- **Z1=Z1.nb**
  Check that the two expressions for Z are equivalent.

```
(* the following function changes sign of kappa[a,b,i,j] so that a<b and i<j *)
kappaGeneral [a_, b_, i_, j_] :=
 Signature[{a, b}] Signature[{i, j}] κ[Min[{a, b}], Max[{a, b}], Min[{i, j}], Max[{i, j}]]
```

- **Constraints for**

  Z1 = {k in A^2_2 : k(u) ∧ v = u ∧ k(v) for all u,v and trace(k)=0 }

```
ConstraintsZ1 = {};

(* first add conditions for ku /\ v = u /\ kv *)
For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++,
  For[a = 1, a ≤ 4, a++,
   For[b = 1, b ≤ 4, b++,
    Exp1 = Sum[Signature[{i, j, p, q}] kappaGeneral[p, q, a, b], {p, 1, 4}, {q, 1, 4}];
    Exp2 = Sum[ Signature[{a, b, p, q}] kappaGeneral[p, q, i, j] , {p, 1, 4}, {q, 1, 4}];
    If[Simplify[Exp1 == Exp2, ConstraintsZ1], (*true*), (*false*), (*otherwise*)
     ConstraintsZ1 = Append[ConstraintsZ1, Exp1 == Exp2]
    ];
   ]
  ]
 ]
]

(* check that condition trace k=
 0 is independent of these conditions. If we obtain True,
then it is a consequence of the above conditions. *)
Simplify[0 == Sum[ kappaGeneral[i, j, i, j] , {i, 1, 4}, {j, 1, 4}], ConstraintsZ1]
```

$$\kappa[2, 3, 2, 3] + \kappa[2, 4, 2, 4] + \kappa[3, 4, 3, 4] == 0$$

```
(* add trace constraint, as we did not get True. *)
ConstraintsZ1 = Append[ConstraintsZ1,
   0 == Sum[ kappaGeneral[i, j, i, j] , {i, 1, 4}, {j, 1, 4}]
  ];

(* List all constraints *)
ConstraintsZ1 // MatrixForm
```

$$\begin{pmatrix} 2\kappa[3, 4, 1, 3] == -2\kappa[2, 4, 1, 2] \\ 2\kappa[3, 4, 1, 4] == 2\kappa[2, 3, 1, 2] \\ 2\kappa[3, 4, 2, 3] == 2\kappa[1, 4, 1, 2] \\ 2\kappa[3, 4, 2, 4] == -2\kappa[1, 3, 1, 2] \\ 2\kappa[3, 4, 3, 4] == 2\kappa[1, 2, 1, 2] \\ -2\kappa[2, 4, 1, 4] == 2\kappa[2, 3, 1, 3] \\ -2\kappa[2, 4, 2, 3] == 2\kappa[1, 4, 1, 3] \\ -2\kappa[2, 4, 2, 4] == -2\kappa[1, 3, 1, 3] \\ -2\kappa[2, 4, 3, 4] == 2\kappa[1, 2, 1, 3] \\ 2\kappa[2, 3, 2, 3] == 2\kappa[1, 4, 1, 4] \\ 2\kappa[2, 3, 2, 4] == -2\kappa[1, 3, 1, 4] \\ 2\kappa[2, 3, 3, 4] == 2\kappa[1, 2, 1, 4] \\ 2\kappa[1, 4, 2, 4] == -2\kappa[1, 3, 2, 3] \\ 2\kappa[1, 4, 3, 4] == 2\kappa[1, 2, 2, 3] \\ -2\kappa[1, 3, 3, 4] == 2\kappa[1, 2, 2, 4] \\ 0 == 2\kappa[1, 2, 1, 2] + 2\kappa[1, 3, 1, 3] + 2\kappa[1, 4, 1, 4] + 2\kappa[2, 3, 2, 3] + 2\kappa[2, 4, 2, 4] + 2\kappa[3, 4, 3 \end{pmatrix}$$

```
Length[ConstraintsZ1]
```

```
16
```

Constraints for

Z2 = {k in A^2_2 : k^ij_lj = 0 }

```
ConstraintsZ2 = {};

For[i = 1, i ≤ 4, i++,
 For[j = 1, j ≤ 4, j++,
  ConstraintsZ2 = Append[ConstraintsZ2,
    0 == Sum[ kappaGeneral[i, P, j, P] , {P, 1, 4}]
   ]
 ]
]
ConstraintsZ2 // MatrixForm
```

$$\begin{pmatrix} 0 == \kappa[1,2,1,2] + \kappa[1,3,1,3] + \kappa[1,4,1,4] \\ 0 == \kappa[1,3,2,3] + \kappa[1,4,2,4] \\ 0 == -\kappa[1,2,2,3] + \kappa[1,4,3,4] \\ 0 == -\kappa[1,2,2,4] - \kappa[1,3,3,4] \\ 0 == \kappa[2,3,1,3] + \kappa[2,4,1,4] \\ 0 == \kappa[1,2,1,2] + \kappa[2,3,2,3] + \kappa[2,4,2,4] \\ 0 == \kappa[1,2,1,3] + \kappa[2,4,3,4] \\ 0 == \kappa[1,2,1,4] - \kappa[2,3,3,4] \\ 0 == -\kappa[2,3,1,2] + \kappa[3,4,1,4] \\ 0 == \kappa[1,3,1,2] + \kappa[3,4,2,4] \\ 0 == \kappa[1,3,1,3] + \kappa[2,3,2,3] + \kappa[3,4,3,4] \\ 0 == \kappa[1,3,1,4] + \kappa[2,3,2,4] \\ 0 == -\kappa[2,4,1,2] - \kappa[3,4,1,3] \\ 0 == \kappa[1,4,1,2] - \kappa[3,4,2,3] \\ 0 == \kappa[1,4,1,3] + \kappa[2,4,2,3] \\ 0 == \kappa[1,4,1,4] + \kappa[2,4,2,4] + \kappa[3,4,3,4] \end{pmatrix}$$

```
Length[ConstraintsZ2]
```

```
16
```

- **Show that all constraints in Z1 are satisfied if constraints in Z2 hold, that is,**

Z2 subset Z1

```
out = {};
For[temp = 1, temp ≤ Length[ConstraintsZ1], temp++,
 out = Append[out,
    Simplify[ConstraintsZ1[[temp]], ConstraintsZ2]];
]
out
```

```
{True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True}
```

- **Show that all constraints in Z2 are satisfied if constraints in Z1 hold, that is,**

  Z1 subset Z2

```
out = {};
For[temp = 1, temp ≤ Length[ConstraintsZ2], temp++,
 out = Append[out,
    Simplify[ConstraintsZ2[[temp]], ConstraintsZ1]];
]
out
```

```
{True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True}
```

- **We have shown that Z1=Z2**