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**Abstract:** *We show how the tools of computational algebra can be used to analyse the configuration space of multibody systems. One advantage of this approach is that the mobility can be computed without using the jacobian of the system. As an example we treat thoroughly the well known Bricard's mechanism, but the same methods can be applied to arbitrary multibody systems. It turns out that the configuration space of Bricard's system is a smooth closed curve which can be explicitly parametrized. Our computations also yield a new formulation of constraints which is better than the original one from the point of view of numerical simulations.*

**AMS subject classifications:** 70B15, 13P10, 70G25

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# 1 Introduction

A basic problem in the study of mechanisms is determining the *mobility* (or the number of degrees of freedom) of the given system. For an open chain this is a rather trivial task, but if the mechanism contains closed loops the situation can be very complicated [1, 9]. For a long time there have been attempts to find a formula which would give the mobility without actually analysing the equations defining the constraints. The names of Kutzbach and Grübler are frequently cited in the literature, but also many others have proposed various formulas, and apparently the first to consider this problem was Chebychev [9]. In spite of all activity it appears that no general formula has been found, and indeed it is not even clear if such a formula can exist.

In this article we show how one can actually compute the mobility, in other words the dimension of the configuration space. The approach is based on computational ideal theory, and the Gröbner bases and the Buchberger algorithm to compute them play a central role. As an example we compute the mobility of the well-known Bricard's mechanism [4]. This system is called overconstrained or paradoxical which means that various formulas do not give the correct mobility: the usual formulas give zero mobility for Bricard's mechanism while it is well-known that the correct mobility is one. Bricard himself was obviously very interested in these kind of mechanisms and in addition to the mechanism analysed here he gave several other examples of paradoxical systems [4]. It is less clear if he thought that they are important in the practical design of machines. Overconstrained mechanisms have been been analyzed previously by means of differential geometry in [14] and [15]. In [15] it was shown that paradoxical mechanisms are "rare" in the space of "all" mechanisms. However, the author still concludes that the study of such mechanisms remains an important topic.

We said that Gröbner bases allow one to compute the mobility of the mechanisms; of course this applies to all mechanisms, not just paradoxical ones. Note that traditionally the jacobian of the map defining the constraints is used in the analysis of the mobility. The idea is to determine the rank of the jacobian and then infer the dimension of the configuration space using the implicit function theorem. This is necessarily a local process since it is quite conceivable, and in fact this frequently occurs in practice, that the rank of the jacobian is not constant. In our approach the jacobian is not needed and the computations make sense globally, not just in a neighborhood of some point in the configuration space.

The points where the rank of the jacobian drops are singular points of the configuration space. Our approach then shows that the singularities are irrelevant in the computation of the mobility. This makes sense also intuitively: the set of singular points is necessarily of lower dimension than the configuration space itself. Hence almost all points in the configuration space are smooth and it is natural that the dimension is determined by them. If one wants to analyse the singularities of the system then the jacobian is needed. We will give below a few remarks about this but do not treat this in any generality because it turned out that there are no singularities in Bricard's mechanism. Incidentally we do not know if the absence of singularities has been shown previously.

But Gröbner bases give even more information than the mobility. In the present case our analysis yields the configuration space of Bricard's mechanism *explicitly*: the essential part of it can be described very simply as a closed curve in 3 dimensional space. Obviously we cannot expect such a strong result in the general case. However, the analysis of the configuration space with the tools presented below might well reveal properties of the configuration space which are not easily available by other means.

To be able to use computational algebra all constraints must be expressed in terms of polynomials. For planar mechanisms this is rather straightforward and in fact Gröbner bases have already been used to analyze planar mechanisms [2, 3, 7]. As far as we know 3 dimensional case has not been treated previously in this way. To be able to formulate constraints in terms of polynomials we represent orientations of bodies in terms of Euler parameters. It turns out that most constraints arising in practice can be formulated using just 3 basic constraints and these are all low order polynomials of Euler parameters and centers of mass.

Our analysis is also useful from the point of view of numerical simulation of multibody systems. In fact Bricard's mechanism has been used as a test problem for numerical codes for multibody systems [10]. The difficulty of solving Bricard's mechanism is directly related to its overconstrained nature: the standard formulation of constraints gives a map whose jacobian is everywhere rank deficient. Now whatever the numerical method used to solve the equations of motion the rank deficient jacobian surely leads to trouble. This is of course the reason for choosing Bricard's system as a benchmark problem. However, our analysis gives a new set of constraints whose jacobian is of full rank. Moreover the structure of the jacobian is very simple: it is quite sparse, most of the nonzero terms are constant and except for a block of size $2 \times 3$ it is in a triangular form. We can expect similar results in more general situations: the preliminary analysis of the configuration space may well lead to a formulation of constraints which are much more suitable for numerical computations than the standard formulation. Note finally that since we give the configuration space of Bricard's mechanism explicitly this can be used to test the kinematical validity of any numerical simulation of the equations of motion.

We think that the idea to use computational algebra and algebraic geometry to analyse configuration spaces is quite natural, yet we were unable to find articles with similar approach in the literature. Our article is also a direct continuation of our previous work [3, 2, 17, 16]. All computations below were done by a publicly available freeware program Singular [12]. We used standard PCs and laptops and none of the computations reported below took more than few minutes, and most of them took just few seconds.

The paper is organized as follows: in Section 2 we first recall some necessary algebraic and geometric notions regarding the correspondence of ideals and varieties. Then we briefly indicate how to algorithmically manipulate ideals and discuss some properties of Gröbner bases. In Section 3 we introduce some basic ideas of multibody systems and show how to write the relevant constraint equations. In Section 4 we then define Bricard's mechanism and formulate the relevant equations whose zero set defines the configuration space. Section 5 is the main part of the article where we decompose the variety defined by the constraints. This decomposition is useful because it reveals that the original equations allow spurious solutions which do not correspond to the physical situation one tries to model. After eliminating all the spurious components we eventually obtain the irreducible part of the variety which is physically relevant. In Section 6 we parametrize this variety and finally in Section 7 we give some conclusions and perspectives for future work.

# 2 Algebraic preliminaries

## 2.1 Notation

The standard orthonormal basis vectors in $\mathbb{R}^3$ are denoted by

$$e^1 = (1,0,0) \quad , \quad e^2 = (0,1,0) \quad , \quad e^3 = (0,0,1)\ .$$

The euclidean inner product of two vectors $x$, $y \in \mathbb{R}^n$ is denoted by $(x, y)$. The length of a vector $x$ is denoted by $|x|$ and the $n$-dimensional unit sphere is denoted by $S^n$. Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map. Its first differential or jacobian is denoted by $dg$, and the jacobian evaluated at $p$ is $dg_p$. The orthogonal and special orthogonal groups are

$$\mathbb{O}(n) = \{A \in \mathbb{R}^{n\times n} \mid\ A^T A = I\ \} \quad , \quad \mathbb{SO}(n) = \{A \in \mathbb{O}(n) \mid \det(A) = 1\ \}\ .$$

## 2.2 Ideals and varieties

For more information on basic ideal theory we refer to [5]. Let $\mathbb{K}$ be one of $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, and let $\mathbb{A} = \mathbb{K}[x_1, \ldots, x_n]$ be the ring of polynomials with coefficients in $\mathbb{K}$. A subset $\mathcal{I} \subset \mathbb{A}$ is an *ideal* if it satisfies

(i) $0 \in \mathcal{I}$.

(ii) If $f, g \in \mathcal{I}$, then $f + g \in \mathcal{I}$.

(iii) If $f \in \mathcal{I}$ and $h \in \mathbb{A}$, then $hf \in \mathcal{I}$.

