. This is somewhat unexpected as usually the skewon part of a medium is described as being related to dissipative effects, that is, energy losses.

Example 3.16. Let $\alpha = \beta_\pm$ and $\beta = \gamma_\pm$, where $\beta_\pm$ and $\gamma_\pm$ are as in Example 3.11 *(iii)* for $k = 1$ and $\cos\phi \neq 0$. Then

$$\alpha \wedge d\beta = \beta \wedge d\alpha = \pm \cos\phi dV.$$

Applying Theorem 3.13 to α, β and $\omega = 1$ gives an electromagnetic field that solves the source-less Maxwell equations in a medium, which by Proposition 3.15 has a skewon part. Moreover, by equations (52)–(53) the Poynting vector $\mathscr{S}$ and energy density $\mathscr{E}$ of the solution are given by

$$\begin{aligned} \mathscr{S} &= \pm \frac{\sin 2t \, \sin\phi}{2} dy \wedge dz, \\ \mathscr{E} &= \pm \frac{\cos\phi}{2} dV, \end{aligned}$$

and energy density $\mathscr{E}$ is time independent. We also see that the sign of $\mathscr{E}$ depends not only on the handedness of rotation in α and β, but also on angle ϕ. For this choice of α and β, equation (16) holds trivially since both sides are zero.

Let us write $E = E_i dx^i$ and $D = D_{ij} dx^i \otimes dx^j$. Then we may write

$$\begin{pmatrix} D_{23} \\ D_{31} \\ D_{12} \end{pmatrix} = ({}^{(1)}\varepsilon + {}^{(2)}\varepsilon) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix},$$

where ${}^{(1)}\varepsilon$ and ${}^{(2)}\varepsilon$ are 3×3 matrices representing the principal and skewon components of L_e from basis $\{dx^i\}$ into basis $\{\frac{1}{2}\varepsilon_{ijk} dx^j \wedge dx^k\}$. Explicitly,

$$\begin{aligned} {}^{(1)}\varepsilon &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \pm \sin x \, \sin(x+\phi) & \frac{1}{2}\sin(2x+\phi) \\ 0 & \frac{1}{2}\sin(2x+\phi) & \pm\cos x \, \cos(x+\phi) \end{pmatrix}, \\ {}^{(2)}\varepsilon &= \begin{pmatrix} 0 & \cos x & \mp \sin x \\ -\cos x & 0 & \frac{1}{2}\sin\phi \\ \pm\sin x & -\frac{1}{2}\sin\phi & 0 \end{pmatrix}. \end{aligned}$$

We have $\det({}^{(1)}\varepsilon + {}^{(2)}\varepsilon) = \pm\cos\phi$. Thus the medium becomes singular when E and H are orthogonal. Since ${}^{(1)}\varepsilon$ and ${}^{(2)}\varepsilon$ are singular matrices for all ϕ there are no limiting cases for which the medium would become purely of principal type or purely of skewon part.

Suppose $\phi = 0$. Then $\alpha = \beta$ and the electromagnetic fields E, D, B, H in Theorem 3.13 coincide with the fields in Theorem 3.7. However, the electromagnetic mediums are qualitatively different. In Theorem 3.7 the medium is purely of principal type (by Proposition 2.4), and in Theorem 3.13 the medium has both a principal part and a skewon part (by Proposition 3.15). □

4. Electromagnetism from symplectic forms

A *symplectic form* on a $2n$-dimensional manifold M is a 2-form $\omega \in \Omega^2(M)$ that satisfies

- *(i)* ω is closed,
- *(ii)* ω is *non-degenerate* in each tangent space. That is, if $u \in T_pM$ and $\omega(u, v) = 0$ for all $v \in T_pM$, then $u = 0$.

The second condition is equivalent to that the $2n$-form $\omega \wedge \cdots \wedge \omega$ is a volume form on M. Alternatively, if locally $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$, then ω is non-degenerate if and only if components $\omega_{ij}(x)$ form an invertible matrix for each $x \in M$. Hence every symplectic form ω induces a linear isomorphism $\flat \colon TM \to T^*M$ given by

$$\flat(y) = \iota_y \omega, \quad y \in TM.$$

If locally $y = y^i \frac{\partial}{\partial x^i}$, we have $\flat(y) = \omega_{ij} y^i dx^j$.

If ω is a symplectic form on M, then for any 2-form $\alpha \in \Omega^2(M)$ we can define a smooth function $\omega^{-1}(\alpha) \in C^\infty(M)$ as follows. If locally $\xi = \frac{1}{2}\xi_{ij} dx^i \wedge dx^j$, and $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$, then we define

$$\omega^{-1}(\xi)(x) = \omega^{ij}(x)\xi_{ij}(x),$$

where $\omega^{ij}(x)$ is the inverse matrix of $\omega_{ij}(x)$, so that $\omega^{ij}(x)\omega_{jk} = \delta^i_k$.

