```
<< VectorAnalysis`
SetCoordinates[Cartesian[x, y, z]];
```

Define TE and TM solutions

```
ee[x_, y_, z_, m_, n_] := Sin[Pi m x / a] Sin[Pi n y / b]

hh[x_, y_, z_, m_, n_] := Cos[Pi m x / a] Cos[Pi n y / b]

kc[m_, n_] := Sqrt[(m Pi / a) ^2 + (n Pi / b) ^2]

ETM[x_, y_, z_, m_, n_] :=
 ({0, 0, 1} ee[x, y, z, m, n] + I β / kc[m, n] ^2 Grad[ee[x, y, z, m, n]]) E^(I (β z - ω t))

ETE[x_, y_, z_, m_, n_] := -I ω Sqrt[ϵ μ] Sqrt[μ / ϵ] / kc[m, n] ^2
  Cross[{0, 0, 1}, Grad[hh[x, y, z, m, n]]] E^(I (β z - ω t))

(* check that both ETM and ETM are solutions to the Helmholtz equation *)
FullSimplify[Curl[Curl[ETM[x, y, z, m, n]]] - (kc[m, n] ^2 + β^2) ETM[x, y, z, m, n]]
FullSimplify[Curl[Curl[ETE[x, y, z, m, n]]] - (kc[m, n] ^2 + β^2) ETE[x, y, z, m, n]]
```

```
{0, 0, 0}

{0, 0, 0}
```

Define Beltrami fields

```
(* lambda = +1/-1 *)
BelTM[x_, y_, z_, m_, n_, λ_] :=
 1 / 2 (ETM[x, y, z, m, n] + λ / Sqrt[kc[m, n] ^2 + β^2] Curl[ETM[x, y, z, m, n]])
BelTE[x_, y_, z_, m_, n_, λ_] :=
 1 / 2 (ETE[x, y, z, m, n] + λ / Sqrt[kc[m, n] ^2 + β^2] Curl[ETE[x, y, z, m, n]])

kgen[m_,n_] := Sqrt[kc[m, n] ^2 + β^2]

FullSimplify[kgen[m, n] BelTE[x, y, z, m, n, +1] - Curl[BelTE[x, y, z, m, n, +1]]]
FullSimplify[kgen[m, n] BelTE[x, y, z, m, n, -1] + Curl[BelTE[x, y, z, m, n, -1]]]
FullSimplify[kgen[m, n] BelTM[x, y, z, m, n, +1] - Curl[BelTM[x, y, z, m, n, +1]]]
FullSimplify[kgen[m, n] BelTM[x, y, z, m, n, -1] + Curl[BelTM[x, y, z, m, n, -1]]]
```

```
{0, 0, 0}

{0, 0, 0}

{0, 0, 0}

{0, 0, 0}
```

TE_11 contact forms

```
(* lambda = +1 or -1 *)Re[BelTE[x, y, z, m, n, λ]].Re[Curl[BelTE[x, y, z, m, n, λ]]] /.
  {ϵ → 1, μ → 1, a → 1, b → 1, β → 1, ω → 1};
nonZero = FullSimplify[ComplexExpand[%]]
```

$$\Big(2\,n^2\,\lambda\,\mathrm{Cos}[m\,\pi\,x]^2\,\mathrm{Sin}[n\,\pi\,y]^2\,\left(\mathrm{Cos}[t-z]^2+\left(1+\left(m^2+n^2\right)\pi^2\right)\mathrm{Sin}[t-z]^2\right)+\lambda\,\mathrm{Cos}[n\,\pi\,y]^2\,\Big(2\,m^2\,\mathrm{Cos}[t-z]^2\,\mathrm{Sin}[m\,\pi\,x]^2+\left(m^2+\left(m^2+n^2\right)\left(2\,m^2+n^2\right)\pi^2+\left(-m^2+n^2\left(m^2+n^2\right)\pi^2\right)\mathrm{Cos}[2\,m\,\pi\,x]\right)\mathrm{Sin}[t-z]^2\Big)\Big)\Big/\left(8\left(m^2+n^2\right)^2\pi^2\sqrt{1+\left(m^2+n^2\right)\pi^2}\right)$$

```
dA = nonZero /. {m → 1, n → 1}
```

$$\frac{1}{32\,\pi^2\sqrt{1+2\,\pi^2}}\Big(2\,\lambda\,\mathrm{Cos}[\pi\,x]^2\,\mathrm{Sin}[\pi\,y]^2\left(\mathrm{Cos}[t-z]^2+\left(1+2\,\pi^2\right)\mathrm{Sin}[t-z]^2\right)+\lambda\,\mathrm{Cos}[\pi\,y]^2\left(2\,\mathrm{Cos}[t-z]^2\,\mathrm{Sin}[\pi\,x]^2+\left(1+6\,\pi^2+\left(-1+2\,\pi^2\right)\mathrm{Cos}[2\,\pi\,x]\right)\mathrm{Sin}[t-z]^2\right)\Big)$$

```
(* rescale *)
dA = dA 32 π^2 Sqrt[1 + 2 π^2]
```

$$2\,\lambda \cos[\pi x]^2 \sin[\pi y]^2 \left(\cos[t-z]^2 + \left(1+2\pi^2\right)\sin[t-z]^2\right) + \lambda \cos[\pi y]^2 \left(2\cos[t-z]^2 \sin[\pi x]^2 + \left(1+6\pi^2+\left(-1+2\pi^2\right)\cos[2\pi x]\right)\sin[t-z]^2\right)$$

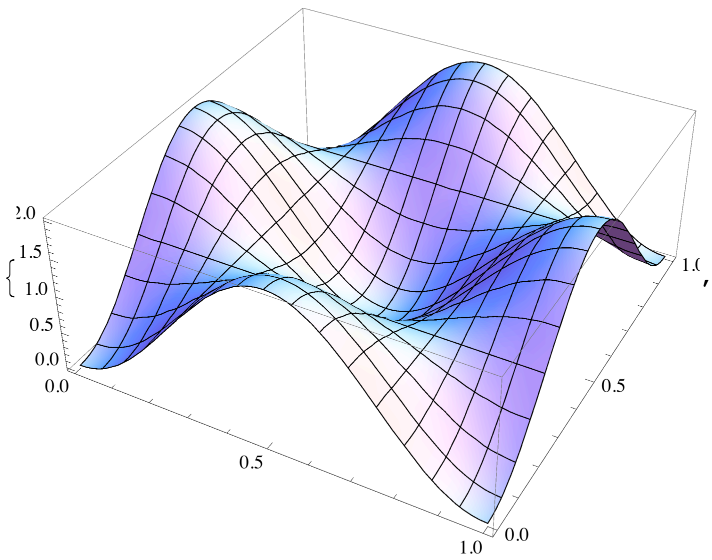
```
Aplus = dA /. λ → 1
Aminus = dA /. λ → -1;
FullSimplify[Aplus + Aminus == 0]
(* the last computation shows that TE+
 is a contact form if and only if TE- is a contact form *)
```

$$2\cos[\pi x]^2 \sin[\pi y]^2 \left(\cos[t-z]^2 + \left(1+2\pi^2\right)\sin[t-z]^2\right) + \cos[\pi y]^2 \left(2\cos[t-z]^2 \sin[\pi x]^2 + \left(1+6\pi^2+\left(-1+2\pi^2\right)\cos[2\pi x]\right)\sin[t-z]^2\right)$$