Given some polynomials $g_1, \ldots, g_k$ we may view them both as a map $g : \mathbb{K}^n \to \mathbb{K}^k$ and as generators of an ideal

$$\mathcal{I} = \langle g_1, \ldots, g_k\rangle = \left\{ \sum_{i=1}^{k} h_i g_i \mid h_1, \ldots, h_k \in \mathbb{A} \right\} \subset \mathbb{A} \tag{1}$$

A set of generators of an ideal is also called a *basis* of an ideal. The common zero set of all $g_i$ is called an (affine) *variety*; if $\mathcal{I}$ is the corresponding ideal, its variety is denoted by $\mathsf{V}(\mathcal{I})$. The *radical* of $\mathcal{I}$ is

$$\sqrt{\mathcal{I}} = \big\{ f \in \mathbb{A} \,|\, f^n \in \mathcal{I} \ \text{ for some } \ n \geq 1 \big\}.$$

Note that $\mathsf{V}(\mathcal{I}) = \mathsf{V}(\sqrt{\mathcal{I}})$. Next we will need to add ideals. Let $I_1 = \langle f_1, \ldots, f_s\rangle$ and $I_2 = \langle g_1, \ldots, g_r\rangle$; then

$$I_1 + I_2 = \langle f_1, \ldots, f_s, g_1, \ldots, g_r\rangle\ .$$

In terms of varieties this means that $\mathsf{V}(I_1 + I_2) = \mathsf{V}(I_1) \cap \mathsf{V}(I_2)$.

An ideal $\mathcal{I}$ is *prime* if $fg \in \mathcal{I}$ imply that either $f \in \mathcal{I}$ or $g \in \mathcal{I}$. We will often in the sequel use the following fact: any radical ideal is a finite intersection of prime ideals:

$$\sqrt{\mathcal{I}} = I_1 \cap \cdots \cap I_r \ . \tag{2}$$

The prime ideals $I_i$ are called the *minimal associated primes* of $\mathcal{I}$. This gives the decomposition of the variety into *irreducible components*:

$$\mathsf{V}(\mathcal{I}) = \mathsf{V}(\sqrt{\mathcal{I}}) = \mathsf{V}(I_1) \cup \cdots \cup \mathsf{V}(I_r) \ .$$

In the analysis below theoretically the most straightforward way to proceed would be to compute this decomposition and then choose the prime ideal/irreducible component one is interested in. However, this would be computationally infeasible, so we find the relevant component in steps. We will now outline the reasoning which will be used several times in the computations below.

Let us consider an ideal $I$ and let us suppose that we are interested in finding a certain component of $\mathsf{V}(I)$. Let us then divide the generators of $I$ into 2 sets: $I = J_1 + J_2$. Suppose now that the computation of prime decomposition of $\sqrt{J_1}$ is possible:

$$\sqrt{J_1} = P_1 \cap \cdots \cap P_r$$

Examining the ideals $P_i$ we conclude that a certain $P_\ell$ corresponds to the situation we want to study and we want to discard other $P_i$. Hence we continue our analysis with $\tilde{I} = P_\ell + J_2$. In terms of varieties this means that

$$\begin{aligned} \mathsf{V}(\tilde{I}) &= \mathsf{V}(P_\ell + J_2) = \mathsf{V}(P_\ell) \cap \mathsf{V}(J_2) \\ &\subset \mathsf{V}(P_1 \cap \cdots \cap P_r) \cap \mathsf{V}(J_2) = \mathsf{V}(\sqrt{J_1}) \cap \mathsf{V}(J_2) = \mathsf{V}(J_1) \cap \mathsf{V}(J_2) = \mathsf{V}(I) \end{aligned} \tag{3}$$

Hence we have eliminated some part of the initial variety $\mathsf{V}(I)$ as desired.

But how to find interesting splittings $I = J_1 + J_2$? Now the main obstacle in computations is that the complexity grows quite fast as a function of the number of variables. Hence it would be nice to have $J_1$ whose generators depend only on few variables. In other words we need *elimination ideals.* Let $I \subset \mathbb{K}[x_1, \ldots, x_k, \ldots, x_n]$. Then $I_k = I \cap \mathbb{K}[x_{k+1}, \ldots, x_n]$ is the $k$th elimination ideal of $I$. Of course the generators of the elimination ideal are usually not immediately available. However, it turns out that it is in fact possible to compute new generators for a given ideal such that the generators of the elimination ideal are a subset of all generators. But this is precisely the situation which we want: we write $I = J_1 + J_2$ where the generators of $J_1$ generate a certain elimination ideal.

## 2.3 Gröbner bases

An essential thing is that all the operations above, especially finding the generators of the elimininination ideals and the prime decomposition can be computed *algorithmically* using the given generators of $\mathcal{I}$. We will only briefly indicate the relevant ideas and refer to [5, 11] for more details.

First we need to introduce *monomial orderings.* All the algorithms handling the ideals are based on some orderings among the terms of the generators of the ideal. An ordering $\succ$ is such that given a set of monomials (e.g. terms of a given polynomial), $\succ$ puts them in order of importance: given any two monomials $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $x^\beta$, where $\alpha \neq \beta$ are different multi-indices, then either $x^\alpha \succ x^\beta$ or $x^\beta \succ x^\alpha$. In addition we require that for all $\gamma$, $x^\gamma \succ 1$ and $x^\alpha \succ x^\beta$ implies $x^{\alpha+\gamma} \succ x^{\beta+\gamma}$.

To compute elimination ideals we need *product orderings*. Let us consider the ring $\mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$ and let $\succ_A$ (resp. $\succ_B$) be an ordering for variables $x$ (resp. $y$). Then we can define the product ordering as follows:

$$x^\alpha y^\beta \succ x^\gamma y^\delta \quad \text{if} \quad \begin{cases} x^\alpha \succ_A x^\gamma & \text{or} \\ x^\alpha = x^\gamma \quad \text{and} \quad y^\beta \succ_B y^\delta & \end{cases} \tag{4}$$

Whenever we use product orderings we indicate it with parenthesis. For example

$$\mathbb{K}[(x_4, x_5, x_7), (x_1, x_2, x_3, x_6)]$$

is the same set as $\mathbb{K}[x_1, \dots, x_7]$ but the parenthesis indicate that we will use $\succ_A$ among the variables $(x_4, x_5, x_7)$, and $\succ_B$ among the variables $(x_1, x_2, x_3, x_6)$.

Now a *Gröbner basis* of a given ideal is a special kind of generating set, with respect to some ordering. An important fact is that given some ordering and some set of generators of an ideal, the corresponding Gröbner basis exists and can be computed. The relevant algorithm for computing Gröbner bases is usually called the *Buchberger algorithm*. The Gröbner bases have many special properties which are important in the analysis of the ideal and the corresponding variety. For us the essential property is the following.

**Lemma 2.1.** Let $\mathcal{I} \subset \mathbb{K}[x, y]$ be an ideal and $\mathcal{I}_n = \mathcal{I} \cap \mathbb{K}[y]$ the $n$th elimination ideal. If $G$ is a Gröbner basis for $\mathcal{I}$ with respect to a product ordering (4), then $G \cap \mathbb{K}[y]$ is a Gröbner basis for $\mathcal{I}_n$.

Hence if the Gröbner basis is available, the generators of the relevant elimination ideal are immediately available.

The drawback of Buchberger algorithm is that it has a very high complexity in the worst case, and in practice the complexity depends quite much on the chosen ordering. Anyway Gröbner bases have proved to be very useful in many different applications. Nowadays there exist many different implementations and improvements of the Buchberger algorithm. We chose to use the program Singular [11, 12] in all the computations in this paper.

## 2.4 Dimension of a variety

There are many different ways to define the dimension of a variety, and it is a priori not at all obvious that these different approaches are in fact equivalent. We will only here explain some basic matters and refer to [5, 8, 13] for precise definitions.

Now a variety is in general composed of many pieces of different sizes. One approach is first to define the dimension for irreducible varieties and then say that the dimension of a general variety is the maximum of dimensions of its irreducible components. However, an important point is that one can compute the dimension without computing the prime decomposition. In fact once the Gröbner basis of an ideal is available, the computation of the dimension is relatively easy. In Singular this algortihm is implemented and we have used it in the computations below.

Note that the dimension refers to complex varieties. For example the dimension of the variety corresponding to polynomial $x_1^2+x_2^2+1$ is one although the real variety is empty. In applications one is mostly interested in real varities, hence one must check separately that the results apply also in the real case. Fortunately this is quite obvious in the computations of the present paper so we will not comment on this further.

