Matrix approach to polynomials 2

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Abstract

We continue our study of the structure initiated in [Arp03]. Our main emphasis is exploring further structure into the formalism introduced in [Arp03]. This formalism reveals beautiful interplay between certain elementary operators, and provides tools for example for checking, handling and generalizing combinatorial identities, as we show in examples. In addition to that we discover a group structure among Vandermonde matrices, a fascinating diophantine equation and new proofs and generalizations of recently found related results.

Keywords: Pascal matrix, Stirling matrix, Tepper identity, Diophantine equation, translation, scaling.
AMS classification: 11Cxx,11Dxx

1 Introduction

We continue the study of our matrix approach to polynomials, initiated in [Arp03]. Our main emphasis is that we have found this matrix formalism very convenient and useful for handling polynomials and combinatorial identities: to check the validity of a claimed (or guessed) identity, or to get new insight, or to generalize them.

We will study further the fascinating interplay between Pascal, Stirling, and Vandermonde matrices, yet we will focus particularly on Vandermondes. We give certain decompositions for them and reveal a group structure among Vandermondes with linearly spaced nodes. A bit surprisingly, though maybe not unexpectedly, these can be derived with the help of two associated operations: translation ($\mathcal{C}$) and scaling ($\mathcal{D}$) who themselves have interesting properties.

Recently we have learnt that the same subject, namely matrix formalism in this context, has been treated also in many recent publications we were unaware of during writing [Arp03]. In particular our matrix M, or its transpose, is apparently widely known as the Pascal matrix. See [BP92, CV93, BT00, AT01, CK01] and references therein. Here we must note that these seem to be unaware of the work of D. Kalman et al. from 1980s, see [Kal83, KU87] and references therein.

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In most of the references Pascal matrix is defined in a *lower* triangular form, while we defined it in [Arp03] as an *upper* triangular. The reason for the latter is that we want to consider matrices as operating on vectors, and it is quite customary that matrices operate from the *left* on column vectors. Hence M turned out to be upper triangular.

Our main result in this paper is the formalism itself as a useful tool, and to support this, we give several applications: first, we discover a fascinating group structure among Vandermonde matrices. Second, we give explicit LDU decompositions of those. Third, new proofs and generalizations of some known results. And last but not least, properties of $\mathcal{C}$ and $\mathcal{D}$ can be visualized as a Diophantine problem.

2 Preliminaries

2.1 Notation

We recall the notation and conventions from [Arp03]. Denote by $\mathbb{R}^\infty$ the direct sum of numerably many $\mathbb{R}$'s, that is, the collection of $\mathbb{R}-$sequences with only finitely many nonzero elements. Clearly a polynomial $p \in \mathbb{R}[x]$ can be interpreted as an element in $\mathbb{R}^\infty$ by its coefficients, i.e. define mapping $ke : \mathbb{R}[x] \to \mathbb{R}^\infty$ by

$$ke: \quad p_0 + p_1 x + p_2 x^2 + \cdots + p_N x^N \mapsto (p_0,\, p_1,\, p_2,\, \ldots,\, p_N,\, 0,\, 0,\, \ldots).$$

If $p \in \mathbb{R}[x]$, we denote by p_j the coefficient of x^j of p.

Some conventions: $0^0 := 1$. We will use the same notation, p, whether $p \in \mathbb{R}[x]$ or $p = ke(p(x)) \in \mathbb{R}^\infty$. It will be clear from context which one is meant. Binomial coefficient is considered as a polynomial in the upper argument (the lower argument will always be a nonnegative integer):

$$\binom{x}{0} = 1, \qquad \binom{x}{k} = \frac{1}{k!} x(x-1)(x-2)\ldots(x-k+1) \quad \text{for } k \in \mathbb{N}.$$

The Stirling numbers of the first kind, denoted here by $s(n,k)$, are defined by

$$x(x-1)\cdots(x-k+1) =: \sum_{n=0}^{k} s(n,k) x^n.$$

Note that some authors, for example [CK01, EFP98], define the Stirling number of first kind to be the absolute value of our $s(n,k)$; in our case $\operatorname{sgn}(s(n,k)) = (-1)^{n-k}$.

The Stirling numbers of the second kind, denoted here by $\mathcal{S}(n,k)$, are defined by

$$x^k =: \sum_{n=0}^{k} \mathcal{S}(n,k)\, x(x-1)\cdots(x-n+1).$$

A matrix or a vector 'is integers' means that its elements are integers. Notation $A(:,j)$ is the j^{th} column of A and $A(j,:)$ is the j^{th} row of A. We will numerate the rows and columns of a matrix beginning from 0, to make the formulas simpler. That is, the top row of an $(N+1) \times (N+1)-$matrix A is $A_{00},\ A_{01},\ A_{02},\ \ldots,\ A_{0N}$. Vectors are considered as elements of $\mathbb{R}^\infty$. That is, when two vectors of different length are added or dot-producted, the shorter one is filled with zeros at the end. Denote by $e_n := (0,\ldots,0,1,0,\ldots) \in \mathbb{R}^\infty$ the n^{th} column of unit matrix, where $n = 0, 1, \ldots$. In other words, $e_n = ke(x^n)$. For a matrix A, $\operatorname{diag}(A)$ means the vector of diagonal elements of A. And, $\operatorname{diag}(a_1,\ldots,a_n)$ means a diagonal matrix with elements $(a_1,\ldots,a_n)$.

2.2 Definitions

Fix a positive integer N. Suppose $p \in \mathbb{R}[x]$ with $\deg(p) \leq N$. The polynomials $\binom{x}{0}, \ldots, \binom{x}{N}$ in x are linearly independent over $\mathbb{R}$, because $\deg\binom{x}{k} = k$. Hence we can write $p(x) =: \sum_{u=0}^{N} b_u \binom{x}{u}$.

Definition 2.1. Mappings $bin, val : \mathbb{R}[x] \to \mathbb{R}^\infty$ by

$$\begin{aligned} bin(p) &:= \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_N & 0 & 0 & \ldots \end{pmatrix} \in \mathbb{R}^\infty \\ val(p) &:= \begin{pmatrix} p(0) & p(1) & p(2) & \cdots & p(N) & 0 & 0 & \ldots \end{pmatrix} \in \mathbb{R}^\infty . \end{aligned}$$

Mappings ke and bin are bijective and linear. So is val when we keep N fixed and restrict val to polynomials of degree $\leq N$. Since we have fixed N, we can present $ke(p)$, $bin(p)$, $val(p)$ as $N+1-$vectors for any $\deg(p) \leq N$.