True

```
polP[x_, y_, z_, t_] := 2 Cos[π x]^2 Sin[π y]^2 (Cos[t - z]^2 + (1 + 2 π^2) Sin[t - z]^2) +
  Cos[π y]^2 (2 Cos[t - z]^2 Sin[π x]^2 + (1 + 6 π^2 + (-1 + 2 π^2) Cos[2 π x]) Sin[t - z]^2)

Table[Plot3D[polP[x, y, 2 Pi k / 16, 0],
  {x, 0, 1}, {y, 0, 1}, PlotRange → Automatic], {k, 0, 16}]
```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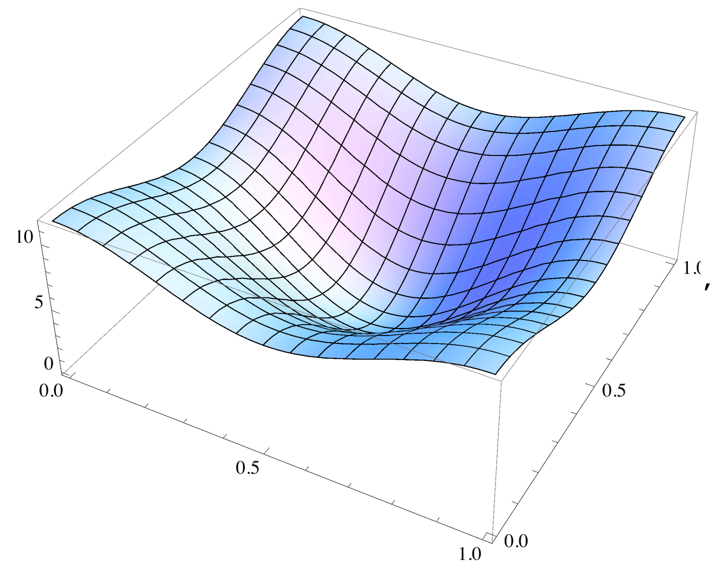
0.5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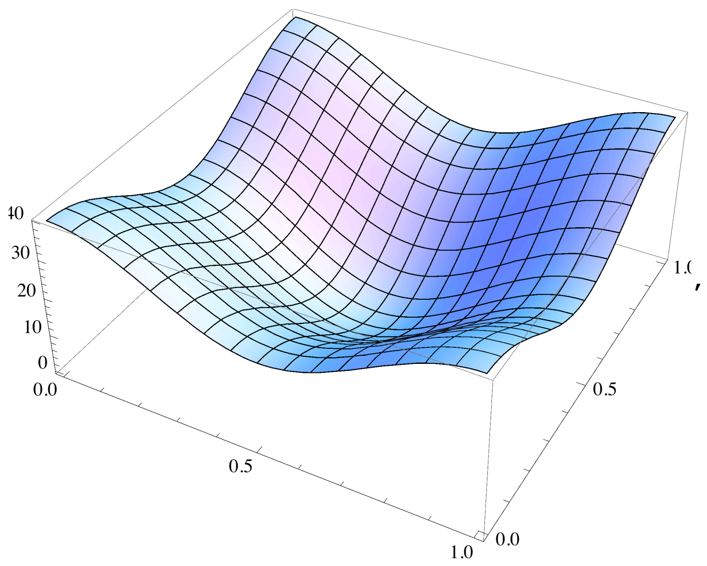
0.5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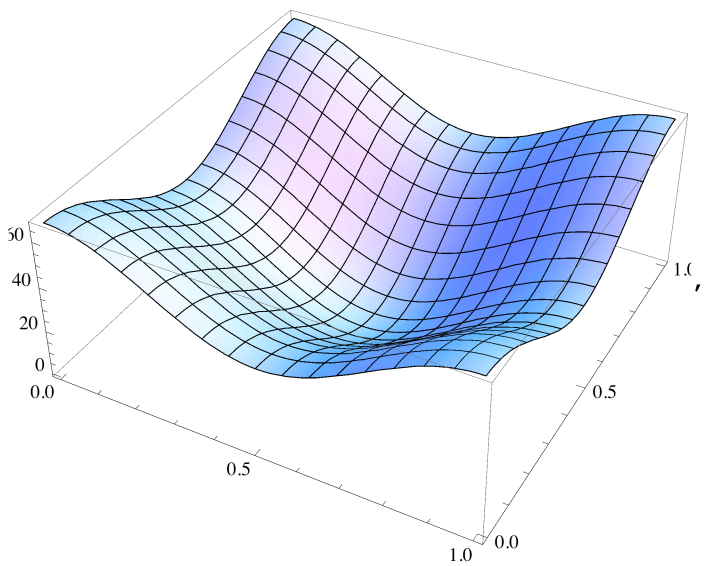
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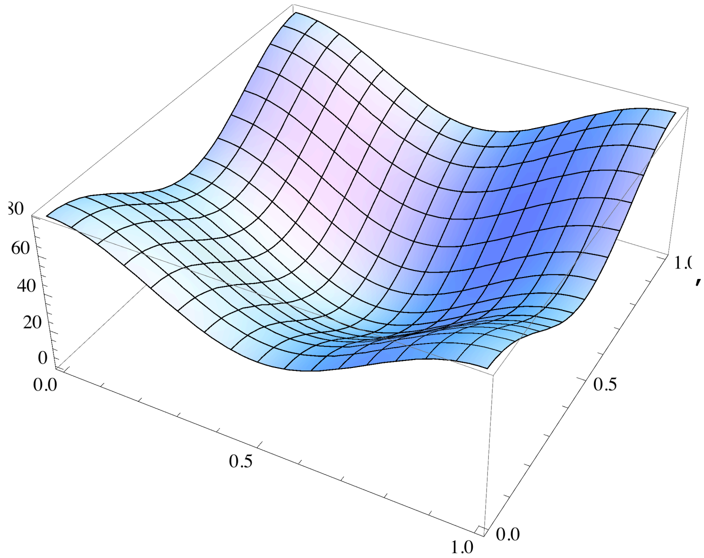
1.(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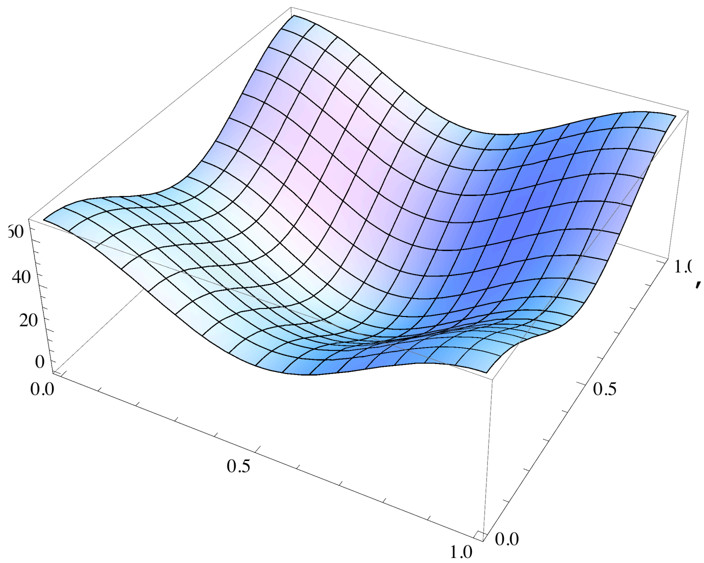
1.0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20
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0.5
1.0
0.0