### 2.5 Singular points of a variety

To study singular points we need Fitting ideals. Let $M$ be a matrix of size $k \times n$ with entries in $\mathbb{A}$. The $\ell$th *Fitting ideal* of $M$, $\mathsf{I}_\ell(M)$, is the ideal generated by the $\ell \times \ell$ minors of $M$. Let us consider the ideal $\mathcal{I} = \langle g_1, \dots, g_k\rangle$ and the corresponding variety $\mathsf{V}(\mathcal{I})$. To define singular and regular points in full generality would require some lengthy explanations which are finally not needed below. Hence we will simply state a special case which we actually use and refer to [6, 13] for more details.

Let $\mathsf{V}(\mathcal{I})$ be an irreducible variety of dimension $n-k$. Then $p \in \mathsf{V}(\mathcal{I})$ is regular, if $dg_p$ is of full rank and singular otherwise. The singular locus in this case is the variety

$$\mathsf{V}\big(\mathcal{I} + \mathsf{I}_k(dg)\big) = \mathsf{V}\big(\mathcal{I}\big) \cap \mathsf{V}\big(\mathsf{I}_k(dg)\big)$$

Intuitively, $\mathsf{V}(\mathcal{I})$ is defined by $k$ equations over $n$ variables, these equations are the generators of $\mathcal{I}$, and singularity means the maximal (size $k \times k$) minors of the jacobian of these equations are zero. In particular if the Gröbner basis of $\mathcal{I} + \mathsf{I}_k(dg)$ is $\{1\}$, then all points of $\mathsf{V}(\mathcal{I})$ are regular.

*Example* 2.1. In this example we will use the mathematical tools presented above and analyze rotations in $\mathbb{R}^2$ using polynomial equations. Writing

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \simeq (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{R}^4$$

we can make identification

$$\mathbb{O}(2) \simeq \big\{(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{R}^4 \mid a_{11}^2+a_{21}^2-1 = 0\,,\ a_{11}a_{12}+a_{21}a_{22} = 0\,,\ a_{12}^2+a_{22}^2-1 = 0\,\big\}\,.$$

Hence the ideal $\mathcal{I} = \langle a_{11}^2 + a_{21}^2 - 1, a_{11}a_{12} + a_{21}a_{22}, a_{12}^2 + a_{22}^2 - 1\rangle$ and the corresponding variety $\mathsf{V}(\mathcal{I})$ fully describes the structure of the orthogonal group $\mathbb{O}(2)$. We then inspect the ideal $\mathcal{I}$ in the ring $\mathbb{Q}[a_{11}, a_{12}, a_{21}, a_{22}]$ and compute the prime decomposition

$$\begin{aligned}
\sqrt{\mathcal{I}} &= P_1 \cap P_2 \\
P_1 &= \langle a_{21}^2 + a_{22}^2 - 1, a_{12} - a_{21}, a_{11} + a_{22}\rangle \\
P_2 &= \langle a_{21}^2 + a_{22}^2 - 1, a_{12} + a_{21}, a_{11} - a_{22}\rangle
\end{aligned}$$

Hence $\mathbb{O}(2) \simeq \mathsf{V}(\sqrt{\mathcal{I}}) = \mathsf{V}(\mathcal{I})$ has two irreducible one dimensional components $\mathsf{V}(P_1)$ and $\mathsf{V}(P_2)$. Moreover the intersection of these components is empty. One way to see this is to note that the Gröbner basis of the ideal $P_1 + P_2$ is $\{1\}$. Another way is to check that

$$\begin{aligned} \det(A) = -1, &\quad (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathsf{V}(P_1) \\ \det(A) = 1, &\quad (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathsf{V}(P_2) \end{aligned}$$

We can thus make the identification $\mathbb{SO}(2) \simeq \mathsf{V}(P_2)$. Let us then consider the map $g$ whose components are the generators of $P_2$. Now computing the Gröbner basis of $P_2 + \mathsf{I}_3(dg)$ we obtain $\{1\}$. Hence $\mathbb{SO}(2)$ is a smooth variety. Below is the Singular script containing the computations.

```
> ring r=0,(a11,a12,a21,a22),dp;
> matrix A[2][2]=a11,a12,a21,a22;
> matrix I[2][2]=1,0,0,1;
> matrix M=transpose(A)*A-I;ideal J=M;
> LIB "all.lib";
> list L=minAssGTZ(J);
> ideal P1=L[1];ideal P2=L[2];
> dim(P1);dim(P2);
> ideal P=P1,P2;
> ideal G=groebner(P);matrix dg=jacob(P2);
> ideal F=fitting(dg,0);ideal K=P2,F;
> ideal G2=groebner(K);
```

# 3 Multibody systems

## 3.1 Configuration space

Usually one describes a rigid body by giving its center of mass and orientation. Hence the configuration space of one rigid body is $Q = \mathbb{R}^3 \times \mathbb{SO}(3)$ and its *mobility* (or the number of degrees of freedom) is 6, i.e. $\dim\big(Q\big) = 6$.

We represent rotations with Euler parameters. Let $a = (a_0, a_1, a_3, a_4) \in S^3 \subset \mathbb{R}^4$. Any $R \in \mathbb{SO}(3)$ can be represented by such an $a$ as

$$\begin{aligned} R = \widetilde{H}H^T &= \begin{pmatrix} -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} -a_1 & a_0 & a_3 & -a_2 \\ -a_2 & -a_3 & a_0 & -a_2 \\ -a_3 & a_2 & -a_1 & a_0 \end{pmatrix}^T \\ &= 2 \begin{pmatrix} a_0^2 + a_1^2 - \frac{1}{2} & a_1a_2 - a_0a_3 & a_1a_3 + a_0a_2 \\ a_1a_2 + a_0a_3 & a_0^2 + a_2^2 - \frac{1}{2} & a_2a_3 - a_0a_1 \\ a_1a_3 - a_0a_2 & a_2a_3 + a_0a_1 & a_0^2 + a_3^2 - \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Note that $a$ and $-a$ correspond to the same $R$.[1] We thus work with the configuration space $\mathbb{R}^3 \times S^3$ with the understanding that the opposite points of $S^3$ correspond to the same physical situation. This is a minor inconvenience compared to the advantages

[1] Topologically this says that $S^3$ is a 2-sheeted covering space of $\mathbb{SO}(3)$.

of this representation. From the point of view of the present paper the essential fact is that all constraints are formulated using polynomial equations which allows us to use the tools introduced above. There are other advantages which are important in numerical computations, see [16] for more details.

## 3.2 Constraints

Let $r$ denote the position of the center of mass of a given rigid body in global coordinates. Then given any point $\chi$ in local coordinates, it can be written in global coordinates as

$$x = r + R\chi\ , \quad R \in \mathbb{SO}(3).$$

From now on we will always place the origin of the local coordinate system of the rigid body to its center of mass. We then introduce two basic constraints. In the following definitions $\chi$, $\eta$ and $\kappa$ will be vectors or points in local coordinate systems.

**Definition 3.1** (Symmetric orthogonality constraint)**.** Let $B_1$ and $B_2$ be rigid bodies. The *symmetric orthogonality constraint* requires that

$$(R^1\chi^1, R^2\chi^2) = 0$$

where $\chi^i$ (resp $R^i$) is a vector (resp. rotation matrix) in local coordinate system of $B_i$.

Hence vectors $\chi^1$ and $\chi^2$ are orthogonal in the *global* coordinate system.

**Definition 3.2** (Coincidence constraint)**.** Let $r^i$ (resp. $\chi^i$) be the center of mass (resp. a point in the local coordinates) of the body $B_i$. The *coincidence constraint* requires that

$$r^1 + R^1\chi^1 - r^2 - R^2\chi^2 = 0\ .$$

Hence $\chi^1$ and $\chi^2$ coincide in the global coordinate system. We can now represent the revolute joint with these two conditions. Let $\chi^i, \eta^i, \kappa^i$ be vectors in the local coordinate system of $B_i$ and let us assume that vectors $\chi^1$ and $\eta^1$ are linearly independent.

