7 B-splines

A piecewise polynomial curve on the interval \([a, b]\) has a B-spline basis representation with similarities with Bézier curves. The representation is based on

- The degree \(p\) so that degree of each segment of the curve \(\leq p\)
- The knot vector \(T = \{t_0, \ldots, t_m\}\) which is a non-decreasing sequence of parameter values, that is \(t_i \leq t_{i+1}, i = 0, \ldots, m - 1\)
- Control points \(b_0, \ldots, b_n\)

7.1 B-spline Basis Functions and Curves

The \(i\)th B-spline basis function of degree \(p\) is denoted by \(N_{i,p}(t)\) and defined recursively as

\[
N_{i,0}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}
\]

\[
N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t)
\]

where \(i = 0, \ldots, n\) and \(p \geq 1\).

Some observations regarding the definition of the basis functions:

- For \(p > 0\), \(N_{i,p}\) is expressed in terms of two basis functions of degree \(p - 1\)
- The interval \([t_i, t_{i+1})\) is referred to as the \(i\)th knot span. It may have zero length since the knots need not to be distinct.
- If there are repeated knots, then a division 0/0 may occur. This is taken to be zero.

Example 7.1. Let \(T = \{0, 0, 0, 1, 1, 1\}\) and \(p = 2\). The B-spline basis functions of degree 0 are

\[
N_{0,0} = N_{1,0} = 0 \quad \text{everywhere}
\]

\[
N_{2,0} = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}
\]

\[
N_{3,0} = N_{4,0} = 0 \quad \text{everywhere}
\]
The B-spline basis functions of degrees 1 and 2 can be written as

\[ N_{0,0} = \frac{t - 0}{0 - 0} N_{0,0} + \frac{0 - t}{0 - 0} N_{1,0} = 0 \quad \text{everywhere} \]

\[ N_{1,1} = \frac{t - 0}{1 - 0} N_{1,0} + \frac{1 - t}{1 - 0} N_{2,0} = \begin{cases} 1 - t, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \]

\[ N_{2,1} = \frac{t - 0}{1 - 1} N_{2,0} + \frac{1 - t}{1 - 1} N_{3,0} = \begin{cases} t, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \]

\[ N_{3,1} = \frac{t - 1}{1 - 1} N_{3,0} + \frac{1 - t}{1 - 1} N_{4,0} = 0 \quad \text{everywhere} \]

and

\[ N_{0,2} = \frac{t - 0}{0 - 0} N_{0,1} + \frac{1 - t}{1 - 0} N_{1,1} = \begin{cases} (1 - t)^2, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \]

\[ N_{1,2} = \frac{t - 0}{1 - 0} N_{1,1} + \frac{1 - t}{1 - 0} N_{2,1} = \begin{cases} 2t(1 - t), & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \]

\[ N_{2,2} = \frac{t - 0}{1 - 1} N_{2,1} + \frac{1 - t}{1 - 1} N_{3,1} = \begin{cases} t^2, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \]

Note that in the previous example the \( N_{i,2} \) on the interval \([0, 1] \) are the quadratic Bernstein polynomials. More generally, the B-spline basis representation with a knot vector \( T = \{0, \ldots, 0, 1, \ldots, 1\} \) corresponds to the Bernstein basis.

A B-spline curve of degree \( p \) with control points \( b_0, \ldots, b_n \) is defined on the interval \([a, b] = [t_p, t_{m-p}] \) as

\[ B(t) = \sum_{i=0}^{n} b_i N_{i,p}(t) \]

**Example 7.2.** Let \( p = 2 \), and \( T = \{0, 0, 0, 1, 2, 3, 3, 4, 4, 4\} \). The quadratic basis functions are shown in Fig. 26 and the B-spline curve corresponding to the control points \((0, 1), (1, 1), (3, 4), (4, 2), (5, 3), (6, 4), (7, 3)\) is shown in Fig. 27.