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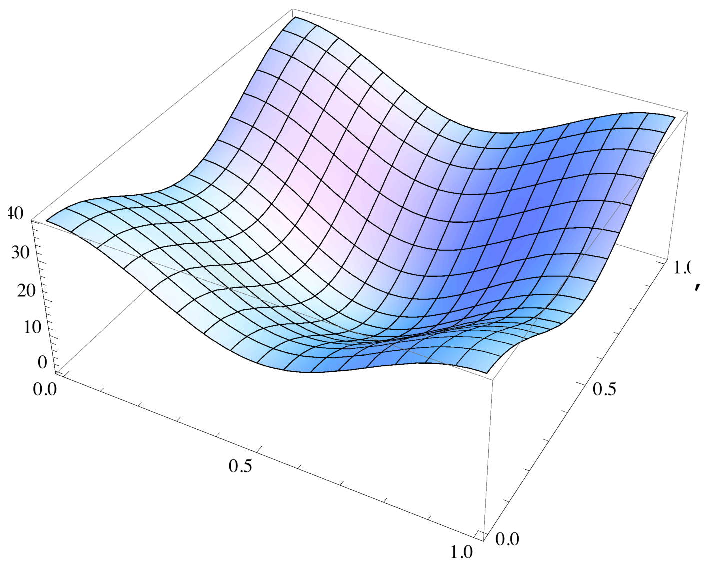
1.0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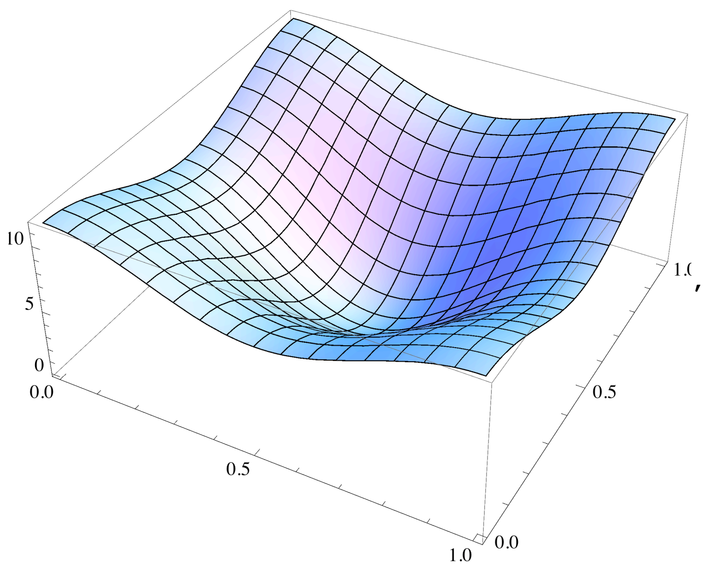
0.0
0.5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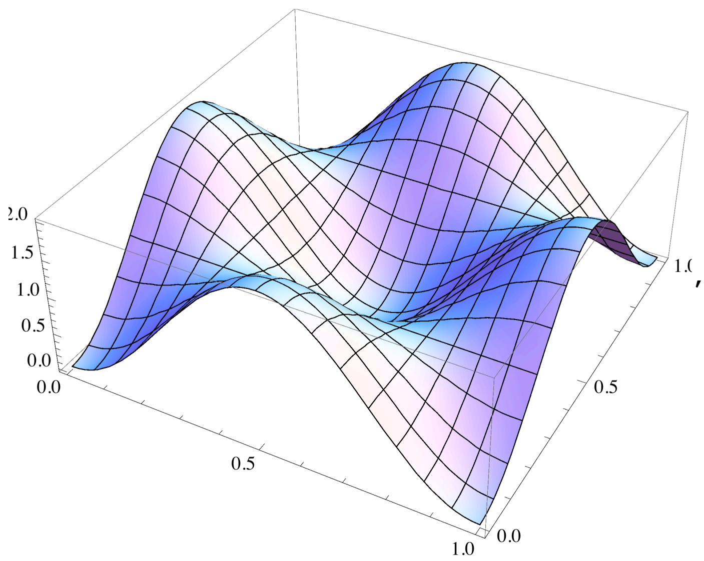
0
0.0
0.5
1.0
0.0
0.5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1.0
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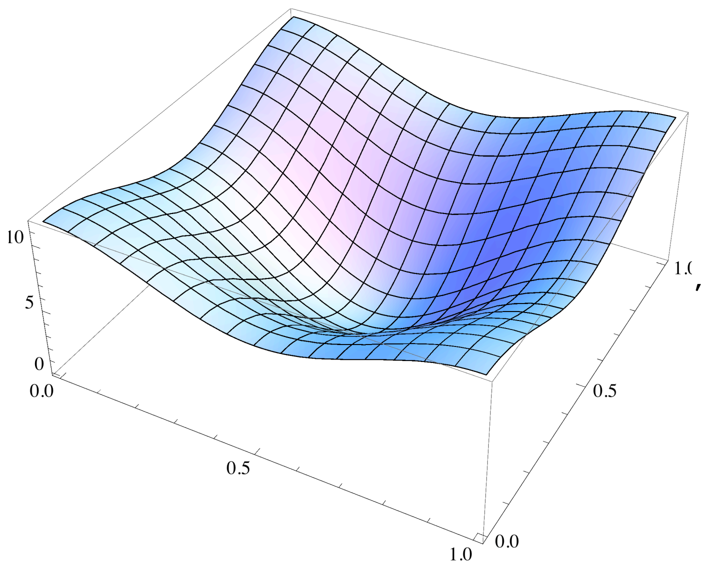
0.0
0.5
1.0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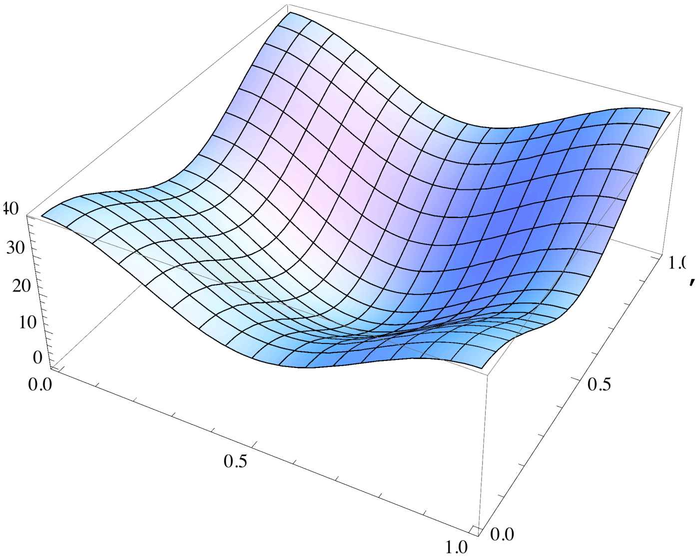
1.0
0.0
0.5
1.0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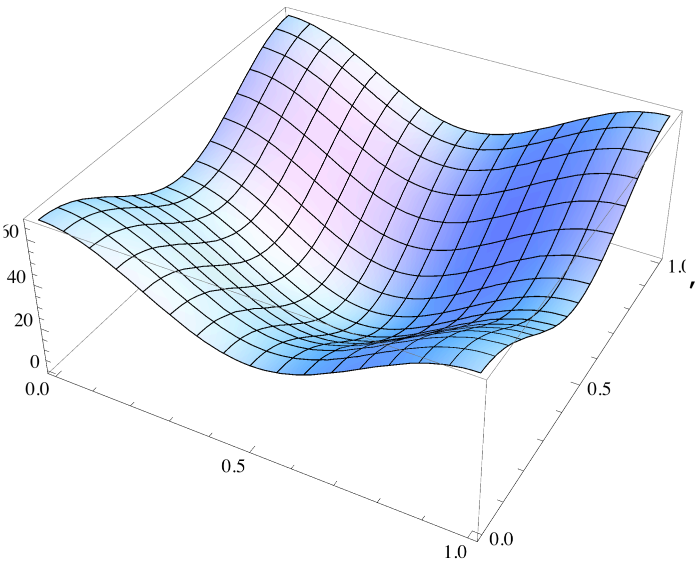
0.5
1.0
0.0
0.5
1.(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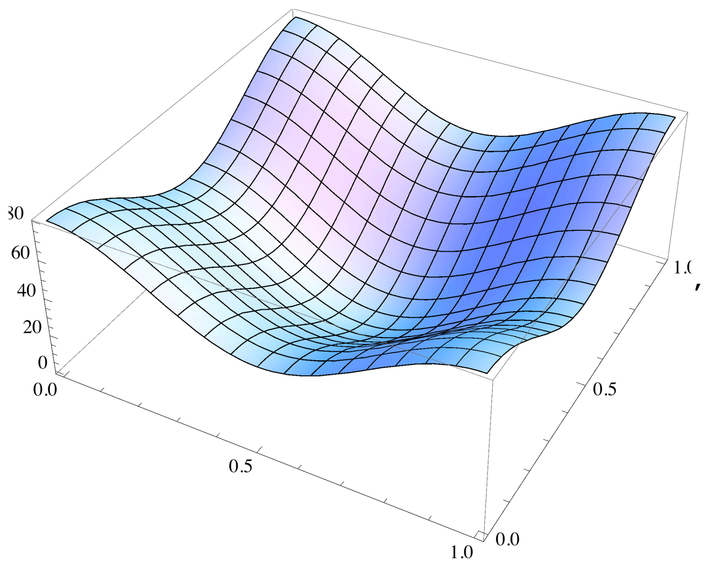
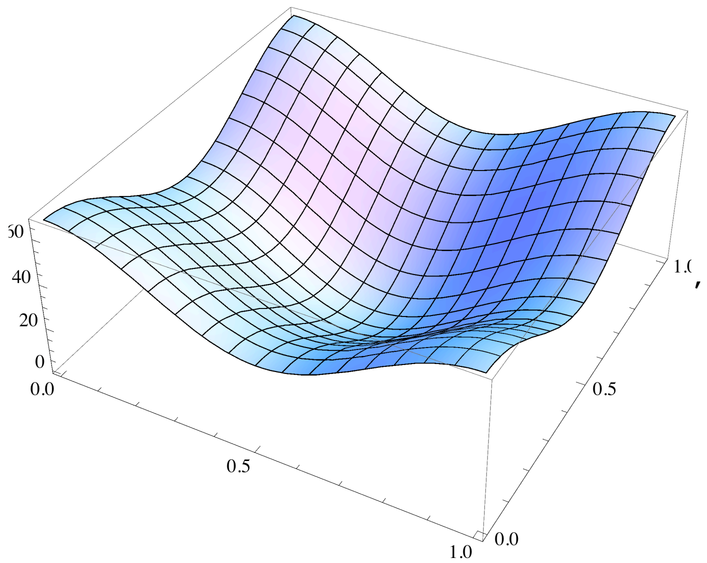
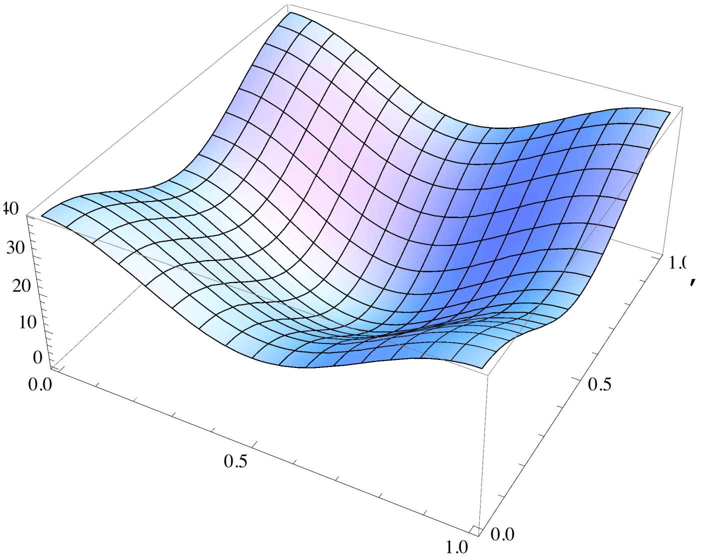
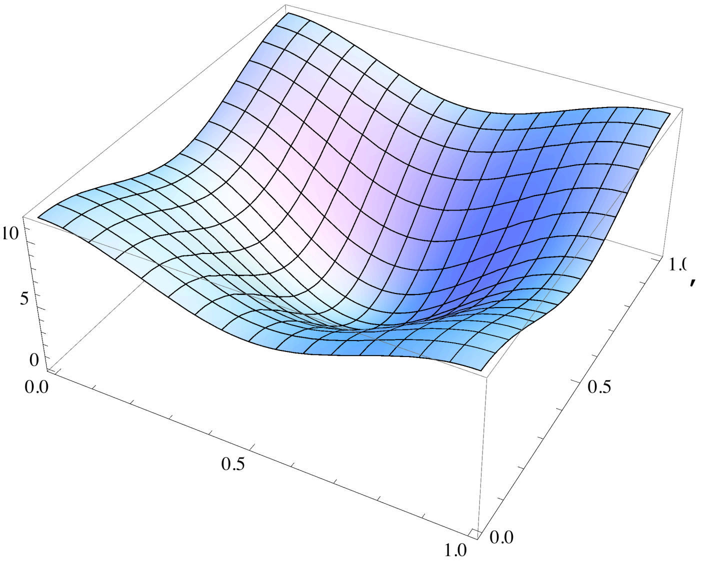
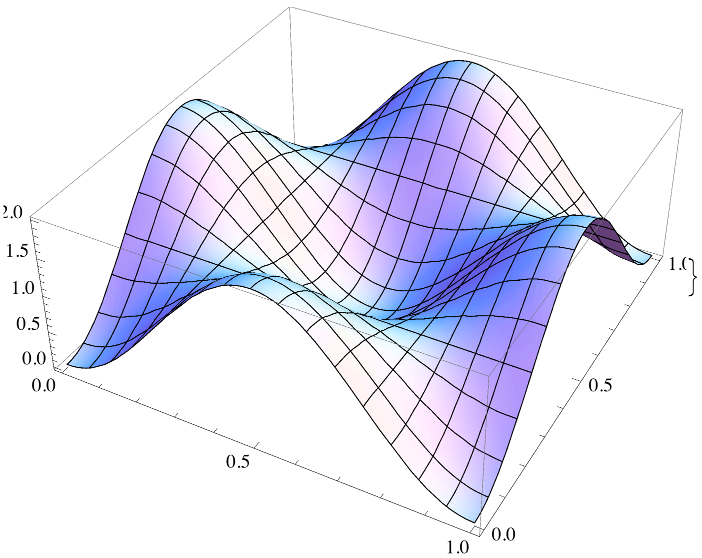
20
0
0.0
0.5
1.0
0.0
0.5
1.(