As in contact geometry, every symplectic form ω induces a Hodge-like operator $*_\omega$ that idenifies p-forms and $(2n-p)$-forms on M. However, unlike the Hodge operator for a Lorentz metric on a 4-manifold, this operator always satisfies $*_\omega^2 = \mathrm{Id}$. For the general definition, see [LM87, p. 43]. Next we specialise to symplectic forms on 4-manifolds. The next theorem summarise properties of the induced Hodge-like operator from 2-forms to 2-forms in this particular case.

Proposition 4.1. *Suppose ω is a symplectic form on a 4-manifold M. Then ω induces an invertible linear map*

$$\kappa\colon \Omega^2(M) \quad \to \quad \Omega^2(M),$$

defined as

$$\kappa(\xi) \quad = \quad -\frac{1}{2}\omega^{-1}(\xi)\omega - \xi, \quad \xi \in \Omega^2(M). \tag{57}$$

Moreover,

- *(i)* $\kappa(\omega) = \omega$.
- *(ii) The principal part ${}^{(1)}\kappa$, skewon part ${}^{(2)}\kappa$ and axion part ${}^{(3)}\kappa$ of κ are given by*

$$\begin{aligned} {}^{(1)}\kappa &= \kappa + \frac{2}{3}\,\mathrm{Id}, \\ {}^{(2)}\kappa &= 0, \\ {}^{(3)}\kappa &= -\frac{2}{3}\,\mathrm{Id}\,. \end{aligned}$$

- *(iii)* $\kappa^2 = \mathrm{id}$.

Proof. If locally $\xi = \frac{1}{2}\xi_{ij}dx^i \wedge dx^j$ and $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$, then

$$\kappa(\xi) = -\frac{1}{2}\left(\frac{1}{2}\omega^{ij}\xi_{ij}\omega_{ab} + \xi_{ab}\right) dx^a \wedge dx^b.$$

Linearity in ξ is clear. Property *(i)* follows since $\omega^{-1}(\omega) = \omega^{ij}\omega_{ij} = -4$. Property *(iii)* is a direct calculation using equation (57), and invertibility follows since $\kappa^{-1} = \kappa$. For *(ii)* we will use Proposition 2.2, so let us first write $\kappa(\xi)$ as

$$\kappa(\xi) = \kappa_{ab}^{ij}\xi_{ij}dx^a \otimes dx^b, \tag{58}$$

where $\kappa_{ab}^{ij} = -\left(\frac{1}{2}\omega^{ij}\omega_{ab} + \delta^i_{[a}\delta^j_{b]}\right)$. Thus $\operatorname{trace}\kappa = \kappa_{ij}^{ij} = -4$, and the expression for ${}^{(3)}\kappa$ follows. Expressions for ${}^{(1)}\kappa$ and ${}^{(2)}\kappa$ follows since $\cancel{\kappa}^i_j = 0$. □

As a corollary, every symplectic form on 4-manifold can be interpreted as an electromagnetic field in a suitable medium.

Theorem 4.2. *Suppose $\omega \in \Omega^2(M)$ is a symplectic form on a 4-manifold M, and suppose that κ is the invertible linear map $\kappa\colon \Omega^2(M) \to \Omega^2(M)$ induced by ω as in Proposition 4.1. Then 2-forms $F, G \in \Omega^2(M)$,*

$$\begin{aligned} F &= \omega, \\ G &= \kappa(F) = \omega, \end{aligned}$$

is a solution to the sourceless Maxwell's equations on M with medium κ. For this solution I_1, I_2 and I_3 coincide and are identically non-zero on M. Moreover, the principal, skewon, and axion parts of medium κ are given in Proposition 4.1 (ii).

Proof. Equations (8)–(9) follow since $F = G = \omega$ and ω is closed, and by equations (10)–(12) we have $I_1 = I_2 = I_3 = \omega \wedge \omega$, which is a volume form on M since ω is non-degenerate. □

If we explicitly write down equation $G = \kappa(F)$ for fields F and G and medium κ in Theorem 4.2 we obtain $G = F$. That is, fields F and G in Theorem 4.2 also solve Maxwell's equations in the purely axion medium $\kappa = \mathrm{Id}$. This is similar to the comment after Proposition 3.15.

4.1. **Space-time solutions from two symplectic forms.** The next theorem generalise Theorem 4.1 to the case where we have two suitably compatible symplectic forms on M.

Theorem 4.3. *Suppose F and G are symplectic forms on a 4-manifold M such that $F \wedge G$ is a volume form on M. Then there exists an invertible linear map*

$$\kappa \colon \Omega^2(M) \quad \to \quad \Omega^2(M)$$

such that

(i) $\kappa(F) = G$.
(ii) The principal, skewon and axion parts of κ are given by

$$\begin{aligned} {}^{(1)}\kappa &= \kappa - \frac{1}{6} F^{-1}(G)\,\mathrm{Id}, \\ {}^{(2)}\kappa &= 0, \\ {}^{(3)}\kappa &= \frac{1}{6} F^{-1}(G)\,\mathrm{Id}\,. \end{aligned}$$

(iii) If $F = G$, then κ coincides with the map in Proposition 4.1.