In [Arp03] we introduced the matrix M which shifts the graph of a polynomial one step to the left, in the sense that $ke(p(x+1)) = M\, ke(p(x))$. It turns out that this is the upper triangular form of a Pascal matrix:

$$M = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{N}{0} \\ & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{N}{1} \\ & & \binom{2}{2} & \cdots & \binom{N}{2} \\ & & & \ddots & \vdots \\ & & & & \binom{N}{N} \end{pmatrix}.$$

Recall that

$$\text{Vandermonde}(x_0,\, x_1,\, x_2,\, x_3,\, \ldots, x_N) := \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ x_0 & x_1 & x_2 & x_3 & \ldots & x_N \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & \ldots & x_N^2 \\ \vdots & & & \ddots & & \vdots \\ x_0^N & x_1^N & x_2^N & x_3^N & \ldots & x_N^N \end{pmatrix}.$$

Definition 2.2. We shall make use of the following Vandermonde matrix:

$$V := \text{ Vandermonde}(0,\, 1,\, \ldots,\, N)$$

and the upper triangular *Stirling* matrices:

$$\begin{aligned} St_1 &:= (s(i,j))_{i,\, j=0\ldots N} \\ St_2 &:= (\mathcal{S}(i,j))_{i,\, j=0\ldots N} . \end{aligned}$$

Further we define, with a a scalar,

$$\begin{aligned} \Lambda &:= \text{diag}(0,\, 1,\, \ldots,\, N) \\ a^\Lambda &:= \text{diag}(1,\, a,\, a^2,\, \ldots,\, a^N) \\ \Lambda! &:= \text{diag}(0!,\, 1!,\, 2!, \ldots,\, N!). \end{aligned}$$

Now $St_1^{-1} = St_2$ and of course we could use St_2^{-1} in place of St_1 (as is quite customary in literature), but in our opinion there are two reasons not to: first, the formulas are simpler when we use both of these, and, the simpler formulas, the clearer result. Second, and more importantly, using notation St_1 instead of St_2^{-1} reminds us that we know the inverse of St_2 explicitly and even recursive rules for its elements, facts worth remembering if we are to develop algorithms from our theory. However, in this paper we are not emphasizing development of practical algorithms.

The shift and derivation matrices, again upper triangular, are denoted by S and D, respectively:

$$S := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}, \qquad D := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & & & \ddots & N \\ & & & & 0 \end{pmatrix}.$$

That D is the derivation matrix means here $ke(Dp) = ke(p')$ for all polynomials p with $\deg(p) \leq N$ where prime indicates the usual derivative.

3 Structure

In this section we first revise our previous relevant results and after that state and prove the new results.

3.1 Revision

In this section we state the results from [Arp03] which are needed in this paper. For proofs as well as further results we refer to [Arp03].

Proposition 3.1.

$$\begin{aligned} ke(p(x+a)) &= M^a ke(p), \quad \text{where } M^a := \exp(aD), &(1)\\ M^a &= a^{-\Lambda} M a^{\Lambda} \quad \forall a \in \mathbb{R} \setminus \{0\} &(2)\end{aligned}$$

especially, when $s \in \mathbb{Z}$, M^s coincides with the s^{th} power of M.

Remark 3.1. An immediate consequence is commutativity: $M^s M^t = M^{s+t}$ as well as $M^{sT} M^{tT} = M^{(s+t)T}$. This property was apparently first proven in [Kal83] and, independently, in [BP92, CV93].

Proposition 3.2. $V^T ke(p) = val(p) = M^T\, bin(p)$.

Proposition 3.3. Let Δ be the difference operator: $\Delta p(x) := p(x+1) - p(x)$. Then

- $ke(\Delta p) = (M - I)\, ke(p)$
- $bin(\Delta p) = S\, bin(p)$.

Proposition 3.4. Decompositions:

- the LDU decomposition of $V = St_2^T\, \Lambda!\, M$
- the Jordan decomposition of $M = (St_1\, \Lambda!^{-1})(I + S)(\Lambda!\, St_2)$.

The LDU decomposition of V has been proven, independently, by several other authors as well, for example in [MS58, CK01, EFP98, AT01]. The Jordan decomposition of M has been proven also in [AT01]. In section 4.3 we give the LDU decomposition of a Vandermonde matrix with linearly spaced nodes, which seems to be a new result.

Proposition 3.5. Intertwining property: $St_2 \Delta = D St_2$ and $\Delta St_1 = St_1 D$, where $\Delta = M - I$ i.e. the matrix for the difference operator in $ke - ke$ bases.

3.2 Further structure

Proposition 3.6. The Pascal matrix M acts as a shifting (or translation) operator in two ways; for any $a \in \mathbb{R}$:

- $M^a ke(p(x)) = ke(p(x+a))$ for all polynomials p
- $M^{aT} \begin{pmatrix} 1 \\ x \\ x^2 \\ \dots \\ x^N \end{pmatrix} = \begin{pmatrix} 1 \\ x+a \\ (x+a)^2 \\ \dots \\ (x+a)^N \end{pmatrix}.$

Proof. The first claim is already stated in proposition 3.1. The second claim comes from binomial theorem: the i^{th} component of the left hand side is

$$\sum_{\nu=0}^{N} (M^a)_{\nu i} x^\nu = \sum_{\nu=0}^{i} \binom{i}{\nu} a^{i-\nu} x^\nu = (a+x)^i$$

which is the i^{th} component of the right hand side. □

Remark 3.2. The latter property appears, in case $x = 0$, in [AT01] (in their notation, $W(a) = P(a)W(0)$). Also, for $a \in \mathbb{Z}$ this is the same as the "swapping lemma" in [BP92].

Lemma 3.1. Some properties of the mapping *bin*:

1. $bin(p) = \begin{pmatrix} p(0) & \Delta p(0) & \dots & \Delta^N p(0) \end{pmatrix}^T$
2. $bin(p) = \Lambda!\, St_2\, ke(p)$
3. $ke(p) = St_1 \Lambda!^{-1}\, bin(p)$.