80
60
40
20
0
0.0
0.5
1.0
0.0
0.5
1.0


60
40
20
0
0.0
0.5
1.0
0.0
0.5
1.0


40
30
20
10
0
0.0
0.5
1.0
0.0
0.5
1.(

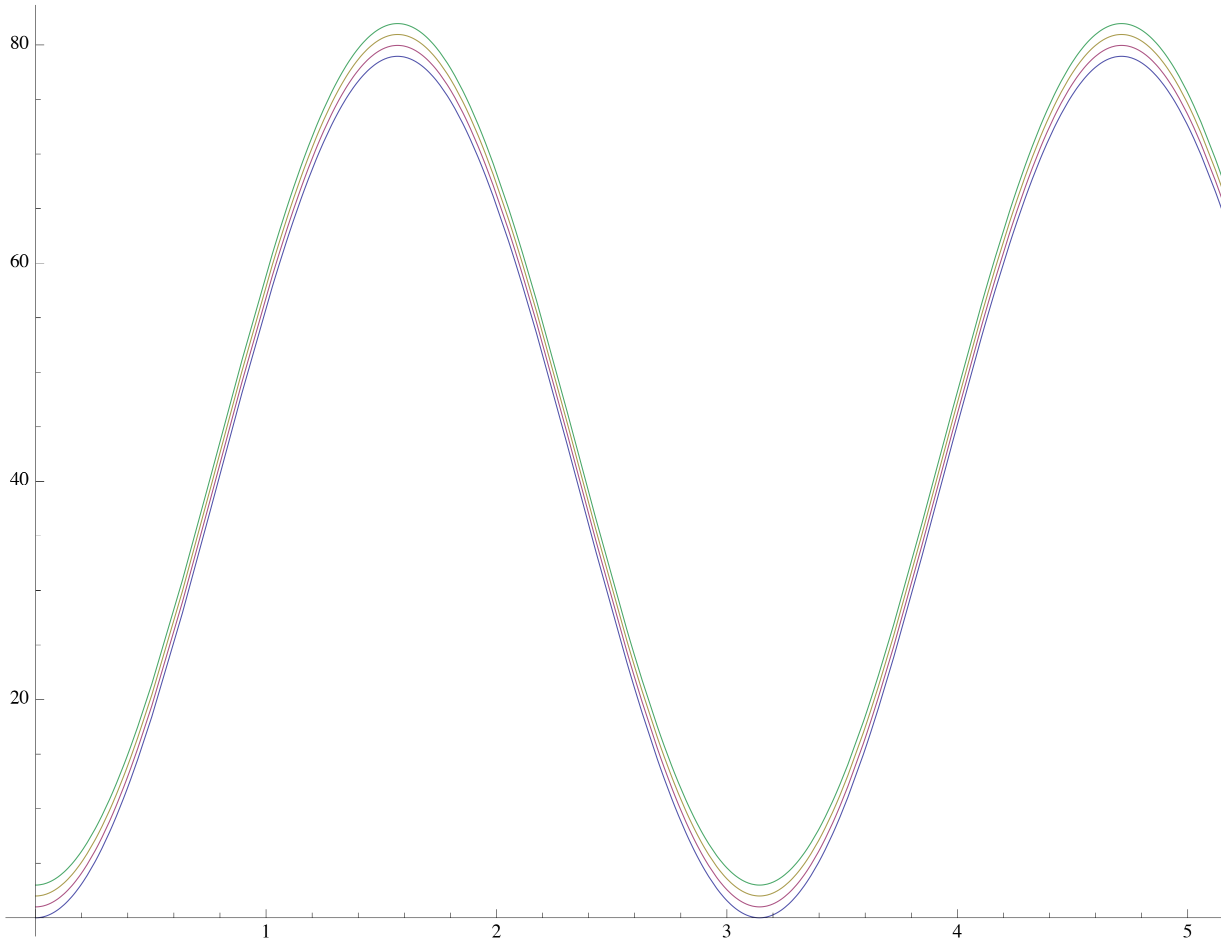

10
5
0
0.0
0.5
1.0
0.0
0.5
1.(


2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
0.0
0.5
1.(

```
(* the contact condition fails in the middle of the
  wave guide for both forms. This can also be seen from the graphs
  above. We only need to check this for polP since polM = -polP.
*)
polP[1 / 2, 1 / 2, z , t]
```

0

```
(* condition fails in all corners when z = 0, z = Pi and z=2Pi and t = 0 *)
Plot[{polP[0, 0, z , 0], polP[0, 1, z , 0] + 1, polP[1, 0, z , 0] + 2,
  polP[1, 1, z , 0] + 3}, {z, 0, 2 Pi}]
```



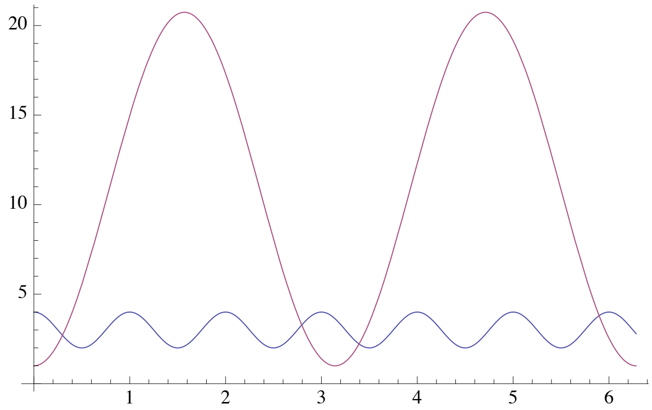
Extra material: a more rigorous proof of location for zeroes of alpha wedge dalpha.

```
(* see derivation below *)
Alt = 2 Cos[π y]^2 Sin[π x]^2 + 2 π^2 (3 + Cos[2 π x]) Cos[π y]^2 Sin[t - z]^2 +
   2 Cos[π x]^2 Sin[π y]^2 (Cos[t - z]^2 + (1 + 2 π^2) Sin[t - z]^2);

Simplify[Alt - dA]
```

0

```
Plot[{(3 + Cos[2 π x]), (Cos[x]^2 + (1 + 2 π^2) Sin[x]^2)}, {x, 0, 2 Pi}]
```



```
(* we can write Alt as: *)
Alt =
 2 Cos[π y]^2 Sin[π x]^2 + posFunc[x] Cos[π y]^2 Sin[t - z]^2 + 2 Cos[π x]^2 Sin[π y]^2 posFuncII[t - z]
(* where posFunc and posFuncII are strictly positive functions *)
```

$2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2 + 2\,\text{Cos}[\pi\,x]^2\,\text{posFuncII}[t-z]\,\text{Sin}[\pi\,y]^2 + \text{Cos}[\pi\,y]^2\,\text{posFunc}[x]\,\text{Sin}[t-z]^2$