**Definition 3.3** (Revolute joint)**.** Let $r^i$ be the center of mass of $B_i$. Bodies $B_1$ and $B_2$ are connected to each other by a revolute joint if

$$\begin{aligned}
&(R^1\chi^1, R^2\chi^2) = 0 \\
&(R^1\eta^1, R^2\chi^2) = 0 \\
&r^2 + R^2\kappa^2 - r^1 - R^1\kappa^1 = 0.
\end{aligned}$$

Thus the revolute joint is defined by 5 equations. Hence the mobility of a system consisting of 2 rods joined together by a revolute joint is $2 \cdot 6 - 5 = 7$.

For completeness let us also mention the third basic constraint.

**Definition 3.4** (Orthogonality constraint)**.** Let $B_1$ and $B_2$ be rigid bodies. The *orthogonality constraint* requires that

$$(R^1\eta, r^1 + R^1\chi^1 - r^2 - R^2\chi^2) = 0$$

where $\eta$ is a given vector in local coordinate system of $B_1$ and $r^1 + R^1\chi^1 - r^2 - R^2\chi^2$ gives the difference of $\chi^1$ and $\chi^2$ in global coordinates.

With these basic constraints most joints occurring in practice can be specified. All constraints are low order polynomials and thus very suitable for the analysis by the methods described above.

# 4 Bricard's mechanism

## 4.1 Initial system of equations

We are now ready to analyze the Bricard's system shown in Figure 1. It consists of 5 rods which are connected to each other with revolute joints and in addition the first and the last rod are connected permanently with respect to the global coordinate system. Bricard himself viewed the mechanism a bit differently: he considered a closed loop of 6 rods and 6 joints [4]. However, from the point of view of kinematic analysis the two formulations are equivalent.

Now a straightforward count says that the mobility of Bricard's system should be zero since the mobility of 5 rods is $5 \cdot 6 = 30$, and 6 revolute joints give $6 \cdot 5 = 30$ constraints. However, it is well-known that the mobility is one, and that is why Bricard called this and similar systems paradoxical. Our purpose below is to provide an explicit description of the configuration space of Bricard's system.

The origin of the global coordinate system coincides with the first joint and is shown in the Figure 1. In each local coordinate system the vector $e^1$ is parallel to the rod. Then for example the rod $B$ is connected to rod $A$, joint 2 allows rod $B$ to move on the plane which is perpendicular to $e^2$ and joint 3 allows rod $B$ to move on the plane which is perpendicular to $e^1$. Analysing similarly other rods we finally arrive at the following system of constraint equations.

$$\begin{array}{ll}
p_1 = (e^1, R^1e^3) = 0 & p_{13\ldots15} = r^1 - \tfrac{1}{2}R^1e^1 = 0 \\
p_2 = (e^2, R^1e^3) = 0 & p_{16\ldots18} = r^1 + \tfrac{1}{2}R^1e^1 - r^2 + \tfrac{1}{2}R^2e^1 = 0 \\
p_3 = (R^1e^1, R^2e^3) = 0 & p_{19\ldots21} = r^2 + \tfrac{1}{2}R^2e^1 - r^3 + \tfrac{1}{2}R^3e^1 = 0 \\
p_4 = (R^1e^3, R^2e^3) = 0 & p_{22\ldots24} = r^3 + \tfrac{1}{2}R^3e^1 - r^4 + \tfrac{1}{2}R^4e^1 = 0 \\
p_5 = (R^2e^1, R^3e^3) = 0 & p_{25\ldots27} = r^4 + \tfrac{1}{2}R^4e^1 - r^5 + \tfrac{1}{2}R^5e^1 = 0 \\
p_6 = (R^2e^3, R^3e^3) = 0 & p_{28\ldots30} = r^5 + \tfrac{1}{2}R^5e^1 = \mathbf{x} = -e^2 \\
p_7 = (R^3e^1, R^4e^3) = 0 & p_{31} = |a|^2 - 1 = 0 \\
p_8 = (R^3e^3, R^4e^3) = 0 & p_{32} = |b|^2 - 1 = 0 \\
p_9 = (R^4e^1, R^5e^3) = 0 & p_{33} = |c|^2 - 1 = 0 \\
p_{10} = (R^4e^3, R^5e^3) = 0 & p_{34} = |d|^2 - 1 = 0 \\
p_{11} = (R^5e^1, e^1) = 0 & p_{35} = |e|^2 - 1 = 0 \\
p_{12} = (R^5e^3, e^1) = 0 &
\end{array} \tag{5}$$

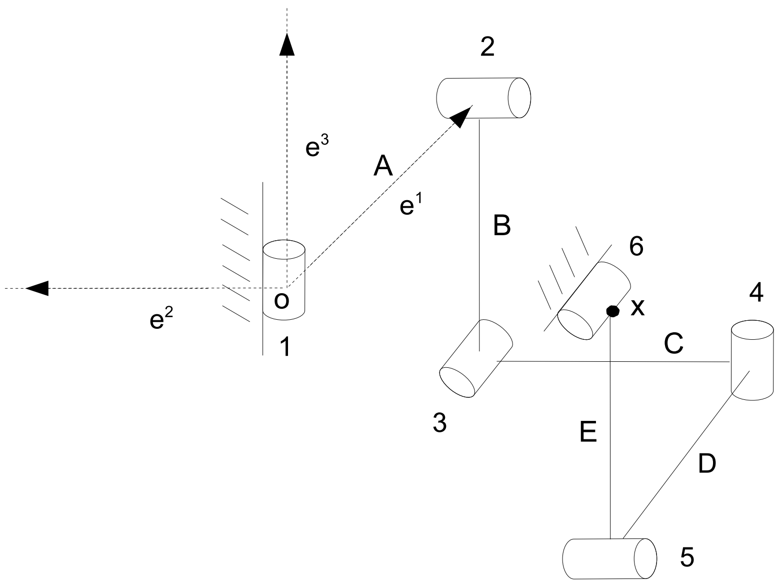


Figure 1: Bricard's system: Cylinders represent the revolute joints 1, 2, 3, 4, 5 and 6.

The rotation matrices are parametrized as $R^1(a)$, $R^2(b), \ldots,$ $R^5(e)$ where $a = (a_0, a_1, a_2, a_3) \in S^3 \subset \mathbb{R}^4$, and so on. The five last equations follow from the fact that $a, b, c, d, e \in S^3$. Hence we have 35 equations depending on 35 variables.

## 4.2 Preliminary simplification

Note that $r^i$ appear linearly in (5), and $p_{13}, \ldots, p_{30}$ are easily rearranged into

$$\begin{aligned}
&r^1 - \tfrac{1}{2}\, R^1 e^1 = 0 \\
&r^2 - R^1 e^1 - \tfrac{1}{2}\, R^2 e^1 = 0 \\
&r^3 - R^1 e^1 - R^2 e^1 - \tfrac{1}{2}\, R^3 e^1 = 0 \\
&r^4 - R^1 e^1 - R^2 e^1 - R^3 e^1 - \tfrac{1}{2}\, R^4 e^1 = 0 \\
&r^5 - R^1 e^1 - R^2 e^1 - R^3 e^1 - R^4 e^1 - \tfrac{1}{2}\, R^5 e^1 = 0 \\
&\big(R^1 + R^2 + R^3 + R^4 + R^5\big) e^1 + e^2 = 0,
\end{aligned} \tag{6}$$

so we can consider the $r^i$ solved and formulate the constraints in terms of orientations alone as follows. The last equation of (6) gives 3 polynomials which we denote by $p_{36}$, $p_{37}$, and $p_{38}$. Hence the ideal which is generated by polynomials containing only orientations and which we are going to analyze is

$$\mathcal{I} = \langle p_1, \ldots, p_{12}, p_{31}, \ldots, p_{38} \rangle \subset \mathbb{Q}[a, b, c, d, e] \ . \tag{7}$$

Hence there are 20 polynomials and 20 variables.