We note that the base changing ($ke \mapsto bin$) matrix is the same as the Jordan transformation matrix of M.

Proof. The first result uses proposition 3.3 and is in the proof of Newton's theorem in [Arp03]: $\Delta^k p(0) = e_0^T bin(\Delta^k p) = e_0^T S^k bin(p) = (S^k bin\ p)_0 = (bin\ p)_k$. The second and third claim are clearly equivalent and come from applying the LDU decomposition of V (proposition 3.4) to proposition 3.2. □

Next we look at some implications of the Jordan decomposition of M: denote first $J := I + S$ so the latter part of proposition 3.4 is

$$M = St_1\, \Lambda!^{-1}\, J\, \Lambda!\, St_2$$

which we can use to *define* J^a for all $a \in \mathbb{R}$:

$$J^a := \Lambda!\, St_2\, M^a\ St_1 \Lambda!^{-1} = \exp(\Lambda!\, St_2\, aD\, St_1\, \Lambda!^{-1}). \tag{3}$$

Especially, when $a \in \mathbb{Z}$, J^a coincides with the a^{th} power of J (this is immediate from the properties of exp, proof is as in proposition 3.1). See also proposition 3.7.

Lemma 3.2.

$$J^a bin(p) = \begin{pmatrix} p(a) \\ \Delta p(a) \\ \vdots \\ \Delta^N p(a) \end{pmatrix} \quad \forall a \in \mathbb{R}. \tag{4}$$

Proof. Applying lemma 3.1 and proposition 3.1:

$$\begin{aligned} J^a bin(p) &= \Lambda!\, St_2\, M^a\ \ St_1 \Lambda!^{-1}\, bin(p) = \Lambda!\, St_2\, M^a\, ke(p) \\ &= \Lambda!\, St_2\, ke(p(\cdot + a)) = bin(p(\cdot + a)) = \begin{pmatrix} p(a) \\ \Delta p(a) \\ \vdots \\ \Delta^N p(a) \end{pmatrix}. \end{aligned} \tag{5}$$

□

As a corollary we get an equivalent definition for J^a:

Proposition 3.7. For all $a \in \mathbb{R}$,

$$(\Lambda!\, St_2\, M^a\ \ St_1 \Lambda!^{-1})_{jk} = \begin{cases} \binom{a}{k-j}, & k \geq j \\ 0, & k < j. \end{cases}$$

Proof. Let p be an arbitrary polynomial (of degree $\leq N$, as before). From lemma 3.2 and the proof of lemma 3.1:

$$\Delta^j p(a) = \sum_{k=j}^{N} (J^a)_{jk} (bin\ p)_k = \sum_{k=j}^{N} (J^a)_{jk} (S^j bin\ p)_{k-j} = \sum_{\nu=0}^{N-j} (J^a)_{j,\nu+j} (bin\ \Delta^j p)_\nu.$$

Since this is true for any p, by definition of bin coefficients (see also proposition 4.2) we get $(J^a)_{j,\nu+j} = \binom{a}{\nu}$ for all $\nu = 0 \dots N - j$. Hence the claim follows. □

Proposition 3.8. Let us define $\log(J) := \Lambda!\, St_2\, D\, St_1\, \Lambda!^{-1}$ compatibly with (3). Then

1. $V^{-1} D^{jT} V = M^{-1} (\log(J))^{jT} M, \quad j \in \mathbb{N}$
2. $\Lambda!^{-1} S \Lambda! = D$
3. $M^a = \Lambda!^{-1} \exp(aS) \Lambda! \quad s \in \mathbb{R}$.

Proof. The first claim:

$$V^{-1} D^{jT} V = (V^{-1} D^T V)^j = (M^{-1}\, \Lambda!^{-1}\, St_1^T\, D^T\, St_2^T\, \Lambda!\, M)^j = M^{-1}\, (\Lambda!\, St_2\, D\, St_1\, \Lambda!^{-1})^{jT}\, M$$

where the expression inside the parenthesis is by definition $\log(J)$.

The second claim is straightforward: only $S_{i,i+1}$ is different from zero, and i^{th} row of S is multiplied by $1/i!$ and $(i + 1)^{st}$ column of S is multiplied by $(i + 1)!$, hence we get $D_{i,i+1}$.

The last claim comes from $M^a = \exp(aD)$ by using the second claim. □

For the following proposition we do not have any particular application, but included it for the sake of peculiarity: the exponentials of M (and hence of D) can essentially be achieved by transforming M itself by Stirling matrices. Denote $e := \exp(1)$.

Proposition 3.9.

$$\begin{aligned} M = \exp(D) &= I + St_1\, D\, St_2 & (6)\\ \exp(M) &= e\, St_1\, M\, St_2 = eI + (St_1)^2\,(eD)\,(St_2)^2 & (7)\\ \exp(\exp(M)) &= (St_1)^2\,(eM)^e\,(St_2)^2. & (8)\end{aligned}$$

Proof. For (6): the first equality is just a restatement of proposition 3.1, and the second equality is immediate from proposition 3.5. To get (7) we evaluate, by using (6),

$$\exp(M) = St_1\, \exp(I + D)\, St_2 = St_1\, \exp(I)\exp(D)\, St_2 = e\, St_1\, M\, St_2$$

where the middle equality is due to I and D commute. For (8): we need to evaluate

$$\exp(eM) = St_1\, \exp(eI + eD)\, St_2 = St_1\, \exp(eI)\exp(eD)\, St_2 = St_1\, e^e M^e\, St_2$$

hence

$$\exp(\exp(M)) = St_1\, \exp(eM)\, St_2 = (St_1)^2\,(eM)^e\,(St_2)^2.$$

□

Remark 3.3. The first equality in (6) has been rediscovered in literature several times but we believe [Kal83] to be the first one.

4 Applications

4.1 Group structures associated with Vandermonde matrices

First we define the basic building block for handling Vandermondes:

Definition 4.1. Let $z \in \mathbb{R}$. Since V is invertible, the following defines the vector $c^N(z)$ uniquely:

$$V c^N(z) := \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^N \end{pmatrix}. \tag{9}$$

As before we assume N an arbitrary, fixed, positive integer. Previously we have supressed it from notations as unnecessary but here, in context with $c^N(z)$, we have found it convenient to include N into the notation. These have a number of interesting properties as shown in the following.