Derivation of expression for dA:

```
dA = nonZero /. {m → 1, n → 1}
```

$2\,\text{Cos}[\pi\,x]^2\,\text{Sin}[\pi\,y]^2\,\left(\text{Cos}[t-z]^2 + \left(1+2\,\pi^2\right)\,\text{Sin}[t-z]^2\right) + \text{Cos}[\pi\,y]^2\,\left(2\,\text{Cos}[t-z]^2\,\text{Sin}[\pi\,x]^2 + \left(1+6\,\pi^2+\left(-1+2\,\pi^2\right)\,\text{Cos}[2\,\pi\,x]\right)\,\text{Sin}[t-z]^2\right)$

```
term1 = 2 Cos[π x]^2 Sin[π y]^2 (Cos[t - z]^2 + (1 + 2 π^2) Sin[t - z]^2);
term2 = Cos[π y]^2 (2 Cos[t - z]^2 Sin[π x]^2 + (1 + 6 π^2 + (-1 + 2 π^2) Cos[2 π x]) Sin[t - z]^2);
```

```
Simplify[dA == term1 + term2]
```

True

```
term2 /. {Cos[2 π x] → Cos[Pi x]^2 - Sin[Pi x]^2}
```

$\text{Cos}[\pi\,y]^2\,\left(2\,\text{Cos}[t-z]^2\,\text{Sin}[\pi\,x]^2 + \left(1+6\,\pi^2+\left(-1+2\,\pi^2\right)\,\left(\text{Cos}[\pi\,x]^2-\text{Sin}[\pi\,x]^2\right)\right)\,\text{Sin}[t-z]^2\right)$

```
Expand[%]
```

$2\,\text{Cos}[\pi\,y]^2\,\text{Cos}[t-z]^2\,\text{Sin}[\pi\,x]^2 + \text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 + 6\,\pi^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 - \text{Cos}[\pi\,x]^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 + 2\,\pi^2\,\text{Cos}[\pi\,x]^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 + \text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2\,\text{Sin}[t-z]^2 - 2\,\pi^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2\,\text{Sin}[t-z]^2$

```
% /. {Cos[t - z]^2 → 1 - Sin[t - z]^2}
```

$\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 + 6\,\pi^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 - \text{Cos}[\pi\,x]^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 + 2\,\pi^2\,\text{Cos}[\pi\,x]^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[t-z]^2 + \text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2\,\text{Sin}[t-z]^2 - 2\,\pi^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2\,\text{Sin}[t-z]^2 + 2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2\,\left(1-\text{Sin}[t-z]^2\right)$

```
Collect[%, Sin[t - z]^2]
```

$2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2 + \left(\text{Cos}[\pi\,y]^2 + 6\,\pi^2\,\text{Cos}[\pi\,y]^2 - \text{Cos}[\pi\,x]^2\,\text{Cos}[\pi\,y]^2 + 2\,\pi^2\,\text{Cos}[\pi\,x]^2\,\text{Cos}[\pi\,y]^2 - \text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2 - 2\,\pi^2\,\text{Cos}[\pi\,y]^2\,\text{Sin}[\pi\,x]^2\right)\,\text{Sin}[t-z]^2$

```
Simplify[(Cos[π y]^2 + 6 π^2 Cos[π y]^2 - Cos[π x]^2 Cos[π y]^2 +
   2 π^2 Cos[π x]^2 Cos[π y]^2 - Cos[π y]^2 Sin[π x]^2 - 2 π^2 Cos[π y]^2 Sin[π x]^2)]
```

$2\,\pi^2\,\left(3+\text{Cos}[2\,\pi\,x]\right)\,\text{Cos}[\pi\,y]^2$

```
term2Alt = 2 Cos[π y]^2 Sin[π x]^2 + 2 π^2 (3 + Cos[2 π x]) Cos[π y]^2 Sin[t - z]^2;
Simplify[term2 == term2Alt]
```

True

```
Simplify[dA == term1 + term2Alt]
```

True

```
term1 + term2Alt
```

$$2\,\mathrm{Cos}[\pi y]^2\,\mathrm{Sin}[\pi x]^2 + 2\pi^2\,(3 + \mathrm{Cos}[2\pi x])\,\mathrm{Cos}[\pi y]^2\,\mathrm{Sin}[t-z]^2 + 2\,\mathrm{Cos}[\pi x]^2\,\mathrm{Sin}[\pi y]^2\,\left(\mathrm{Cos}[t-z]^2 + \left(1 + 2\pi^2\right)\mathrm{Sin}[t-z]^2\right)$$

TM_11 plus field contact forms

```
(* lambda = +1 or -1 *)Re[BelTM[x, y, z, m, n, λ]].Re[Curl[BelTM[x, y, z, m, n, λ]]] /.
  {ϵ → 1, μ → 1, a → 1, b → 1, β → 1, ω → 1};
FullSimplify[ComplexExpand[%]]
```

$$-\frac{1}{32\left(m^2+n^2\right)^2\pi^2}\sqrt{1+\left(m^2+n^2\right)\pi^2}\;\lambda\,\Big(\left(m^2+n^2\right)\left(-2-2\left(m^2+n^2\right)\pi^2+\mathrm{Cos}[2\pi\,(m x-n y)]+\mathrm{Cos}[2\pi\,(m x+n y)]-2\left(m^2+n^2\right)\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)+2\,\mathrm{Cos}[2 n\pi y]\left(m^2-n^2+m^2\left(m^2+n^2\right)\pi^2+m^2\left(m^2+n^2\right)\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)+2\,\mathrm{Cos}[2 m\pi x]\left(-m^2+n^2+n^2\left(m^2+n^2\right)\pi^2+n^2\left(m^2+n^2\right)\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)\Big)$$

```
dA = % /. {m → 1, n → 1}
```

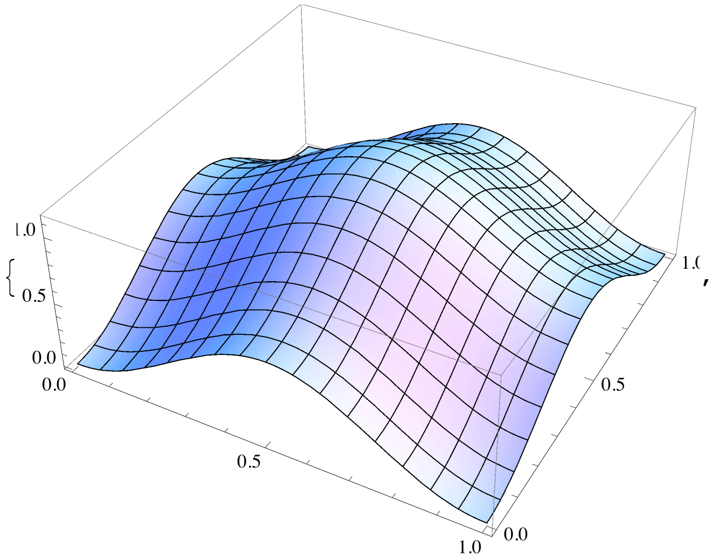
$$-\frac{1}{128\pi^2}\sqrt{1+2\pi^2}\;\lambda\,\Big(2\left(-2-4\pi^2+\mathrm{Cos}[2\pi\,(x-y)]+\mathrm{Cos}[2\pi\,(x+y)]-4\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)+2\,\mathrm{Cos}[2\pi x]\left(2\pi^2+2\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)+2\,\mathrm{Cos}[2\pi y]\left(2\pi^2+2\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)\Big)$$

```
Aplus = dA /. λ → 1
Aminus = dA /. λ → -1;
FullSimplify[Aplus + Aminus]
(* the last computation shows that TE+
 is a contact form if and only if TE- is a contact form *)
```

$$-\frac{1}{128\pi^2}\sqrt{1+2\pi^2}\;\Big(2\left(-2-4\pi^2+\mathrm{Cos}[2\pi\,(x-y)]+\mathrm{Cos}[2\pi\,(x+y)]-4\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)+2\,\mathrm{Cos}[2\pi x]\left(2\pi^2+2\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)+2\,\mathrm{Cos}[2\pi y]\left(2\pi^2+2\pi^2\,\mathrm{Cos}[2\,(t-z)]\right)\Big)$$