## 4.3 Initial configuration

Our aim is to decompose the variety $\mathsf{V}(\mathcal{I})$ into irreducible components. It turns out that there are a lot of components, and some of them are not physically relevant. In other words the equations admit "spurious" solutions which are not compatible with the configuration shown in Figure 1. Hence we need a test if the initial configuration actually belongs to a particular component of $\mathsf{V}(\mathcal{I})$.

We construct an ideal which specifies the initial configuration. The position of the joint 2 in the initial configuration satisfies $R^1 e^1 - e^1 = 0$ which gives an ideal $I_a = \langle R^1_{11} - 1, R^1_{21}, R^1_{31} \rangle$. Similarly if we look at the position of the joint 3 we get $R^2 e^1 + e^3 = 0$ which yields the ideal $I_b = \langle R^2_{11}, R^2_{21}, R^2_{31} + 1 \rangle$. Continuing in this manner for joints $4, 5$ and 6 we get

$$I_c = \langle R^3_{11}, R^3_{21} + 1, R^3_{31} \rangle \quad , \quad I_d = \langle R^4_{11} + 1, R^4_{21}, R^4_{31} \rangle \quad , \quad I_e = \langle R^5_{11}, R^5_{21}, R^5_{31} - 1 \rangle.$$

Now defining

$$\mathcal{I}_{init} = I_a + I_b + I_c + I_d + I_e$$

it is seen that the initial configuration belongs to the variety $\mathsf{V}(\mathcal{I}_{init})$. Now suppose that a particular component $V_{comp} \subset \mathsf{V}(\mathcal{I})$ is given by an ideal $\mathcal{I}_{comp}$: $V_{comp} = \mathsf{V}(\mathcal{I}_{comp})$. We can discard $\mathsf{V}(\mathcal{I}_{comp})$ if $\mathsf{V}(\mathcal{I}_{comp}) \cap \mathsf{V}(\mathcal{I}_{init}) = \emptyset$. This is certainly the case if $\mathcal{I}_{comp} + \mathcal{I}_{init} = \mathbb{Q}[a, b, c, d, e]$ because

$$\mathsf{V}(\mathcal{I}_{comp}) \cap \mathsf{V}(\mathcal{I}_{init}) = \mathsf{V}(\mathcal{I}_{comp} + \mathcal{I}_{init})$$

Now Singular computes a *minimal* Gröbner basis and one can show that if the ideal is the whole ring the minimal Gröbner basis is $\{1\}$. Hence if the Gröbner basis of $\mathcal{I}_{comp} + \mathcal{I}_{init}$ is $\{1\}$ we can discard $\mathsf{V}(\mathcal{I}_{comp})$.

# 5 Decomposition of the configuration space of Bricard's mechanism

Let us consider the ideal $\mathcal{I}$ given in (7) and start implementing the strategy in (3). Let us write $\mathcal{I} = S_1 + \tilde{S}_1$ where $S_1 = \langle p_1, p_2, p_{31}\rangle$ and $\tilde{S}_1$ is generated by other generators of $\mathcal{I}$. Calculating the prime decomposition of $\sqrt{S_1}$ we get

$$\begin{aligned}\sqrt{S_1} &= S_{1,1} \cap S_{1,2}\\ S_{1,1} &= \langle a_1^2 + a_2^2 - 1, a_3, a_0\rangle\\ S_{1,2} &= \langle a_0^2 + a_3^2 - 1, a_2, a_1\rangle.\end{aligned}$$

Hence either $a_3 = a_0 = 0$ or $a_2 = a_1 = 0$. Similarly writing $\mathcal{I} = S_2 + \tilde{S}_2$ where $S_2 = \langle p_{11}, p_{12}, p_{35}\rangle$ we get the prime decomposition of $\sqrt{S_2}$,

$$\begin{aligned}\sqrt{S_2} &= S_{2,1} \cap S_{2,2}\\ S_{2,1} &= \langle 2e_2^2 + 2e_3^2 - 1, e_1 - e_2, e_0 + e_3\rangle\\ S_{2,2} &= \langle 2e_2^2 + 2e_3^2 - 1, e_1 + e_2, e_0 - e_3\rangle.\end{aligned}$$

In this case either $e_2 = e_1, e_3 = -e_0$ or $e_2 = -e_1, e_3 = e_0$. Now we have four different possibilities from which to continue.

1. $(a_1, a_2, e_2, e_3) = (0, 0, e_1, -e_0)$
2. $(a_1, a_2, e_2, e_3) = (0, 0, -e_1, e_0)$
3. $(a_0, a_3, e_2, e_3) = (0, 0, e_1, -e_0)$
4. $(a_0, a_3, e_2, e_3) = (0, 0, -e_1, e_0)$

We will investigate the case 1. further, i.e. consider the ideal $\mathcal{I}_2 = \mathcal{I} + S_{1,1} + S_{2,1}$. There is no loss of generality in choosing just one of the above cases. Recall that Euler parameters $a$ and $-a$ correspond to the same physical situation. Hence the first and second cases are equivalent, as well as the third and fourth. Moreover choosing between $a_1 = a_2 = 0$ and $a_0 = a_3 = 0$ corresponds to choosing different local coordinates for the first rod, and this has obviously no effect on what happens physically in global coordinates.

We now write $\mathcal{I}_2 = S_3 + \tilde{S}_3$ where

$$S_3 = S_{1,1} + \langle p_1, p_2, p_3.p_4, p_{31}, p_{32}\rangle$$

Computing the prime decomposition of $\sqrt{S_3}$ in $\mathbb{Q}[(a_1, a_2), (b_0, b_3), (b_1, b_2), (a_0, a_3)]$ we get

$$\begin{aligned}\sqrt{S_3} &= S_{3,1} \cap S_{3,2}\\ S_{3,1} &= \langle 2b_1^2 + 2b_2^2 - 1, a_0^2 + a_3^2 - 1, b_3 - 2a_0a_3b_1 - 2a_3^2b_2 + b_2, b_0 + 2a_3^2b_1 - 2a_0a_3b_2 - b_1\rangle\\ S_{3,2} &= \langle 2b_1^2 + 2b_2^2 - 1, a_0^2 + a_3^2 - 1, b_3 + 2a_0a_3b_1 + 2a_3^2b_2 - b_2, b_0 - 2a_3^2b_1 + 2a_0a_3b_2 + b_1\rangle\ .\end{aligned}$$

Similarly we write $\mathcal{I}_2 = S_4 + \tilde{S}_4$ where

$$S_4 = S_{2,1} + \langle p_9, p_{10}, p_{11}, p_{12}, p_{34}, p_{35} \rangle.$$

Computing the prime decomposition of $\sqrt{S_4}$ in $\mathbb{Q}[(e_3, e_2), (d_0, d_1), (d_2, d_3), (e_0, e_1)]$ we get

$$\sqrt{S_4} = S_{4,1} \cap S_{4,2}$$

where for example

$$\begin{aligned} S_{4,1} = \langle & 2e_0^2 + 2e_1^2 - 1, 4d_2e_1^2 - d_2 - 4d_3e_0e_1 + d_3, d_2e_0 + d_2e_1 - d_3e_0 + d_3e_1, \\ & 2d_1^2 + 2d_3^2 - 4e_0e_1 - 1, 4d_0e_1^2 - d_0 - 4d_1e_0e_1 + d_1, d_0e_0 + d_0e_1 - d_1e_0 + d_1e_1, \\ & d_0d_3 - d_1d_2, 2d_0d_1 + 2d_2d_3 + 4e_1^2 - 1, 2d_0^2 + 2d_2^2 + 4e_0e_1 - 1 \rangle \ . \end{aligned}$$