Theorem 1. Properties of $c^N(z)$. We remind the reader of our $\mathbb{R}^\infty$ convention, which is needed in the second statement.

1. $c^N(z)$ is integers if and only if z is an integer
2. $c^N(z) = c^{N-1}(z) + M^{-1}(:, N)\binom{z}{N}$
3. $c^N(z) = M^{-1} \begin{pmatrix} \binom{z}{0} \\ \vdots \\ \binom{z}{N} \end{pmatrix}$

4. if $z \in \{0, \dots, N\}$, then $c^N(z) = e_z$
5. $c^N(z_0), \dots, c^N(z_N)$ are linearly independent if and only if $z_0, \dots, z_N \in \mathbb{R}$ are mutually distinct.

Proof. We prove first property 3: denote $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{R}^{N+1}$. Using LDU decomposition of V and definition of $c^N(z)$:

$$V\, c^N(z) = St_2^T\, \Lambda!\, M\, c^N(z) = z^\Lambda\, \mathbf{1} \tag{10}$$

hence

$$M\, c^N(z) = \Lambda!^{-1}\, St_1^T\, z^\Lambda\, \mathbf{1}. \tag{11}$$

Now

$$(\Lambda!^{-1}\, St_1^T\, z^\Lambda\, \mathbf{1})_j = \frac{1}{j!} \sum_{n=0}^{N} s(n,j)\, z^n = \frac{1}{j!} \sum_{n=0}^{j} s(n,j)\, z^n = \binom{z}{j} \tag{12}$$

where the last equality follows from definition of $s(n,j)$. Hence $(M\, c^N(z))_j = \binom{z}{j}$ and property 3 is proven.

Property 2: write open property 3:

$$\begin{aligned} c^N(z) &= M^{-1} \begin{pmatrix} \binom{z}{0} \\ \dots \\ \binom{z}{N-1} \\ 0 \end{pmatrix} + M^{-1} \begin{pmatrix} 0 \\ \dots \\ 0 \\ \binom{z}{N} \end{pmatrix} = M^{-1} \begin{pmatrix} \binom{z}{0} \\ \dots \\ \binom{z}{N-1} \end{pmatrix} + M^{-1}(:, N)\binom{z}{N} \\ &= c^{N-1}(z) + M^{-1}(:, N)\binom{z}{N} \end{aligned} \tag{13}$$

where the second equality follows from our $\mathbb{R}^\infty$ convention and the last from property 3.

Property 1: note first that $M^{-1} = (-1)^\Lambda\, M\, (-1)^\Lambda$ is integers since M and Λ are. If $z \in \mathbb{Z}$ then $\binom{z}{j} \in \mathbb{Z}$ for all j. By property 3, $c^N(z)$ is then integers. On the other way, suppose $c^N(z)$ is integers. Now $(c^N(z))_N = \binom{z}{N}$ (since M^{-1} has unit diagonal) hence $\binom{z}{N} \in \mathbb{Z}$. By property 2,

$$c^{N-1}(z) = c^N(z) - M^{-1}(:, N)\binom{z}{N}$$

hence $c^{N-1}(z)$ is integers. Arguing as before, $(c^{N-1}(z))_{N-1} = \binom{z}{N-1} \in \mathbb{Z}$. By induction, $(c^1(z))_1 = \binom{z}{1} \in \mathbb{Z}$ hence $z \in \mathbb{Z}$.

Property 4 is trivial since $\begin{pmatrix} 1 & j & j^2 & \dots & j^N \end{pmatrix}^T = V(:, j)$ for $j \in \{0, \dots, N\}$.

Property 5: since V is invertible the claim is equivalent with

$$\begin{pmatrix} V c^N(z_0) & | & \dots & | & V c^N(z_N) \end{pmatrix}$$

being an invertible matrix. But this is, by definition of c^N, the Vandermonde$(z_0, \dots, z_N)$ which is well known to be invertible if and only if the nodes $z_0, \dots, z_N$ are distinct. $\square$

Definition 4.2. Let $N \in \mathbb{N}$ and $z \in \mathbb{R}$ be given.

$$\mathcal{C}_z := \left(\begin{array}{c|c|c|c} c^N(z) & c^N(z+1) & \dots & c^N(z+N) \end{array} \right)$$

$$\mathcal{D}_z := \left(\begin{array}{c|c|c|c} c^N(0) & c^N(z) & \dots & c^N(Nz) \end{array} \right).$$

Most fascinating for us is that these act on $c^N(z)$ as shifting and scaling operators, respectively.

Proposition 4.1. Let $a \in \mathbb{R}$ and N, z as before. Then

1. $\mathcal{D}_a c^N(z) = c^N(a\,z)$ (scaling),
2. $\mathcal{C}_a c^N(z) = c^N(a+z)$ (shifting),
3. $V\mathcal{C}_a = M^{aT}V$,
4. $V\mathcal{D}_a = a^\Lambda V$,
5. Vandermonde$(a,\, a+b,\, a+2b,\, \ldots,\, a+Nb) = V\mathcal{C}_a\mathcal{D}_b$.

Proof. First,

$$V\, c^N(az) = (az)^\Lambda\, \mathbf{1} = a^\Lambda\, z^\Lambda\, \mathbf{1} = a^\Lambda\, V\, c^N(z). \tag{14}$$

Especially,

$$V\, c^N(aj) = a^\Lambda\, V(:,j) \quad \forall j \in \{0, \ldots, N\} \tag{15}$$

hence

$$V\,\mathcal{D}_a = \big(a^\Lambda\, V(:,0) \;\;|\;\; \ldots \;\;|\;\; a^\Lambda\, V(:,N)\big) = a^\Lambda V \tag{16}$$

which proves the property 4.

From (14) we get $c^N(az) = V^{-1}a^\Lambda\, V\, c^N(z)$. By property 4, $\mathcal{D}_a = V^{-1}a^\Lambda V$ hence property 1 is proven.