0

```
(* expression for lam = +1*)
pol[x_, y_, z_, t_] :=
  -1/(128 π^2) Sqrt[1 + 2 π^2] (2 (-2 - 4 π^2 + Cos[2 π (x - y)] + Cos[2 π (x + y)] - 4 π^2 Cos[2 (t - z)]) +
    2 Cos[2 π x] (2 π^2 + 2 π^2 Cos[2 (t - z)]) + 2 Cos[2 π y] (2 π^2 + 2 π^2 Cos[2 (t - z)]))
Table[Plot3D[pol[x, y, 2 Pi k / 20, 0], {x, 0, 1}, {y, 0, 1}], {k, 0, 20}]
```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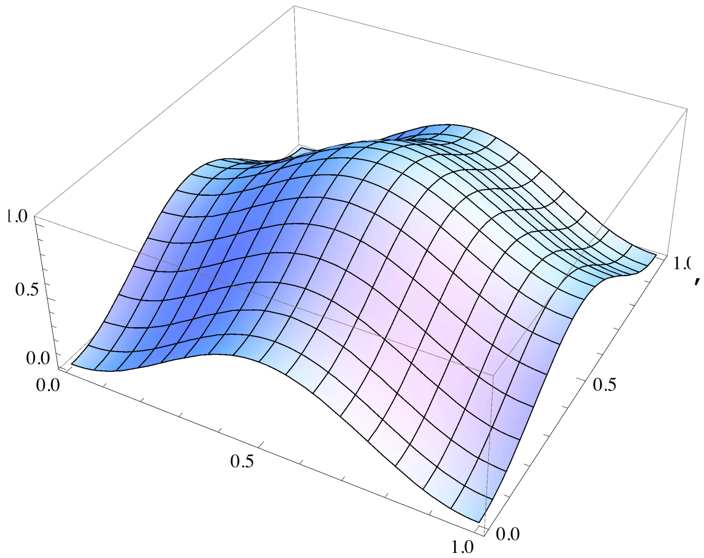
0.5
1.0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1.0
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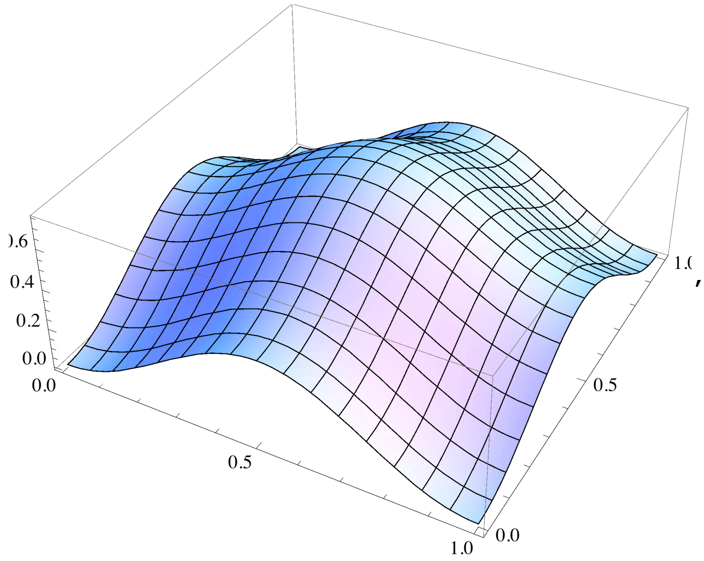
0.5
1.0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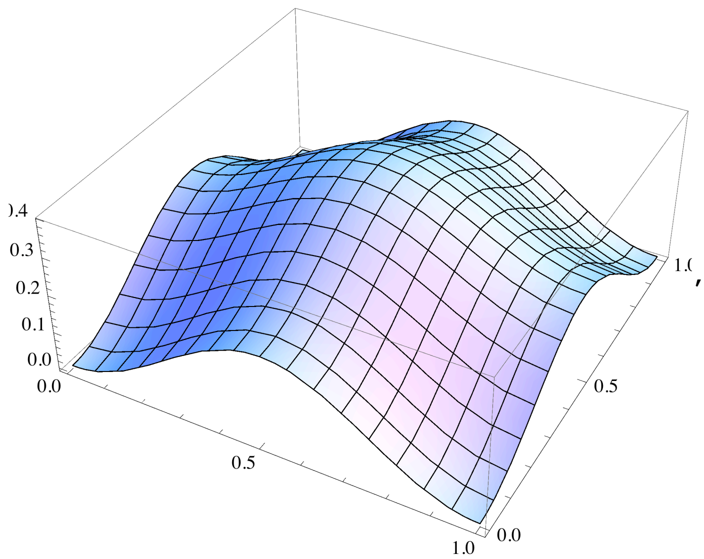
0.2
0.4
0.0
0.5
1.0
0.0
0.5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0.1
0.2
0.3
0.0
0.5
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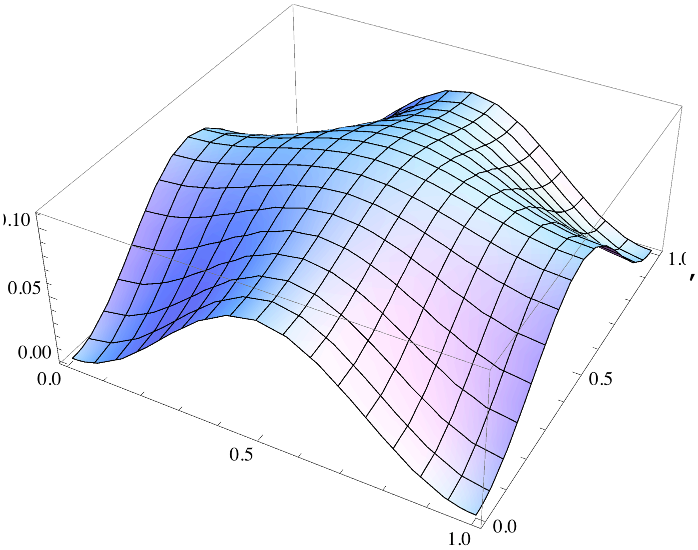
0.0
0.5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0.5
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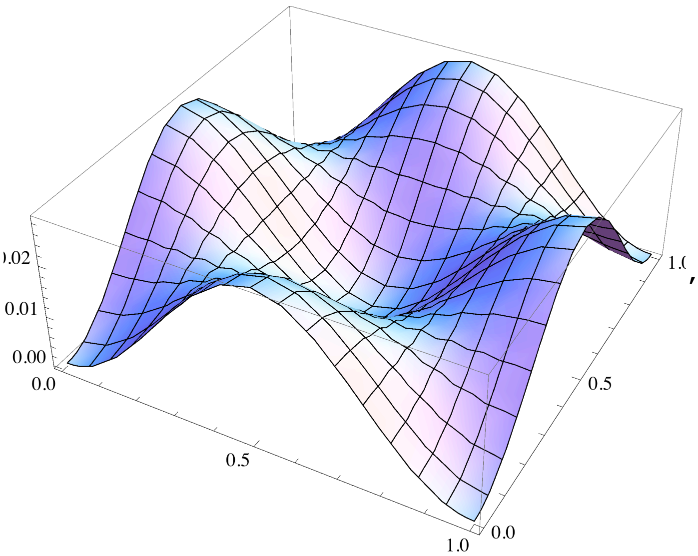
0.5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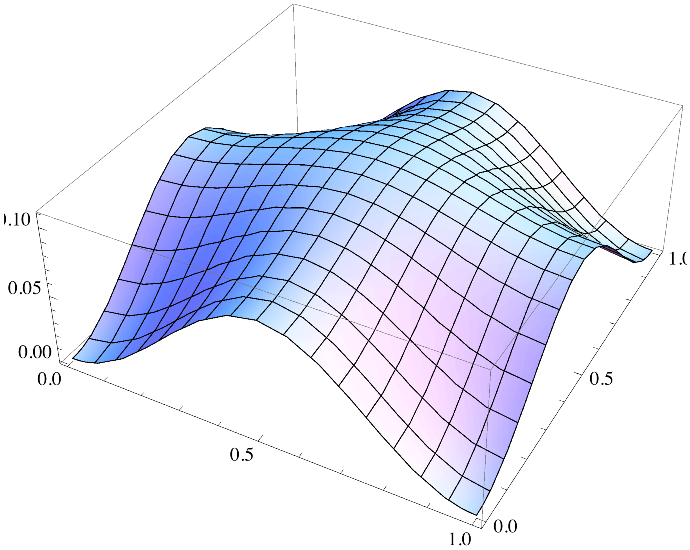

0.10
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0.0
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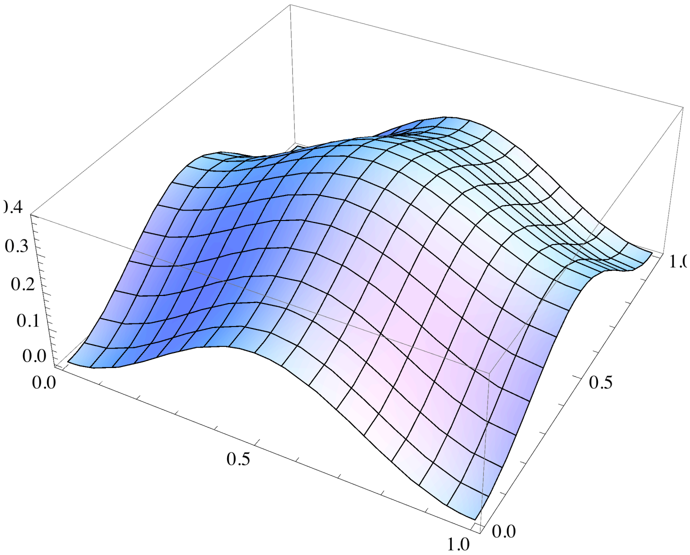
1.0
0.0
0.5
1.0

