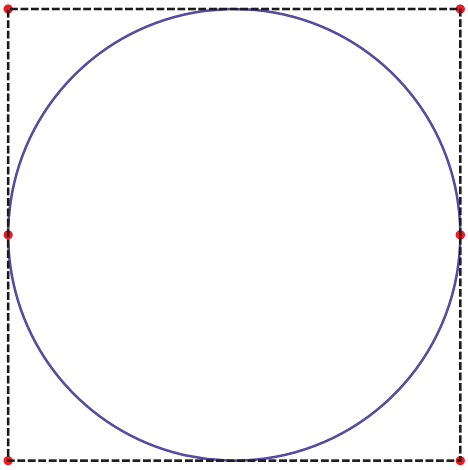

Figure 28: An example of a NURBS circle.

8 Surfaces

8.1 Introduction

Like curves, surfaces can also be represented by using the parametric and implicit forms. The implicit equation of a surface is of the form $f(x, y, z) = 0$ for some function f. The simplest example of an implicit surface is a plane determined by the equation $ax + by + cz + d = 0$ for some constants $a, b, c, d \in \mathbb{R}$. A *parametric surface* is a mapping $\mathcal{S} : U \to \mathbb{R}^3$ of the form

$$\mathcal{S}(s, t) = (x(s, t), y(s, t), z(s, t)), \quad (s, t) \in U \subset \mathbb{R}^2$$

The subset $S = \mathcal{S}(U) \subset \mathbb{R}^3$ is referred to as the *surface* and the function $\mathcal{S}$ is said to *parametrize* S. In practice, the geometry of the set U is chosen to be as simple as possible (rectangle).

Example 8.1. If $f : U \to \mathbb{R}^2$ is a function with two variables, then its graph is a surface with the equation

$$z = f(x, y)$$

The natural parametrization is given by

$$\mathcal{S}(s, t) = (s, t, f(s, t)), \quad (s, t) \in A.$$

Sometimes such surfaces are called non-parametric explicit surfaces or Monge patches.

Example 8.2. The coordinates of the points (x, y, z) on the unit sphere centered at the origin satisfy the equation

$$x^2 + y^2 + z^2 = 1$$

A parametric representation can be obtained by using the spherical coordinates

$$\begin{aligned} x(s,t) &= \sin s \cos t \\ y(s,t) &= \sin s \sin t \\ z(s,t) &= \cos s, \quad (s,t) \in [0,\pi] \times [0,2\pi] \end{aligned}$$

The curves

$$\mathcal{C}(s) = \mathcal{S}(s,t_0) \quad \text{and} \quad \mathcal{C}(t) = \mathcal{S}(s_0,t)$$

obtained by freezing one parameter value and letting the other one to vary are referred to as the coordinate curves. At a point $\mathcal{S}(s,t)$, the tangents to the coordinate curves are given by

$$\mathcal{S}_s(s,t) = (x_s(s,t), y_s(s,t), z_s(s,t)) \quad \text{and} \quad \mathcal{S}_t(s,t) = (x_t(s,t), y_t(s,t), z_t(s,t))$$

At any point of the surface where the vector $\mathcal{S}_s \times \mathcal{S}_t$ does not vanish, the unit normal vector to the surface is given by

$$\boldsymbol{\nu}(s,t) = \frac{\mathcal{S}_s(s,t) \times \mathcal{S}_t(s,t)}{|\mathcal{S}_s(s,t) \times \mathcal{S}_t(s,t)|} \tag{25}$$

The plane through the point $P = \mathcal{S}(s,t)$ with the normal $\boldsymbol{\nu}(s,t)$ is the tangent plane of the surface at P.

Example 8.3. For the parametric surface of Example 8.1, we have

$$\mathcal{S}_s(s,t) \times \mathcal{S}_t(s,t) = (1,0,f_s(s,t)) \times (0,1,f_t(s,t)) = (-f_s(s,t), -f_t(s,t), 1) \neq 0$$

and consequently

$$\boldsymbol{\nu}(s,t) = \frac{(-f_s(s,t), -f_t(s,t), 1)}{\sqrt{1 + f_s(s,t)^2 + f_t(s,t)^2}}.$$

Example 8.4. For the parametrization of the unit sphere in Example 8.2, we have

$$\begin{aligned} \mathcal{S}_s(s,t) &= (\cos s \cos t, \cos s \sin t, -\sin s) \\ \mathcal{S}_t(s,t) &= (-\sin s \sin t, \sin s \cos t, 0) \end{aligned}$$

Because $\mathcal{S}_t(0,t) = \mathcal{S}_t(\pi,t)$, Eq. 25 does not yield the normal to the sphere at the north and south poles. Clearly there is nothing special about those points, but the considered parametrization is not sufficiently regular at the poles.