From proposition 3.6 we get $(M^{aT}V)_{ij} = (a+j)^i \quad \forall a \in \mathbb{R}$. Now, $(V\mathcal{C}_a)_{ij} = (a+j)^i = (M^{aT}V)_{ij}$ which proves property 3.

The latter part of proposition 3.6 can be written as

$$M^{aT} z^\Lambda\, \mathbf{1} = (z+a)^\Lambda\, \mathbf{1}$$

which is equivalent with

$$M^{aT} V c^N(z) = V c^N(a+z) = V\mathcal{C}_a c^N(z)$$

where the last equality comes from property 3. Multiplying by V^{-1} gives property 2.

Property 5:

$$\begin{aligned}
&\text{Vandermonde}(a,\, a+b,\, a+2b,\, \ldots,\, a+Nb)\\
&\qquad = V\begin{pmatrix} & | & & | & & | & \\ c^N(a) & | & c^N(a+b) & | & \ldots & | & c^N(a+Nb) \\ & | & & | & & | & \end{pmatrix}\\
&\qquad = V\mathcal{C}_a\begin{pmatrix} & | & & | & & | & \\ c^N(0) & | & c^N(b) & | & \ldots & | & c^N(Nb) \\ & | & & | & & | & \end{pmatrix} = V\mathcal{C}_a\mathcal{D}_b.
\end{aligned} \tag{17}$$

□

Theorem 2. Both $\mathcal{C}_z$ and $\mathcal{D}_z$ have a group structure:

- $\{\mathcal{C}_z\}_{z\in\mathbb{R}}$ is a (commutative) group: $\mathcal{C}_x\mathcal{C}_y = \mathcal{C}_{x+y}$. In particular, $\mathcal{C}_0 = I$ and $\mathcal{C}_x^{-1} = \mathcal{C}_{-x}$. Moreover, it has $\{\mathcal{C}_m\}_{m\in\mathbb{Z}}$ as a subgroup.

- $\{\mathcal{D}_z\}_{z\in\mathbb{R},\, z\neq 0}$ is a (commutative) group: $\mathcal{D}_x\mathcal{D}_y = \mathcal{D}_{xy}$. In particular, $\mathcal{D}_1 = I$ and $\mathcal{D}_x^{-1} = \mathcal{D}_{1/x}$.

Proof. That $\{\mathcal{C}_z\}_{z\in\mathbb{R}}$ is a group stems from the similarity of $\mathcal{C}_z$ to M^{zT}, shown in proposition 4.1. The commutativity is due to remark 3.1. In a similar fashion, $\mathcal{D}_z$ gets its group properties from similarity with z^Λ. Details:

$$\begin{aligned}
\mathcal{C}_x\mathcal{C}_y &= (V^{-1}M^{xT}V)(V^{-1}M^{yT}V) = V^{-1}M^{(x+y)T}V = \mathcal{C}_{x+y} \\
\mathcal{D}_x\mathcal{D}_y &= (V^{-1}x^\Lambda V)(V^{-1}y^\Lambda V) = V^{-1}(xy)^\Lambda V = \mathcal{D}_{xy}.
\end{aligned}$$

If $z = m \in \mathbb{Z}$, then $\mathcal{C}_m^{-1} = \mathcal{C}_{-m}$ and the subgroup property is clear. □

Let $N \in \mathbb{N}$ be fixed, as before. Denote by $\mathcal{V}$ the set of $(N+1)\times(N+1)$ Vandermonde matrices with linearly spaced nodes. That is, if V_1, $V_2 \in \mathcal{V}$ they are of the form

$$\begin{aligned}
V_1 &= \text{Vandermonde}(a,\, a+b,\, a+2b,\, \dots,\, a+Nb) \\
V_2 &= \text{Vandermonde}(c,\, c+d,\, c+2d,\, \dots,\, c+Nd).
\end{aligned} \tag{18}$$

Theorem 3. There is a group structure in $\mathcal{V}$ with respect to the operation:

$$\mu : (V_1,\, V_2) \mapsto V_1V^{-1}V_2.$$

The unit of this group is V and the inverse elements are given by

$$inv(V_1) = VV_1^{-1}V \in \mathcal{V} \quad \forall V_1 \in \mathcal{V}$$

where V_1^{-1} is the usual matrix inverse. Furthermore, if V_1 and V_2 are as in (18), their product in this group is equal to

$$\mu(V_1,\, V_2) = \text{Vandermonde}(a+bc,\, a+bc+bd, a+bc+2bd,\, \dots,\, a+bc+Nbd).$$

Proof. By property 5 of proposition 4.1

$$\begin{aligned}
V_1 &= V\mathcal{C}_a\mathcal{D}_b \\
V_2 &= V\mathcal{C}_c\mathcal{D}_d
\end{aligned}$$

and the claim is

$$V_1V^{-1}V_2 = V\mathcal{C}_{a+bc}\mathcal{D}_{bd} \in \mathcal{V}.$$

Expanding the product, this is equivalent with

$$V\mathcal{C}_a\mathcal{D}_b\mathcal{C}_c\mathcal{D}_d = V\mathcal{C}_{a+bc}\mathcal{D}_{bd}.$$

By proposition 4.1 and theorem 2:

$$\begin{aligned}
&\Leftrightarrow\ V\mathcal{C}_a\mathcal{D}_b\mathcal{C}_c\mathcal{D}_d = V\mathcal{C}_a\mathcal{C}_{bc}\mathcal{D}_b\mathcal{D}_d \\
&\Leftrightarrow\ \mathcal{D}_b\mathcal{C}_c = \mathcal{C}_{bc}\mathcal{D}_b \\
&\Leftrightarrow\ (V^{-1}b^\Lambda V)(V^{-1}M^{cT}V) = (V^{-1}M^{bcT}V)(V^{-1}b^\Lambda V) \\
&\Leftrightarrow\ b^\Lambda M^{cT} = M^{bcT}b^\Lambda
\end{aligned} \tag{19}$$

and the last claim is true by proposition 3.1. Hence $\mu(V_1,\, V_2) \in \mathcal{V}$. Next, $VV_1^{-1}V \in \mathcal{V}$ follows again from proposition 4.1 and theorem 2:

$$VV_1^{-1}V = V(\mathcal{D}_{1/b}\mathcal{C}_{-a}V^{-1})V = V\mathcal{D}_b^{-1}\mathcal{C}_a^{-1} \in \mathcal{V}.$$

Other group properties are easily checked:

- $\mu(V_1, V) = V_1 = \mu(V, V_1)$ (V is the unit)
- $\mu(V_1, \mu(V_2, V_3)) = \mu(\mu(V_1, V_2), V_3)$ (associativity)
- $\mu(V_1,\, VV_1^{-1}V) = V$, hence $inv(V_1) = VV_1^{-1}V \in \mathcal{V}$.