,



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1.0
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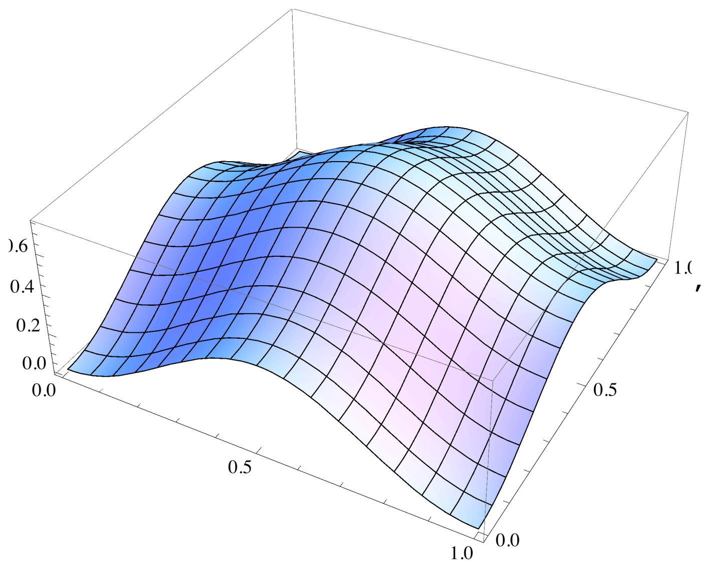
0.5
1.0

,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0.5
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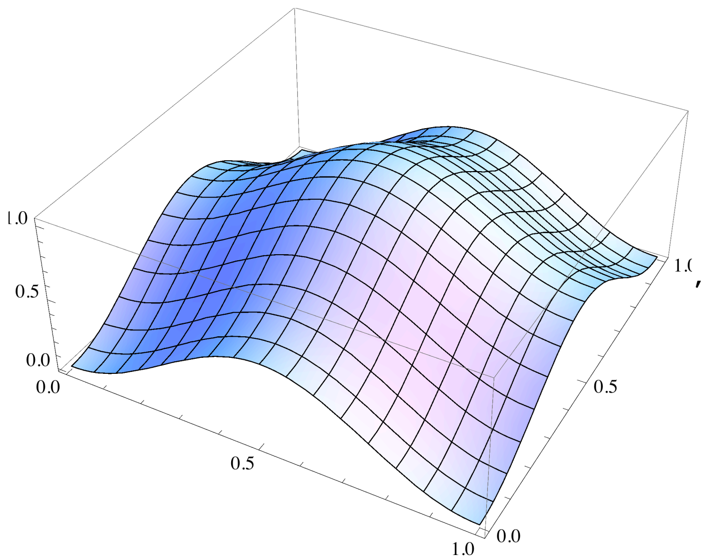
1.(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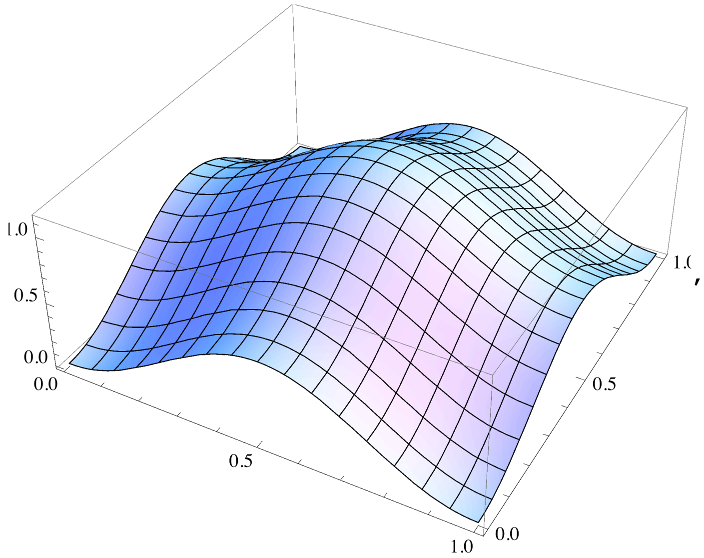
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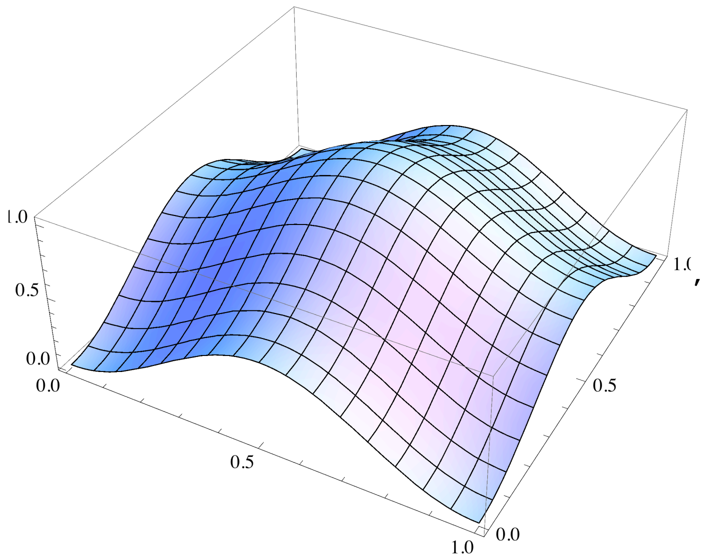
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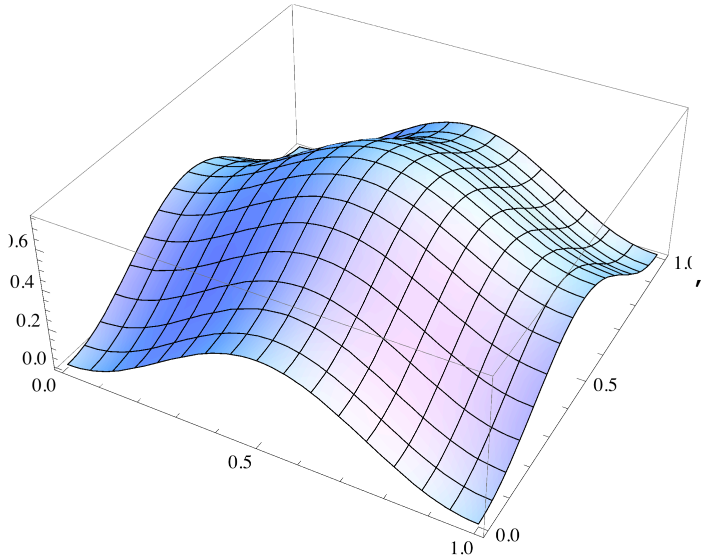
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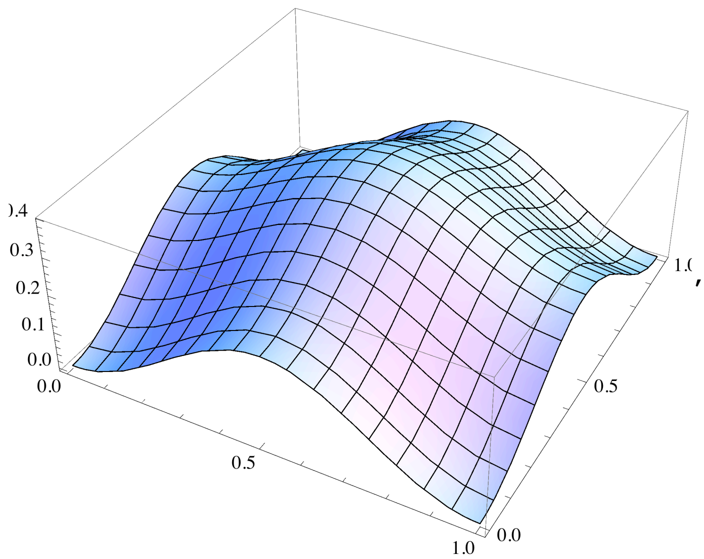
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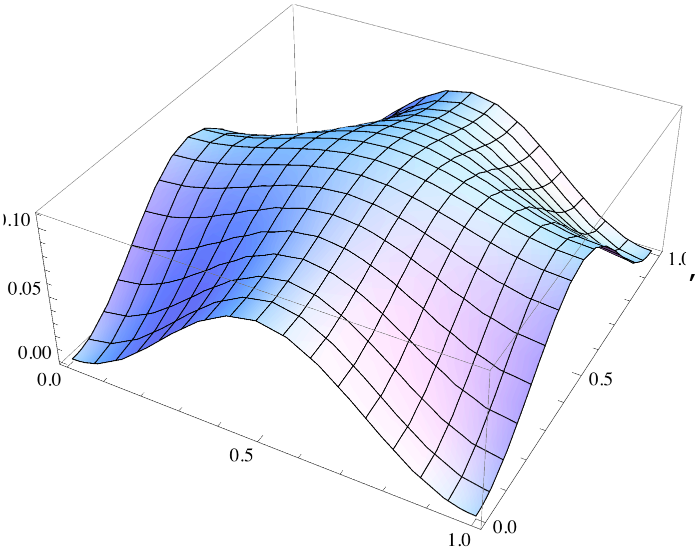
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0.0
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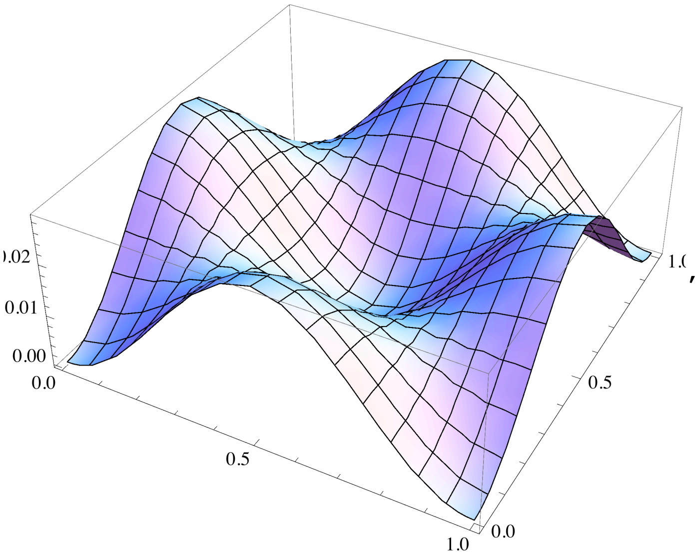
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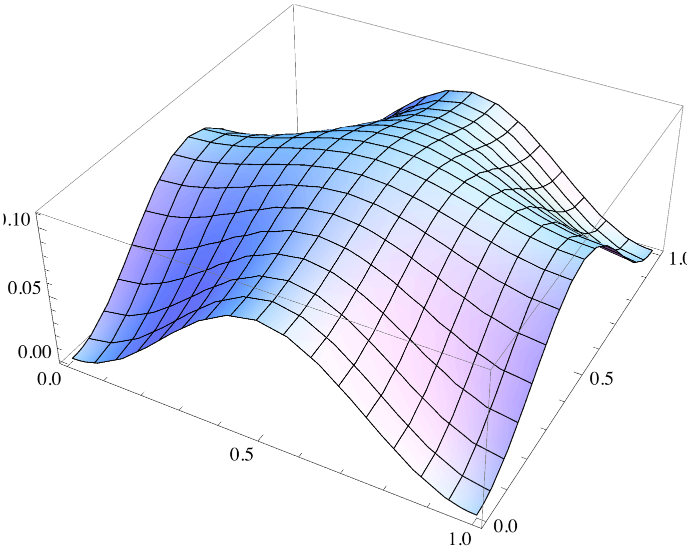
0.0
0.5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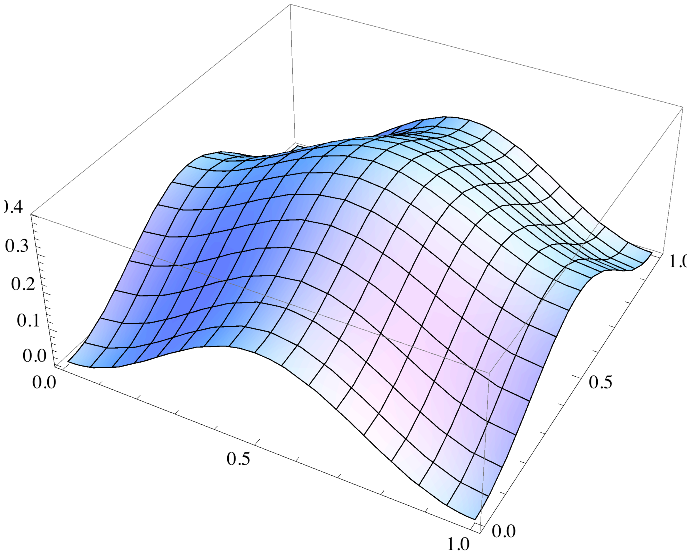
0.00
0.0
0.5
1.0
0.0
0.5
1.(