Proof. If locally

$$F = \frac{1}{2} F_{ij}(x) dx^i \wedge dx^j, \quad G = \frac{1}{2} G_{ij}(x) dx^i \wedge dx^j,$$

then we define κ by setting

$$\kappa(\xi) \quad = \quad -\frac{1}{4}\left(F^{ij}\xi_{ij}G_{pq} + \xi_{pi}F^{ij}G_{jq} - \xi_{qi}F^{ij}G_{jp}\right) dx^p \wedge dx^q, \tag{59}$$

where $\xi = \frac{1}{2}\xi_{ij} dx^i \wedge dx^j$ is any 2-form $\xi \in \Omega^2(M)$. It is clear that κ is linear in ξ, globally defined. Direct calculations using $F^{ij}F_{jk} = \delta^i_k$ and $F^{ij}F_{ij} = -4$ give *(i)* and *(iii)*. To prove that κ is invertible, suppose that $\kappa(\xi) = 0$ for some $\xi \in \Omega^2(M)$. We only need to show that $\xi = 0$ at one point, so we can work locally around some point $x \in M$. Then

$$F^{ij}\xi_{ij}G_{pq} + \xi_{pi}F^{ij}G_{jq} - \xi_{qi}F^{ij}G_{jp} \quad = \quad 0 \tag{60}$$

and contracting by G^{pq} yields

$$F^{ij}\xi_{ij} \quad = \quad 0, \tag{61}$$

whence equation (60) simplifies into

$$\xi_{pi}F^{ij}G_{jq} = \xi_{qi}F^{ij}G_{jp}. \tag{62}$$

Using Darboux' theorem [LM87, p. 51], we may assume that in local coordinates around x, components of F and G are

$$F_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{ij} \qquad G_{ij} = \begin{pmatrix} 0 & G_{12} & G_{13} & G_{14} \\ -G_{12} & 0 & G_{23} & G_{24} \\ -G_{13} & -G_{23} & 0 & G_{34} \\ -G_{14} & -G_{24} & -G_{34} & 0 \end{pmatrix}_{ij}.$$

Writing out equation (61) using computer algebra shows that in these coordinates ξ is of the form

$$\xi_{ij} = \begin{pmatrix} 0 & A & B & C \\ -A & 0 & D & -B \\ -B & -D & 0 & E \\ -C & B & -E & 0 \end{pmatrix}_{ij}$$

for some constants $A, \dots, E$ and that equation (62) reads

$$\begin{aligned}
&A(G_{13} + G_{24}) = 0, \\
&AG_{34} + 2BG_{13} + CG_{23} + DG_{14} + EG_{12} = 0, \\
&C(G_{13} + G_{24}) = 0, \\
&D(G_{13} + G_{24}) = 0, \\
&AG_{34} - 2BG_{24} + CG_{23} + DG_{14} + EG_{12} = 0, \\
&E(G_{13} + G_{24}) = 0.
\end{aligned}$$

Writing out $F \wedge G \neq 0$ using computer algebra gives $G_{13} + G_{24} \neq 0$, so $\xi = 0$. For *(ii)* let us rewrite equation (59) for $\kappa(\xi)$ as

$$\kappa(\xi) = \kappa^{ab}_{pq} \xi_{ab} dx^p \otimes dx^q,$$

where

$$\kappa^{ab}_{pq} = -\frac{1}{2} \left(F^{ab} G_{pq} + \delta^a_{[p} \delta^b_{i]} F^{ij} G_{jq} - \delta^a_{[q} \delta^b_{i]} F^{ij} G_{jp} \right).$$

It follows that $\kappa^{am}_{pm} = \frac{1}{4} F^{ij} G_{ij} \delta^a_p$ and $\kappa^{ab}_{ab} = F^{ij} G_{ij}$. The expression for ${}^{(3)}\kappa$ follows. Since $\not\kappa^i_j = 0$, it follows that ${}^{(2)}\kappa = 0$ and ${}^{(1)}\kappa = \kappa -{}^{(3)} \kappa$. □

The next example shows that there are non-proportional symplectic forms F and G that satisfy the compatibility assumption $F \wedge G \neq 0$ in Proposition 4.3.

Example 4.4. Let (x, y, P, Q) be coordinates for $\mathbb{R}^4$, let F be the standard symplectic form

$$F = dx \wedge dP + dy \wedge dQ, \tag{63}$$

and let G be the 2-form

$$G = C_1\, dx \wedge dy + C_2\, dx \wedge dP + C_3\, dx \wedge dQ + C_4\, dy \wedge dP + C_5\, dy \wedge dQ + C_6\, dP \wedge dQ,$$

where $C_1, \dots, C_6$ are constants. Then G is a symplectic form if and only if $C_1 C_6 + C_3 C_4 - C_2 C_5 \neq 0$. Moreover, F and G satisfy $F \wedge G \neq 0$ if and only if $C_2 + C_5 \neq 0$.

For example, for any $\theta \in \mathbb{R}$, form F in equation (63) and G given by

$$G = dx \wedge (\cos\theta dP - \sin\theta dQ) + dy \wedge (\sin\theta dP + \cos\theta dQ)$$

are both symplectic forms and $F \wedge G \neq 0$ if and only if $\cos\theta \neq 0$. □

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