$\square$

Remark 4.1. From theorem 2 follows immediately that $\mathcal{V}$ has (at least) two commutative subgroups, namely $\{V\mathcal{C}_z\}_{z\in\mathbb{R}}$ and $\{V\mathcal{D}_z\}_{z\in\mathbb{R},\, z\neq 0}$. For example $inv(V\mathcal{C}_z) = V\mathcal{C}_{-z}$ and $inv(V\mathcal{D}_z) = V\mathcal{D}_{1/z}$.

Remark 4.2. From (19) we get as a corollary, by setting $c = 1$,

$$\mathcal{C}_b = \mathcal{D}_b\mathcal{C}_1\mathcal{D}_{b^{-1}}$$

which has a nice geometrical interpretation, operating on an arbitrary $c^N(z)$: "shift b units" is equal to "scale by b^{-1}, shift one unit, rescale by b".

This should be compared to a similar property of the translation $M^s = s^{-\Lambda}Ms^{\Lambda}$, operating on an arbitrary $ke(p)$: "shift s units = scale by s, shift one unit, rescale by s^{-1}".

4.2 Further structure with $\mathcal{C}_a, \mathcal{D}_b$

Proposition 4.2. Denote $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^{N+1}$. Polynomial evaluation can be expressed by ke, bin, and val coefficients: Here we remind the reader about our $\mathbb{R}^\infty$ convention.

$$\begin{aligned} p(z) &= ke(p)^T z^{\Lambda}\mathbf{1} \\ &= val(p)^T c^n(z) \\ &= bin(p)^T \begin{pmatrix} \binom{z}{0} \\ \vdots \\ \binom{z}{n} \end{pmatrix}. \end{aligned} \tag{20}$$

Proof. The first and last equalities follow trivially from definitions, but the second one needs proposition 3.2 to see that

$$val(p)^T c^n(z) = (V^T ke(p))^T c^n(z) = ke(p)^T V c^n(z) = ke(p)^T z^{\Lambda}\mathbf{1}.$$

$\square$

Proposition 4.3.

$$\begin{pmatrix} p(a) & p(a+b) & \ldots & p(a+Nb) \end{pmatrix}^T = \mathcal{D}_b^T\, \mathcal{C}_a^T val(p). \tag{21}$$

Proof. From propositions 4.2 and 4.1:

$$\begin{pmatrix} p(a) \\ p(a+b) \\ \vdots \\ p(a+Nb) \end{pmatrix} = \begin{pmatrix} c^N(a)^T val(p) \\ c^N(a+b)^T val(p) \\ \vdots \\ c^N(a+Nb)^T val(p) \end{pmatrix} = \begin{pmatrix} c^N(0)^T \\ c^N(b)^T \\ \vdots \\ c^N(Nb)^T \end{pmatrix} \mathcal{C}_a^T val(p) = \mathcal{D}_b^T\, \mathcal{C}_a^T val(p).$$

$\square$

4.3 Decompositions

Proposition 4.4. We have the explicit LDU decompositions:

1. Vandermonde$(a,\, a+b,\, a+2b,\, \dots,\, a+Nb) = (M^{aT}\, b^{\Lambda}\, St_2^T\, b^{-\Lambda})\, (b^{\Lambda}\, \Lambda!)\, M$
2. Vandermonde$(a,\, a+1,\, a+2,\, \dots,\, a+N) = (M^{aT}\, St_2^T)\, \Lambda!\, M$
3. Vandermonde$(0,\, b,\, 2b,\, \dots,\, Nb) = (b^{\Lambda}\, St_2^T\, b^{-\Lambda})\, (b^{\Lambda}\, \Lambda!)\, M$.

Proof. The first one becomes, using propositions 4.1 and 3.4:

$$V\mathcal{C}_a\mathcal{D}_b = M^{aT} V\mathcal{D}_b = M^{aT}\, b^{\Lambda}\, V = M^{aT}\, b^{\Lambda}\, St_2^T\, \Lambda!\, M$$

which is almost the LDU decomposition, except for the lower triangular part: since both M^{aT} and St_2^T have unit diagonal, we have $\mathrm{diag}(M^{aT}\, b^{\Lambda}\, St_2^T) = \mathrm{diag}(b^{\Lambda})$. Hence we need to multiply it by $b^{-\Lambda}$ and we get the first claim.

The second and third claims are clearly special cases of the first one with $b = 1$, $a = 0$, respectively. □

The second claim has been proven, in a different way, also in [CK01, thm 2.4].

Let us define the *general binomial matrix* as:

$$M(x_0,\, x_1,\, \dots, x_N) := \begin{pmatrix} \binom{x_0}{0} & \binom{x_1}{0} & \cdots & \binom{x_N}{0} \\ \binom{x_0}{1} & \binom{x_1}{1} & \cdots & \binom{x_N}{N} \\ \vdots & & \cdots & \vdots \\ \binom{x_0}{N} & \binom{x_1}{N} & \cdots & \binom{x_N}{N} \end{pmatrix}.$$

Especially, $M(0, 1, \dots, N) = M$. It is related to the general Vandermonde matrix by

Proposition 4.5.

$$\text{Vandermonde}(x_0,\, x_1,\, x_2, \dots, x_N) = St_2^T\, \Lambda!\, M(x_0,\, x_1,\, x_2,\, \dots, x_N). \tag{22}$$

Proof. In the following, the first, third, and fourth equalities are due to definition 4.1, theorem 1 and proposition 3.4, respectively.

$$\begin{aligned} &\text{Vandermonde}(x_0,\, x_1,\, x_2, \dots, x_N) \\ &\quad = \left(\begin{array}{c|c|c|c} Vc^N(x_0) & Vc^N(x_1) & \dots & Vc^N(x_N) \end{array} \right) \\ &\quad = V \left(\begin{array}{c|c|c|c} c^N(x_0) & c^N(x_1) & \dots & c^N(x_N) \end{array} \right) \\ &\quad = V\, M^{-1} \begin{pmatrix} \binom{x_0}{0} & \binom{x_1}{0} & \cdots & \binom{x_N}{0} \\ \vdots & & \cdots & \vdots \\ \binom{x_0}{N} & \binom{x_1}{N} & \cdots & \binom{x_N}{N} \end{pmatrix} \\ &\quad = St_2^T\, \Lambda!\, M(x_0,\, x_1,\, x_2,\, \dots, x_N). \end{aligned} \tag{23}$$

□

Note that if $x_j = j$, this reduces to the LDU decomposition of V. A similar result is proven in [EFP98].