Offset Surfaces

Given a surface $\mathcal{S}(s,t)$ with non-vanishing unit normal vector $\boldsymbol{\nu}(s,t)$, the *offset surface* $\mathcal{O}_\delta(s,t)$ of $\mathcal{S}$ at a distance δ is defined as

$$\mathcal{O}_\delta(s,t) = \mathcal{S}(s,t) + \delta\,\boldsymbol{\nu}(s,t).$$

Offset surfaces are used in *shelling* and *thickening* operations in CAD to transform solids or surfaces to thin-walled shells.

8.2 Bézier and B-spline surfaces

Let $B_{i,l}(s)$ and $B_{j,n}(t)$ be the Bernstein basis functions of degrees l and n in the variables s and t. A *Bézier surface* with control points $\mathbf{p}_{ij}$, $0 \leq i \leq l$, $0 \leq j \leq n$ is the parametric surface

$$\mathcal{S}(s,t) = \sum_{i=0}^{l} \sum_{j=0}^{n} \mathbf{p}_{ij} B_{i,l}(s) B_{j,n}(t), \quad (s,t) \in [0,1] \times [0,1]$$

A rational Bézier surface with control points $\mathbf{p}_{ij}$ and weights w_{ij} is given by

$$\mathcal{S}(s,t) = \frac{\sum_{i=0}^{l} \sum_{j=0}^{n} w_{ij} \mathbf{p}_{ij} B_{i,l}(s) B_{j,n}(t)}{\sum_{i=0}^{l} \sum_{j=0}^{n} w_{ij} B_{i,l}(s) B_{j,n}(t)}, \quad (s,t) \in [0,1] \times [0,1]$$

The coordinate curves of Bézier surfaces are Bézier curves. The curves $\mathcal{S}(s,0)$, $\mathcal{S}(s,1)$, $\mathcal{S}(0,t)$, and $\mathcal{S}(1,t)$ form the four edges of a Bézier surface. The $(l+1) \times (n+1)$ control points form the *control point polyhedron.*

In a similar fashion, we may define B-spline and NURBS surfaces associated to basis functions $N_{i,q}(s)$ and $N_{j,p}(t)$ of degree q and p with knot vectors $\{s_0, \ldots, s_k\}$ and $\{t_0, \ldots, t_m\}$. A B-spline surface with the control points p_{ij}, $0 \leq i \leq l = k-q-1$, $0 \leq j \leq n = m-p-1$, is defined by

$$\mathcal{S}(s,t) = \sum_{i=0}^{l} \sum_{j=0}^{n} \mathbf{p}_{ij} N_{i,p}(s) N_{j,q}(t), \quad (s,t) \in [s_q, s_{k-q}] \times [t_p, t_{m-p}]$$

and the corresponding NURBS surface with non-constant weights w_{ij} is defined by

$$\mathcal{S}(s,t) = \frac{\sum_{i=0}^{l} \sum_{j=0}^{n} w_{ij} \mathbf{p}_{ij} N_{i,p}(s) N_{j,q}(t)}{\sum_{i=0}^{l} \sum_{j=0}^{n} w_{ij} N_{i,p}(s) N_{j,q}(t)}, \quad (s,t) \in [s_q, s_{k-q}] \times [t_p, t_{m-p}] \qquad (26)$$

8.3 Surface Constructions

Extruded Surface

An extruded surface is constructed by translating a generating curve $\mathcal{B}(s)$ in the direction of a trajectory line. If a generating NURBS curve

$$\mathcal{B}(s) = \sum_{i=0}^{l} u_i \mathbf{b}_i N_{i,p}(s)$$

with knot vector $\{s_0, \ldots, s_m\}$ is sweeped in the direction of a unit vector $\mathbf{n}$, through a distance d, the result is the extruded NURBS surface

$$\mathcal{S}(s,t) = \frac{\sum_{i=0}^{l} \sum_{j=0}^{1} w_{ij} \mathbf{p}_{ij} N_{i,p}(s) N_{j,1}(t)}{\sum_{i=0}^{l} \sum_{j=0}^{1} w_{ij} N_{i,p}(s) N_{j,1}(t)}$$

with the knot vector $\{s_0, \ldots, s_m\}$ in the s-direction and $\{0, 0, 1, 1\}$ in the t-direction. The control points and weights are given by

$$\begin{aligned} &\mathbf{p}_{i,0} = \mathbf{b}_i \quad \text{and} \quad \mathbf{p}_{i,1} = \mathbf{b}_i + d\mathbf{n}, \\ &w_{i,0} = w_{i,1} = u_i \end{aligned}$$

for $i = 0, \ldots, l$.