Again we have four choices from which to continue and again we can without loss of generality to choose just one of them. Let us choose the ideals $S_{3,1}$ and $S_{4,1}$ and define $\mathcal{I}_3 = \mathcal{I}_2 + S_{3,1} + S_{4,1}$. We then write $\mathcal{I}_3 = S_5 + \tilde{S}_5$ where the generators of $S_5$ depend only on the variables $a$, $b$ and $c$:

$$S_5 = S_{3,1} + \langle p_1, \ldots, p_6, p_{31}, p_{32}, p_{33} \rangle.$$

We investigate $S_5$ in the ring

$$\mathbb{Q}[(c_1, c_2, a_1, a_2), (b_0, b_3), (c_3, c_0), (b_1, b_2), (a_0, a_3)] \tag{8}$$

in order to eliminate some variables. First we compute the Gröbner basis $G_5$ of $S_5$. It turns out that $G_5$ has 33 generators, but only first three of these contain variables $a_0, a_3, b_1, b_2, c_0, c_3$. Hence we can write

$$S_5 = \langle G_5 \rangle = E_5 + \tilde{E}_5$$

where the generators of $E_5$ are

$$\begin{aligned} E_5(1) =& a_0^2 + a_3^2 - 1 \quad , \quad E_5(2) = 2b_1^2 + 2b_2^2 - 1 \\ E_5(3) =& ((-32b_2^3 + 8b_2)b_1a_3^3 + (16b_2^3 - 4b_2)b_1a_3)a_0 + (16b_2^2 - 32b_2^4 - 1)a_3^4 \\ & + (-16b_2^2 + 32b_2^4 + 1)a_3^2 - 4b_2^4 + 2b_2^2 + c_0^4 + (2c_3^2 - 1)c_0^2 + c_3^4 - c_3^2. \end{aligned}$$

We now compute the prime decomposition of $\sqrt{E_5}$ using the ordering $\mathbb{Q}[(b_1, b_2), (a_0, a_3), (c_3, c_0)]$. This gives 2 prime ideals whose Gröbner bases, in the ordering (8), are

$$\begin{aligned} & \sqrt{E_5} = E_{5,1} \cap E_{5,2} \\ & E_{5,1} = \langle a_0^2 + a_3^2 - 1, 2b_1^2 + 2b_2^2 - 1, c_0^2 + c_3^2 + 4b_1b_2a_0a_3 + 4b_2^2a_3^2 - 2b_2^2 - a_3^2 \rangle \\ & E_{5,2} = \langle a_0^2 + a_3^2 - 1, 2b_1^2 + 2b_2^2 - 1, c_0^2 + c_3^2 - 4b_1b_2a_0a_3 - 4b_2^2a_3^2 + 2b_2^2 + a_3^2 - 1 \rangle \ . \end{aligned}$$

Again we have two choices from which to continue, and we will choose the ideal corresponding to $E_{5,1}$ for further analysis and define $\mathcal{I}_4 = \mathcal{I}_3 + E_{5,1}$. We then compute the Gröbner basis of $\mathcal{I}_4$, denoted $G_4$, which has 918 generators. The generators $G_4(32)$ and $G_4(34)$ are particularly interesting:

$$\begin{aligned} G_4(32) &= e_1(c_3 + c_0)(b_1 + b_3) \\ G_4(34) &= (d_0 - d_1)(c_3 + c_0)(b_1 + b_3) \ . \end{aligned}$$

Hence there are 3 essentially different cases

1. $c_0 + c_3 = 0$

2. $b_1 + b_3 = 0$

3. $d_0 - d_1 = e_1 = 0$.

Now computing the Gröbner basis of $\mathcal{I}_{init} + \mathcal{I}_4 + \langle d_0 - d_1, e_1 \rangle$ we get $\{1\}$. Hence the initial configuration does not belong to the variety corresponding to this case. However, the Gröbner bases of

$$\mathcal{I}_{init} + \mathcal{I}_4 + \langle b_1 + b_3 \rangle \quad \text{and} \quad \mathcal{I}_{init} + \mathcal{I}_4 + \langle c_0 + c_3 \rangle$$

are not $\{1\}$, so these cases must be examined further. It turns out that the case $b_1 + b_3 = 0$ can be discarded. To show this let us define $\tilde{I} = \mathcal{I}_4 + \langle b_1 + b_3 \rangle$ and let us use the factorizing Gröbner basis algorithm.[2] This gives a list of ideals $\tilde{F}_i$ such that

$$\sqrt{\tilde{I}} = \sqrt{\tilde{F}_1 \cap \cdots \cap \tilde{F}_\ell} = \sqrt{\tilde{F}_1} \cap \cdots \cap \sqrt{\tilde{F}_\ell}$$

but the factors $\tilde{F}_i$ are not necessarily prime ideals. In the present case we obtain 150 factors for $\tilde{I}$. Now only 5 factors are positive dimensional and none of them contain the initial configuration. Hence the initial configuration is contained only in a zero dimensional component of $\mathsf{V}(\tilde{I})$ and we need not analyse this case any further.

This leaves us with the case $c_0 + c_3 = 0$ and we continue our analysis with $\mathcal{I}_5 = \mathcal{I}_4 + \langle c_0 + c_3 \rangle$. Computing the Gröbner basis for $\mathcal{I}_5$ we can then determine the dimension of $\mathsf{V}(\mathcal{I}_5)$. Somewhat surprisingly this gives $\dim(\mathsf{V}(\mathcal{I}_5)) = 2$. This is because $\mathsf{V}(\mathcal{I}_5)$ still contains spurious components. In fact just by looking at Figure 1 one can convince oneself that there must be 2 dimensional components in $\mathsf{V}(\mathcal{I}_5)$. Namely the equations allow solutions where rods B and E are identified as well as C and D. Clearly the mobility of the system now is 2 because B and E can rotate independently of C and D. Another 2 dimensional component is obtained by identifying A and D, and B and C. Obviously these solutions do not contain the initial configuration.

To get rid of the spurious components we again use factorizing Gröbner basis algorithm. This time we get a list of ideals $F_1, \ldots, F_{93}$. Only one of them, $F_{65}$ (whose Gröbner basis contains 182 elements), is both one dimensional and contains the initial configuration. Hence we set $\mathcal{I}_6 = F_{65}$. Now $\mathcal{I}_6$ is not necessarily a prime ideal so that the variety $\mathsf{V}(\mathcal{I}_6)$ might still have zero dimensional components. Also it may still have several one dimensional components which describe the same physical configuration. Consequently we investigate this ideal further in the ring

$$\mathbb{Q}[(e_2, e_3, a_1, a_2, c_3, c_2, c_1, e_1), (b_0, b_3), (d_0, d_1, d_2, d_3, b_1, b_2), (e_0, c_0, a_0, a_3)]$$

Computing the Gröbner basis of $\mathcal{I}_6$ we find that the second generator is $e_0^2 - c_0^2$. Again we have 2 choices which correspond to the same physical situation. We choose $\mathcal{I}_7 = \mathcal{I}_6 + \langle e_0 - c_0 \rangle$ and inspect $\mathcal{I}_7$ in the ring

$$\mathbb{Q}[(d_3, d_2, d_1, d_0, b_3, b_2, b_1, b_0, e_3, e_2), (e_1, e_0, c_3, c_2, c_1, a_2, a_1 c_0, a_0, a_3)].$$

[2] This is implemented as the command facstd in Singular.

The first 19 generators give the relevant elimination ideal $E_7 = \mathcal{I}_7 \cap \mathbb{Q}[e_1, e_0, c_3, c_2, c_1, a_2, a_1, c_0, a_0, a_3$
The prime decomposition of $\sqrt{E_7}$ has 12 components

$$\sqrt{E_7} = E_{7,1} \cap \ldots \cap E_{7,12}$$

Only 2 of the prime ideals combined with ideal $\mathcal{I}_7$ contain the initial position and are one dimensional. Again these 2 correspond to same physical situation and we choose one of them, say $E_{7,1}$, and continue with $\mathcal{I}_8 = \mathcal{I}_7 + E_{7,1}$. Now computing the prime decomposition of $\sqrt{\mathcal{I}_8}$ we find that it has 4 components $\sqrt{\mathcal{I}_8} = \mathcal{I}_{8,1} \cap \mathcal{I}_{8,2} \cap \mathcal{I}_{8,3} \cap \mathcal{I}_{8,4}$. But all cases lead to same physical situation because of the reflection invariance of Euler parameters. In fact the symmetries to change $\mathcal{I}_{8,1}$ to other ideals $\mathcal{I}_{8,i}$ are $b \to -b$, $d \to -d$ and $(b,d) \to (-b,-d)$. The ideal $\mathcal{I}_9 = \mathcal{I}_{8,1}$ is given by the generators