4.4 A generalization of Tepper's identity

Tepper conjectured in 1965 the following combinatorial identity

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}(x+l)^n = n!. \tag{24}$$

This is proved and generalized in [BT00, rem.1] as in the following proposition. We will give a new proof and a generalization.

Proposition 4.6. For any polynomial f with $\deg(f) =: n$, for all x,

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}f(x+l) = \quad 0 \qquad (n < p) \tag{25}$$

$$\sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}f(x+l) = \quad f_n n! \qquad (n = p). \tag{26}$$

Proof. Take our $N > n$. The result (25) is due to the following property of *any* upper triangular matrix: if U is an upper triangular matrix and v a vector with $v_p = 0$ for all $p > n$, then $(Uv)_p = 0$ for all $p > n$ as well.

Here, we claim, we have $U = \Lambda!\, St_2\, M^x$ which indeed is upper triangular and $v = ke(f)$. That is, we claim the left hand sides of (25), (26) to be

$$\left(\Lambda!\, St_2\, M^x\, ke(f)\right)_p. \tag{27}$$

To show this, let us put the left hand side into our formalism: first, $(-1)^{p-l}\binom{p}{l} = (M^{-T})_{pl}$ by proposition 3.1. The translational property of M^x (see proposition 3.1) gives $f(x+l) = (val(M^x f))_l$ which by proposition 3.2 expands to $\left(V^T M^x\, ke(f)\right)_l$. Hence the left hand side of (25) is equal to

$$\left(M^{-T}\, V^T\, M^x\, ke(f)\right)_p.$$

Now, the LDU decomposition of V (proposition 3.4) gives

$$M^{-T}\, V^T\, M^x = \Lambda!\, St_2\, M^x \tag{28}$$

which gives (27) and proves (25) since $(ke(f))_p = 0$ for all $p > n = deg(f)$. For (26), note that in (28), both St_2 and M^x have unit diagonal hence do not effect $(ke(f))_n$. But $\Lambda!$ multiplies $(ke(f))_n$ with $n!$ hence (26) is proven. □

We can immediately generalize this to:

Theorem 4. Assume f a given polynomial and $n := \deg(f)$. Denote

$$A_p := \sum_{l=0}^{p}(-1)^{p-l}\binom{p}{l}f(x+l).$$

Then $A_p = 0$ for $p > n$, and is a polynomial in x of degree $n - p$ for $p \leq n$. In the latter case the coefficients of A_p depend only on coefficients f_n, f_{n-1}, …, f_p (and n).

Proof. The cases $p \geq n$ are proposition 4.6. Assume $p \leq n$. Now A_p is still, as in the proof of proposition 4.6, given by (27). Let $v := M^x\, ke(f)$. Since $\Lambda!\, St_2$ is upper triangular, the value of A_p depends only on v_p, v_{p+1}, …, v_n. But v_{p+j} is a linear combination of x^j and f_{p+j} with $j = 0, \ldots, n-p$. Multiplying by $\Lambda!\, St_2$ brings in only constants. □

For example,

$$\begin{pmatrix} A_{n-2} \\ A_{n-1} \\ A_n \end{pmatrix} = \begin{pmatrix} (n-2)! & & \\ & (n-1)! & \\ & & n! \end{pmatrix} \begin{pmatrix} 1 & \mathcal{S}(n-2,n-1) & \mathcal{S}(n-2,n) \\ & 1 & \mathcal{S}(n-1,n) \\ & & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & \binom{n-1}{n-2}x & \binom{n}{n-2}x^2 \\ & 1 & \binom{n}{n-1}x \\ & & 1 \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}. \tag{29}$$

As corollaries we can get new combinatorial identities by multiplying (27) from left by another upper triangular matrix. For example, multiplying (27) by M^{-1} gives:

$$\begin{aligned} \sum_{p=0}^{N} \sum_{l=0}^{p} (-1)^{i-l} \binom{p}{i} \binom{p}{l} f(x+l) &= 0 \qquad (n < i) \quad \forall N \geq n \\ \sum_{p=0}^{N} \sum_{l=0}^{p} (-1)^{i-l} \binom{p}{i} \binom{p}{l} f(x+l) &= f_n n! \qquad (n = i) \quad \forall N \geq n. \end{aligned}$$

The conclusions of theorem 4 hold for these as well.

4.5 Relations to results by Aceto and Trigiante

In a recent paper [AT01] Aceto and Trigiante deduce, using differential equation approach, several interesting results related to our approach. Though, we must mention that they were not the first ones to use this kind of approach, see also [KU87] and references therein.

We give new proofs of, using our point of view, and generalize, some of their results. Let B_k denote the k^{th} Bernoulli polynomial and define

$$\mathcal{B}(t) := \begin{pmatrix} B_0(t) & B_0(t+1) & \cdots & B_0(t+N) \\ B_1(t) & B_1(t+1) & \cdots & B_1(t+N) \\ \vdots & & & \\ B_N(t) & B_N(t+1) & \cdots & B_N(t+N) \end{pmatrix} \quad \forall t \in \mathbb{R}. \tag{30}$$

We recall from [Arp03] the definition of Bernoulli matrix

$$B := \left(\begin{array}{c|c|c|c} & & & \\ \text{ke } B_0 & \text{ke } B_1 & \ldots & \text{ke } B_N \\ & & & \end{array} \right).$$

Remark 4.3. B is constant and defined through the *coefficients* of B_k's, while $\mathcal{B}$ is defined through the *values* of B_k's.