,


).4
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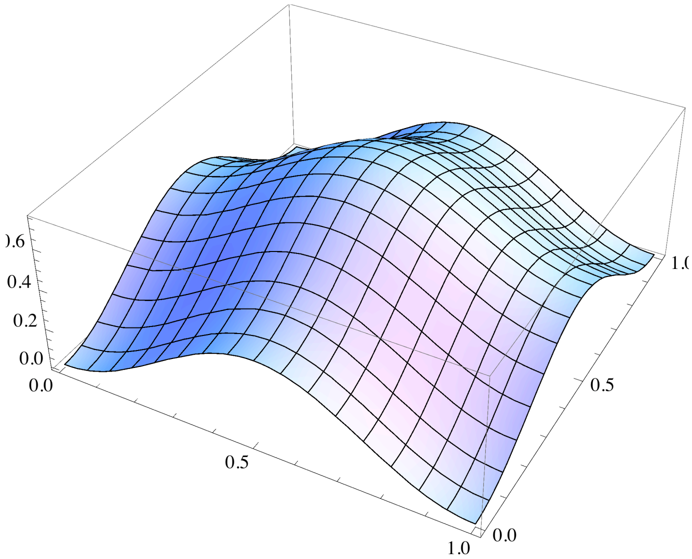
0.2
0.1
0.0
0.0
0.5
1.0
0.0
0.5
1.(

,



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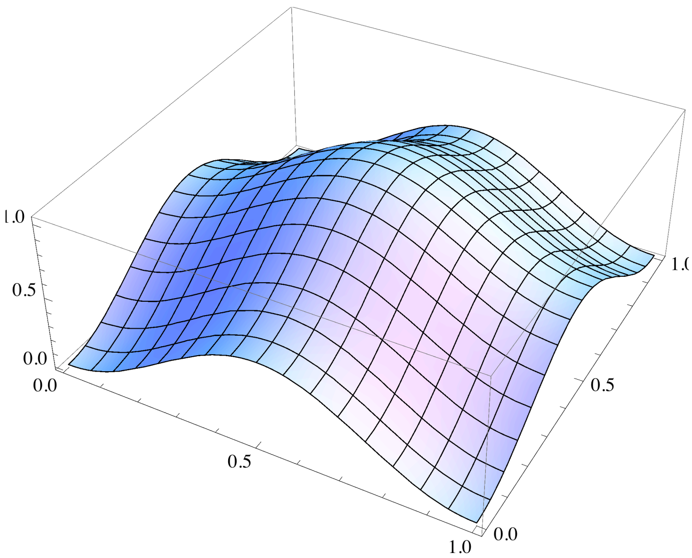

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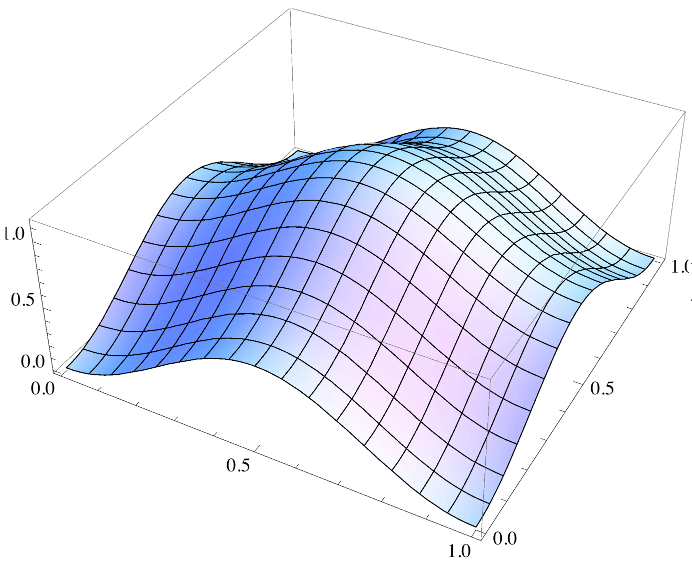
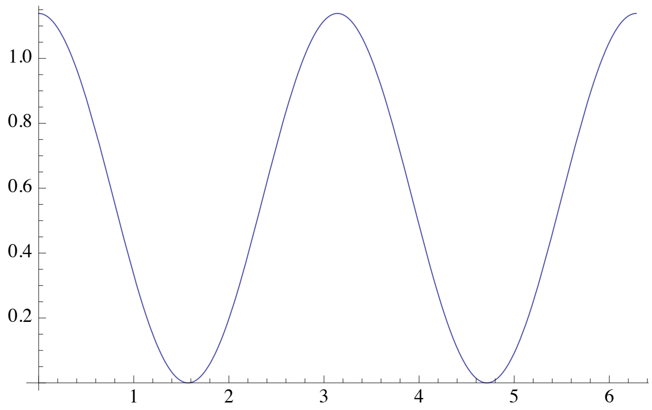


}

```
Plot[pol[1 / 2, 1 / 2, z, 0], {z, 0, 2 Pi}]
```




```
(* condition fails in all corners when z = kPi, t = 0 *)
Plot[{pol[0, 0, z , 0], pol[0, 1, z , 0] + 1, pol[1, 0, z , 0] + 2,
  pol[1, 1, z , 0] + 3}, {z, 0, 2 Pi}]
```