$$
\begin{aligned}
\tilde{q}_1 &= a_0^2 + a_3^2 - 1 & \tilde{q}_{11} &= e_2 + 2c_0^2 a_0 + 2c_0^2 a_3 - a_0 - a_3 \\
\tilde{q}_2 &= 8c_0^2 a_0 a_3 + 4c_0^2 - 4a_0 a_3 - 1 & \tilde{q}_{12} &= e_3 + e_0 \\
\tilde{q}_3 &= 8c_0^2 a_3^3 - 4c_0^2 a_0 - 8c_0^2 a_3 - 4a_3^3 + a_0 + 4a_3 & \tilde{q}_{13} &= 4b_0 - 8c_0^2 a_3^2 + 4c_0^2 + 4c_0 a_3 + 4a_3^2 - 3 \\
\tilde{q}_4 &= a_1 & \tilde{q}_{14} &= 4b_1 - 8c_0^2 a_3^2 + 4c_0^2 - 4c_0 a_3 + 4a_3^2 - 3 \\
\tilde{q}_5 &= a_2 & \tilde{q}_{15} &= 4b_2 + 8c_0^2 a_3^2 - 4c_0^2 + 4c_0 a_0 - 4a_3^2 + 1 \\
\tilde{q}_6 &= c_1 + 2c_0^2 a_0 + 2c_0^2 a_3 - a_3 & \tilde{q}_{16} &= 4b_3 + 8c_0^2 a_3^2 - 4c_0^2 - 4c_0 a_0 - 4a_3^2 + 1 \\
\tilde{q}_7 &= c_2 + 2c_0^2 a_0 + 2c_0^2 a_3 - a_0 & \tilde{q}_{17} &= 4d_0 - 8c_0^2 a_3^2 + 4c_0^2 - 4c_0 a_3 + 4a_3^2 - 1 \\
\tilde{q}_8 &= c_3 + c_0 & \tilde{q}_{18} &= 4d_1 + 8c_0^2 a_3^2 - 4c_0^2 - 4c_0 a_3 - 4a_3^2 + 1 \\
\tilde{q}_9 &= e_0 - c_0 & \tilde{q}_{19} &= 4d_2 - 8c_0^2 a_3^2 + 4c_0^2 + 4c_0 a_0 + 4a_3^2 - 3 \\
\tilde{q}_{10} &= e_1 + 2c_0^2 a_0 + 2c_0^2 a_3 - a_0 - a_3 & \tilde{q}_{20} &= 4d_3 + 8c_0^2 a_3^2 - 4c_0^2 + 4c_0 a_0 - 4a_3^2 + 3
\end{aligned}
$$

Ideal $\mathcal{I}_9 = \langle \tilde{q}_1, \ldots, \tilde{q}_{20} \rangle$ still has 20 polynomials for 20 unknowns. However, $\langle \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \rangle = \langle \tilde{q}_1, \tilde{q}_2 \rangle$ so $\tilde{q}_3$ can be dropped. Then simplifying we get the following system of polynomials which fully describes the configuration space of Bricard's system with Euler parameters.

$$
\begin{aligned}
q_1 &= a_0^2 + a_3^2 - 1 & q_{10} &= e_2 - e_1 \\
q_2 &= 4c_0^2(2a_0 a_3 + 1) - 4a_0 a_3 - 1 & q_{11} &= e_3 + e_0 \\
q_3 &= a_1 & q_{12} &= 4b_0 + 4c_0^2(1 - 2a_3^2) + 4a_3(c_0 + a_3) - 3 \\
q_4 &= a_2 & q_{13} &= b_1 - b_0 - 2c_0 a_3 \\
q_5 &= c_1 + 2c_0^2(a_0 + a_3) - a_3 & q_{14} &= 2b_2 + 2b_0 + 2c_0(a_3 + a_0) - 1 \\
q_6 &= c_2 - c_1 + a_3 - a_0 & q_{15} &= b_3 - b_2 - 2c_0 a_0 \\
q_7 &= c_3 + c_0 & q_{16} &= 2d_0 - 2b_1 + 1 \\
q_8 &= e_0 - c_0 & q_{17} &= 2d_1 + 2b_0 - 1 \\
q_9 &= e_1 - c_1 - a_0 & q_{18} &= 2d_2 + 2b_3 - 1 \\
& & q_{19} &= 2d_3 - 2b_2 + 1 \; .
\end{aligned}
\tag{9}
$$

Note that this is in triangular form. If we look at the map $q : \mathbb{R}^{20} \mapsto \mathbb{R}^{19}$ the Jacobian of $q$ is a $19 \times 20$ matrix and has 380 elements but only 49 of these are nonzero and only 17 are nonconstant. From the equations we see that the subsystem

$$\begin{aligned} q_1 &= a_0^2 + a_3^2 - 1 = 0 \\ q_2 &= 4c_0^2(2a_0a_3 + 1) - 4a_0a_3 - 1 = 0, \end{aligned} \tag{10}$$

is the one of essential importance because after solving this system every other variable can be solved immediately. Therefore let us inspect the ideal $\mathcal{I}_{\text{ess}} = \langle q_1, q_2 \rangle \subset \mathbb{Q}[c_0, a_0, a_3]$. We define the map $\hat{q} : \mathbb{R}^3 \mapsto \mathbb{R}^2$, $\hat{q} = (q_1, q_2)$ and find that the Gröbner basis of $\mathcal{I}_{\text{ess}} + \mathsf{I}_2(d\hat{q})$ is $\{1\}$. Hence the variety $\mathsf{V}(\mathcal{I}_{\text{ess}})$ is smooth and one dimensional. We have proven

**Theorem 1.** The configuration space of the Bricard's system is a one-dimensional smooth variety. Moreover, the polynomials (10) give an essential description of the configuration space in terms of 3 variables $a_0$, $a_3$, and $c_0$. Every other Euler parameter is then explicitly given by the triangular system (9).

# 6 Parametrization of the configuration space

We want to parametrize the variety $\mathsf{V}(\mathcal{I}_{\text{ess}})$ in order to present the variety of the whole system with this parameter. We make substitutions

$$a_0 = \cos(\alpha/2) \quad , \quad a_3 = \sin(\alpha/2).$$

With this choice the equation $q_1 = 0$ is identically satisfied. Note that $\alpha$ gives the rotation angle of the first joint. Substituting the expressions of $a_0$ and $a_3$ to $q_2 = 0$ we get

$$c_0^2 = \frac{1}{4}\Big(\frac{1 + 4a_0a_3}{1 + 2a_0a_3}\Big) = \frac{1}{4}\Big(\frac{1 + 2\sin(\alpha)}{1 + \sin(\alpha)}\Big).$$

Because $c_0^2 \geq 0$ we see that $\alpha \in [-\pi/6, 7\pi/6]$. From this it easily follows that topologically the variety $\mathsf{V}(\mathcal{I}_{\text{ess}})$ is a union of two smooth Jordan curves, see Figure 2. One of the curves is given by

$$\gamma_{\pm}(\alpha) = \left( \cos(\alpha/2)\ ,\ \sin(\alpha/2)\ ,\ \pm\tfrac{1}{2}\sqrt{\frac{1 + 2\sin(\alpha)}{1 + \sin(\alpha)}} \right) \quad \alpha \in [-\pi/6, 7\pi/6],$$

and the other is $-\gamma_{\pm}$. Note that both curves represent the same physical situation.

