A rigidity theorem for two continuous functions

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Let $S = \{v \in \mathbb{R}^3 : \|v\| = 1\}$.

Proposition 0.1. *Suppose $f_1, f_2 \colon S \to \mathbb{R}$ are two continuous functions such that $f_1(x) = f_2(x)$ only for finitely many $x \in S$. Suppose further that $g_1, g_2 \colon S \to \mathbb{R}$ are continuous functions such that*

$$\{f_1, f_2\} = \{g_1, g_2\} \text{ on } S.$$

Then $f_1 = g_1$, $f_2 = g_2$ or $f_1 = g_2$, $f_2 = g_1$.

Proof. Let $S' \subset S$ be the set where $f_1 \neq f_2$. By assumption there exists a function $\pi \colon S' \to \{1, 2\}$ such that

$$g_1(x) = f_{\pi(x)}(x), \quad x \in S'. \tag{1}$$

To prove that π is constant on S', suppose that $x, y \in S'$ are such that $\pi(x) \neq \pi(y)$. Since S' is path connected, there exists a curve $\gamma \colon [0,1] \to S'$ connecting x to y. By the intermediate value theorem, there exists a $t \in [0,1]$ where $\pi \circ \gamma \colon [0,1] \to \mathbb{R}$ is not continuous; for all $\delta > 0$ there exists an $s \in [0,1]$ such that $|s - t| < \delta$ and $\pi \circ \gamma(t) \neq \pi \circ \gamma(s)$. In this way we can construct a sequence $x_1, x_2, \ldots \in S'$ converging to $\gamma(t)$, but such that $\pi(x_i) \neq \pi \circ \gamma(t)$ for all i. Then

$$\begin{aligned} f_{\pi\circ\gamma(t)}(\gamma(t)) &= g_1(\gamma(t)) \\ &= \lim_{i\to\infty} g_1(x_i) \\ &= \lim_{i\to\infty} f_{\pi(x_i)}(x_i) \\ &= f_{\pi(x_1)}(\gamma(t)), \end{aligned}$$

so π is must be constant on S'. On $S \setminus S'$, we have

$$f_1 = f_2 = g_1 = g_2$$

and we can extend π to a constant function on S such that equation 1 holds. $\square$