Proposition 4.7. Denote $W(t) := V\mathcal{C}_t$. Then

$$\begin{aligned} W(t)^{-1} M^{aT} W(t) &= \mathcal{C}_a \quad \forall a \in \mathbb{R}, & (31) \\ W(t)^{-1} D^{jT} W(t) &= M^{-1} (\log(J))^{jT} M \quad \forall j \in \mathbb{N}, & (32) \\ W(t) \mathcal{B}(t)^{-1} &= B^{-T}, & (33) \end{aligned}$$

in particular, the right hand sides are *independent* of t.

Proof. The left hand side of (31) becomes, by using proposition 4.1 and theorem 2 in second, third and last equalities:

$$(V\mathcal{C}_t)^{-1}M^{jT}V\mathcal{C}_t = (V\mathcal{C}_t)^{-1}(V\mathcal{C}_j)\mathcal{C}_t = (\mathcal{C}_{-t}V^{-1})V\mathcal{C}_{t+j} = \mathcal{C}_{-t}\mathcal{C}_{t+j} = \mathcal{C}_j$$

which proves (31) for all real numbers j.

The left hand side of (32) is, by proposition 4.1,

$$(V\mathcal{C}_t)^{-1}D^{jT}V\mathcal{C}_t = (M^{tT}V)^{-1}D^{jT}M^{tT}V = V^{-1}M^{-tT}D^{jT}M^{tT}V.$$

Now D^{jT} and M^{tT} commute due to proposition 3.1 hence this becomes

$$W(t)^{-1}D^{jT}W(t) = V^{-1}D^{jT}V$$

and by the first item of proposition 3.8 the claim (32) follows.

To prove (33), using proposition 4.3 and $val = V^Tke$ (proposition 3.2) we evaluate a row of $\mathcal{B}(t)$:

$$\mathcal{B}(t)(j,:) = \left(\mathcal{D}_1^T\,\mathcal{C}_t^T val(B_j)\right)^T = \left(\mathcal{C}_t^T V^T ke(B_j)\right)^T = B(:,j)^T V\mathcal{C}_t = (B^TW(t))(j,:),$$

hence B and $\mathcal{B}$ are related by

$$\mathcal{B}(t) = B^TW(t)$$

and $W(t)\mathcal{B}(t)^{-1} = B^{-T}$ as claimed. □

4.6 A Diophantine equation of Vandermonde type

We are interested in the following Diophantine problem: given $n \in \mathbb{N}$ and $a, b, z \in \mathbb{R}$, do there exist integers ξ_j, $j = 0, \ldots, n$ such that

$$\sum_{j=0}^{n}(a+jb)^k\xi_j = z^k \quad \forall k = 0, \ldots, n\ ? \tag{34}$$

which we call *Vandermonde type* due to its representability with a Vandermonde matrix. Namely, (34) is equivalent with the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ a & a+b & a+2b & \ldots & a+nb \\ a^2 & (a+b)^2 & (a+2b)^2 & \ldots & (a+nb)^2 \\ \vdots & \vdots & & & \vdots \\ a^n & (a+b)^n & (a+2b)^n & \ldots & (a+nb)^n \end{pmatrix} \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^n \end{pmatrix}. \tag{35}$$

We will solve this equation by applying results of the previous sections.

Theorem 5. Let $n \in \mathbb{N}$ and $a, b, z \in \mathbb{R}$. The equation (34) is solvable over integers if and only if $\frac{z-a}{b} \in \mathbb{Z}$. When such a solution exists, it is unique.

Proof. Denote $\xi := (\xi_0, \ldots, \xi_n)^T$. The matrix of (35) is Vandermonde(a, $a+b$, $a+2b$, …, $a+nb$) which is equal to VC_aD_b. Hence the claim is equivalent with the matrix equation

$$VC_aD_b\xi = Vc^n(z) \tag{36}$$

which, by proposition 4.1, and theorems 1 and 2 above, simplifies to

$$\xi = D_{1/b}C_{-a}c^n(z) = c^n(\frac{z-a}{b}). \tag{37}$$

which is, again by theorem 1, integers if and only if $\frac{z-a}{b} \in \mathbb{Z}$. □

5 Conclusions

We have continued the work initiated in [Arp03]. Main result is the structure of the formalism itself, this paper further expands its applicability. We have given several applications to show the usefulness of this formalism: reproving and generalizing combinatorial identities from [AT01, BT00, EFP98], decompositions from [CK01, EFP98], a group structure in the set of Vandermonde matrices, and a surprising Diophantine equation arising from our tools.

We believe that the tool c^N introduced in definition 4.1 will reveal interesting (number theoretical) structure and is therefore a promising subject of research in itself, but that is beyond the scope of the present paper and will be considered in future research.

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References

[Arp03] T. Arponen. A matrix approach to polynomials. *Lin. Alg. Appl.*, 359:181–196, 2003.

[AT01] L. Aceto and D. Trigiante. The matrices of Pascal and other greats. *Amer. Math. Mon.*, 108(3):232–245, 2001.

[BP92] R. Brawer and M. Pirovino. The linear algebra of the Pascal matrix. *Lin. Alg. Appl.*, 174:13–23, 1992.

[BT00] M. Bayat and H. Teimoori. Pascal k-eliminated functional matrix and its property. *Lin. Alg. Appl.*, 308:65–75, 2000.

[CK01] G.-S. Cheon and J.-S. Kim. Stirling matrix via Pascal matrix. *Lin. Alg. Appl.*, 329:49–59, 2001.

[CV93] G.S. Call and D.J. Velleman. Pascal's matrices. *Amer. Math. Mon.*, 100(4):372–376, 1993.

[EFP98] A. Eisinberg, G. Franzé, and P. Pugliese. Vandermonde matrices on integer nodes. *Numer. Math.*, 80:75–85, 1998.

[Kal83] D. Kalman. Polynomial translation groups. *Math. Mag.*, 56(1):23–25, 1983.

[KU87] D. Kalman and A. Ungar. Combinatorial and functional identities in one-parameter matrices. *Amer. Math. Mon.*, 94(1):21–35, 1987.

[MS58] N. Macon and A. Spitzbart. Inverses of vandermonde matrices. *Amer. Math. Mon.*, 65(2):95–100, 1958.