Example 8.5. The unit quarter circle can be represented as a rational quadratic Bézier curve with control points $\mathbf{b}_0 = (1, 0)$, $\mathbf{b}_1 = (1, 1)$, $\mathbf{b}_2 = (0, 1)$ and weights $w_0 = w_1 = 1$, $w_2 = 2$. Making the control points spatial and extruding in the direction $(0, 0, 1)$ a distance 2 yields a representation of a cylindrical surface with radius 1 and height 2. The control points are

$$\begin{aligned} \mathbf{p}_{0,0} &= (1, 0, 0), & \mathbf{p}_{1,0} &= (1, 1, 0), & \mathbf{p}_{2,0} &= (0, 1, 0) \\ \mathbf{p}_{0,1} &= (1, 0, 2), & \mathbf{p}_{1,1} &= (1, 1, 2), & \mathbf{p}_{2,1} &= (0, 1, 2) \end{aligned}$$

and the weights are

$$\begin{aligned} w_{0,0} &= w_{1,0} = 1, & w_{2,0} &= 2 \\ w_{0,1} &= w_{1,1} = 1, & w_{2,1} &= 2 \end{aligned}$$

The cylindrical surface is shown in Fig. 29.

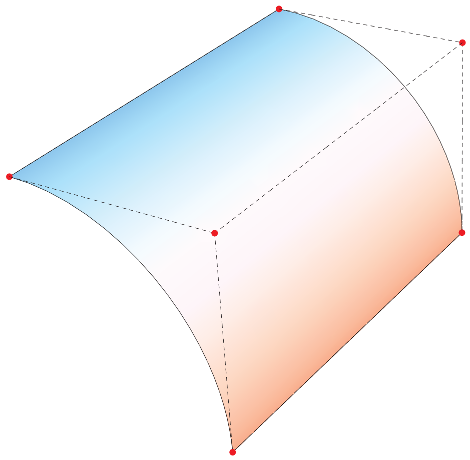

Figure 29: Cylindrical surface as a rational quadratic Bézier surface.

Ruled Surfaces

A ruled surface is formed by two spatial curves $\mathcal{B}(s)$ and $\mathcal{C}(s)$ when points on each curve corresponding to parameter value s are joined by a line. Consider two B-spline curves

$$\mathcal{B}(s) = \sum_{i=0}^{n} \mathbf{b}_i N_{i,p}(s)$$
$$\mathcal{C}(s) = \sum_{i=0}^{n} \mathbf{c}_i N_{i,p}(s)$$

assumed to have the same knot vectors $\{s_0, \ldots, s_m\}$. The ruled surface is

$$\mathcal{S}(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{1} \mathbf{p}_{ij} N_{i,p}(s) N_{j,1}(t)$$

where the knot vector in the s-direction $\{s_0, \ldots, s_m\}$, and $\{0, 0, 1, 1\}$ in the t-direction. The control points are

$$\mathbf{p}_{i,0} = \mathbf{b}_i \quad \text{and} \quad \mathbf{p}_{i,1} = \mathbf{c}_i$$

where $i = 0, \ldots, n$. Obviously, an extruded surface is a special case of a ruled surface.

Final Assignment

1. The *Aalto vase* shown in Fig. 30 is often considered to be the flagship of Finnish design. It was created by Alvar Aalto and his wife Aino in 1936. The vase is sometimes referred to as the *Savoy vase* because of its appearance in the historical Savoy restaurant in Helsinki that opened in 1937.

   The first glass vase was manufactured by Aalto using a wooden mould and consequently the surface had more textures than the modern smooth versions. Currently, the vase is being manufactured industrially in a variety of colours, sizes and materials.

Figure 30: Aalto vase and its profile.

   Construct a NURBS surface model of the vase. Model first the profile of the vase as a NURBS (or B-spline) curve $\mathcal{B}(t)$ and then construct an extruded surface by sweeping in the direction determined by a suitable unit vector $\mathbf{n}$, through some distance d. A more refined model might be obtained by forming a ruled surface from the two curves $\mathcal{B}_{\text{bottom}}(t)$ and $\mathcal{B}_{\text{top}}(t)$ representing the profile at the bottom and the top of the vase. The vase can also be thickened by using an offset surface at a suitable distance δ.