In [10] the position of joint 3 was chosen to compare the performance of different numerical solvers, so let us give an explicit description of the set of possible positions. The position of joint 3 is given by $R^1e^1 + R^2e^1$. We first express the rotation matrices $R^1$ and $R^2$ in terms of parameters $a_0$, $a_3$ and $c_0$ using the system (9). This gives

$$R^1 = \begin{pmatrix} 1 - 2a_3^2 & -2a_0a_3 & 0 \\ 2a_0a_3 & 1 - 2a_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad R^2 = \begin{pmatrix} (4c_0^2 - 1)(2a_3^2 - 1) & 2c_0(a_3 - a_0) & 2a_0a_3 \\ 4c_0^2 - 2a_0a_3 - 1 & 2c_0(1 - 4c_0^2)(a_0 + a_3) & 2a_3^2 - 1 \\ 4c_0(1 - 2c_0^2)(a_0 + a_3) & 1 - 4c_0^2 & 0 \end{pmatrix}$$

Hence the position of joint 3 is given by

$$\begin{aligned} x &= 8c_0^2a_3^2 - 4c_0^2 - 4a_3^2 + 2 \quad , \quad y = 4c_0^2 - 1 \\ z &= -8c_0^3a_0 - 8c_0^3a_3 + 4c_0a_0 + 4c_0a_3. \end{aligned} \tag{11}$$

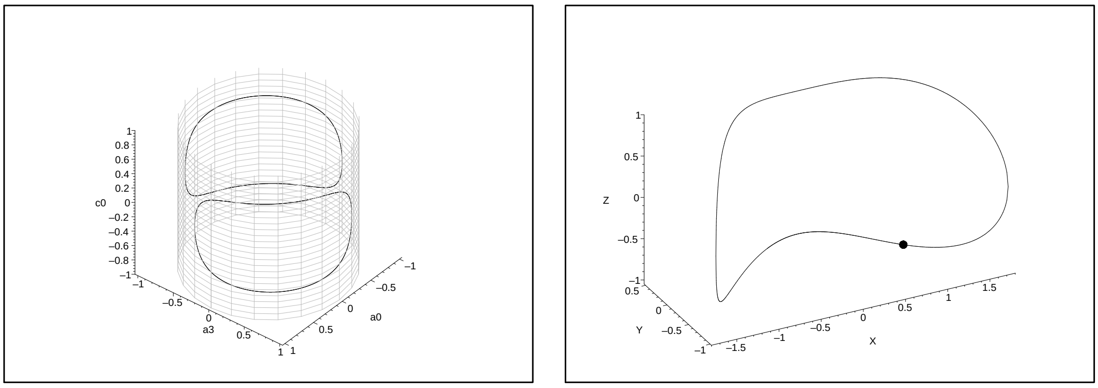


Figure 2: On the left the variety $\mathsf{V}(\mathcal{I}_{\text{ess}}) \subset \mathbb{R}^3$. On the right the possible positions of joint 3 in global coordinates. The dot marks the initial position $(1, 0, -1)$ of the joint 3 shown in Figure 1.

Now consider the ideal

$$J = \langle -x+8c_0^2a_3^2-4c_0^2-4a_3^2+2,\ -y+4c_0^2-1,\ -z-8c_0^3a_0-8c_0^3a_3+4c_0a_0+4c_0a_3,\ q_1,\ q_2\rangle.$$

Computing the Gröbner basis of $J$ in $\mathbb{Q}[(c_0, a_0, a_3), (x, y, z)]$ yields the elimination ideal

$$E_J = \langle y^2 + z^2 - 1,\ x^2 + 2y - 1\rangle\ .$$

Hence $\mathsf{V}(E_J)$ gives the possible positions of the joint 3. It is easy to check that this is again a smooth Jordan curve in $\mathbb{R}^3$. Substituting the parametrizations of $a_0$, $a_3$ and $c_0$ into (11) it is seen that the curve $\mathsf{V}(E_J)$ can be parametrized by $\beta_\pm : [-\pi/6, 7\pi/6] \mapsto \mathbb{R}^3$

$$\beta_\pm(\alpha) = \left( \frac{1-\sin(\alpha)}{\cos(\alpha)}\ ,\ \frac{\sin(\alpha)}{1+\sin(\alpha)}\ ,\ \frac{\pm\sqrt{1+2\sin(\alpha)}}{1+\sin(\alpha)} \right)\ .$$

Similarly we can plot the whole Bricard's system, see Figure 3.

As one can expect the end points $\alpha = -\pi/6$ and $\alpha = 7\pi/6$ are points where the Bricard's system is in the plane $z = 0$. In these configurations the Bricard's system is in the form of an equilateral triangle.

# 7 Conclusions

We have shown using the tools of modern algebraic geometry and computational algebra that the mobility of Bricard's mechanism is indeed one, and moreover we have explicitly parametrized the configuration space. The key parameter is the anticlokwise angle $\alpha$ between the vector representing rod A and the global basis vector $e^1$. In the process we have seen that the variety defined by the initial equations contains many spurious components. This is probably the case also more generally:

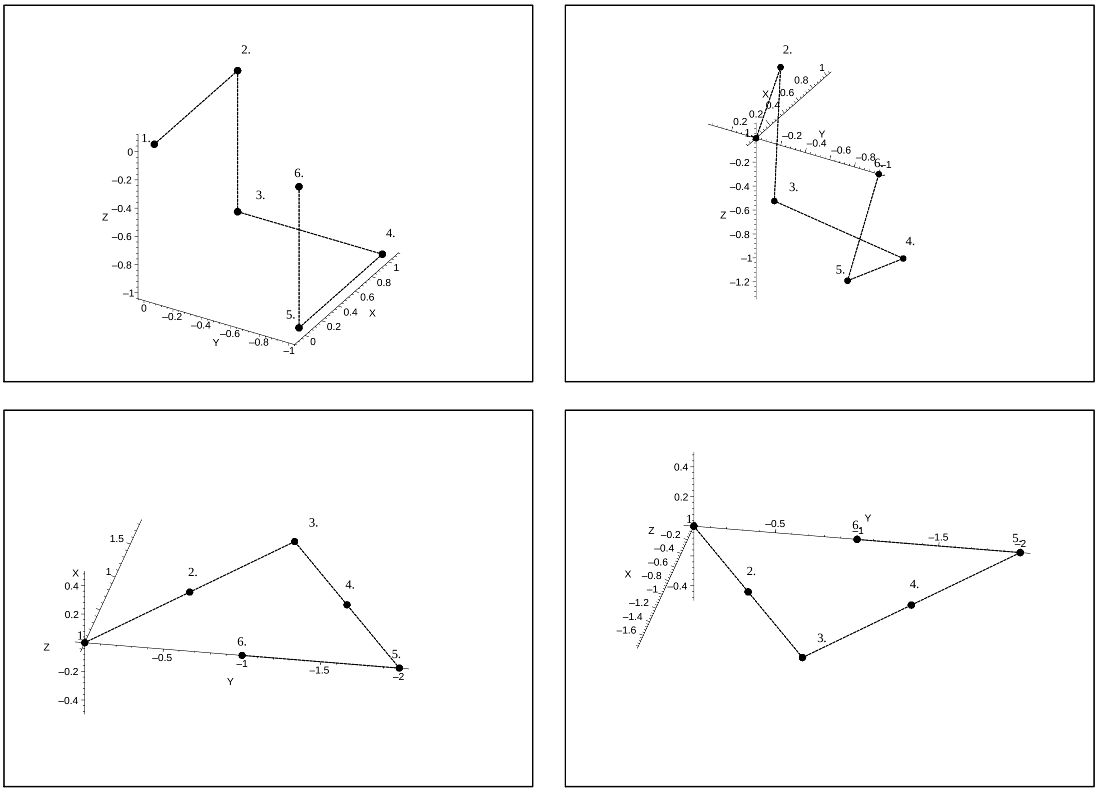

Figure 3: From upper left to lower right: Bricard's system when $\alpha = 0,\ \pi/9,\ -\pi/6$ and $7\pi/6$. Dots represent the joints $1, 2, 3, 4, 5, 6$.

the ideal defined by constraints may be "far from being prime". This suggests that the analysis performed above for Bricard's system will be useful for general multibody systems because it is likely to lead to a better understanding of the structure of the configuration space. Indeed we can argue that the "real" configuration space is the relevant irreducible component and not the whole variety defined by the constraints.

In addition our analysis gave a formulation for constraints which is more suitable for numerical computations. In fact the reason for choosing Bricard's mechanism as a benchmark problem disappears because using our final system based on the generators of the relevant prime ideal the numerical problem becomes a standard well-posed problem in multibody dynamics.

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