A B-spline curve of degree \( p \) can have any number of control points provided that the knot vector is specified accordingly. Each basis function \( N_{i,p} \) is determined by the \( p+2 \) knots \( t_i, \ldots, t_{i+p+1} \). If \( n + 1 \) control points (and basis functions) are required, then \( n + p + 2 \) knots must be specified. Therefore the number of knots \( m + 1 \) must equal the number of control points plus the degree plus one. In other words,

\[ m = n + p + 1. \]

A knot may be repeated in the knot vector. The number of times a knot value occurs is called the multiplicity of the knot.
7.2 Properties of B-spline Basis Functions and Curves

**Theorem 6.** The B-spline basis functions have the following properties

- **Positivity:** \( N_{i,p}(t) > 0, \ t \in (t_i, t_{i+p+1}) \)
- **Local support:** \( N_{i,p}(t) = 0, \ t \notin (t_i, t_{i+p+1}) \)
- **Piecewise polynomial:** \( N_{i,p}(t) \) are piecewise polynomial functions of degree \( p \).
- **Partition of unity:** \( \sum_{i=r-p}^{r} N_{i,p}(t) = 1, \ t \in [t_r, t_{r+1}) \)
- **Continuity:** If the interior knot \( t_i \) has multiplicity \( k_i \), then \( N_{i,p}(t) \) is \( C^{p-k_i} \) at \( t = t_i \) and \( C^\infty \) elsewhere.

**Proof.** (Partial) The first three properties can be proved by using induction. The step \( p = 0 \) is clear from the definition of the basis. Suppose now that the basis functions \( N_{i,0}(t), \ldots, N_{i,p}(t) \) satisfy the properties and consider

\[
N_{i,p+1}(t) = \frac{t - t_i}{t_{i+p+1} - t_i} N_{i,p}(t) + \frac{t_{i+p+2} - t}{t_{i+p+2} - t_{i+1}} N_{i+1,p}(t),
\]

Figure 26: B-spline basis of degree \( p = 2 \) for the knot vector \( T = \{0, 0, 1, 2, 3, 4, 4, 4\} \).

Figure 27: B-spline curve of Example 7.2
Suppose \( t \notin (t_i, t_{i+p+2}) \). Then \( N_{i,p}(t) = N_{i+1,p}(t) = 0 \) because they are assumed to satisfy the local support property. Hence, \( N_{i,p+1}(t) = 0 \). Suppose now that \( t \in (t_i, t_{i+p+2}) \). Then
\[
\frac{t - t_i}{t_{i+p+1} - t_i}, \frac{t_{i+p+2} - t}{t_{i+p+2} - t_{i+1}} > 0
\]
Moreover, we have
\[
N_{i,p}(t) > 0, \quad N_{i+1,p} \geq 0, \quad \text{when } t \in (t_i, t_{i+p+1})
\]
\[
N_{i,p}(t) \geq 0, \quad N_{i+1,p} > 0, \quad \text{when } t \in (t_{i+1}, t_{i+p+2})
\]
which imply \( N_{i,p+1}(t) > 0 \) in both cases.

Since the sum of piecewise polynomials is a piecewise polynomial and the product of a polynomial and a piecewise polynomial is a piecewise polynomial, \( N_{i,p+1} \) is a piecewise polynomial.

The partition of unity property can be also shown by using induction. We skip the details.

The continuity property follows from the formula for the derivative of B-spline basis functions (proof by induction):
\[
N'_{i,p}(t) = \frac{p}{t_{i+p} - t_i} N_{i,p-1}(t) - \frac{p}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t)
\] (24)
Namely, if \( N_{i,p-1} \) are \( C^{p-1-k_i} \), then so is \( N'_{i,p} \). Thus, \( N_{i,p} \) is \( C^{p-k_i} \). Since \( N_{i,1} \) are \( C^0 \), the continuity property follows by induction.

The properties of B-spline basis functions yield the following properties of B-spline curves.

**Theorem 7.** A B-spline curve \( \mathcal{B}(t) = \sum_{i=0}^{n} b_i N_{i,p}(t) \) of degree \( p \) associated to the knot vector \( \{t_0, \ldots, t_m\} \) satisfies

- **Local control:** Each segment of the curve depends on \( p+1 \) control points. If \( t \in [t_r, t_{r+1}) \) with \( p \leq r \leq m - p - 1 \), then
  \[ \mathcal{B}(t) = \sum_{i=r-p}^{r} b_i N_{i,p}(t) \]

- **Convex Hull property:** If \( t \in [t_r, t_{r+1}) \) (\( p \leq r \leq m - p - 1 \)), then
  \[ \mathcal{B}(t) = CH(b_{r-p}, \ldots, b_r) \]

- **Continuity:** if \( k_i \) is the multiplicity of the breakpoint \( t = t_i \) then \( \mathcal{B}(t) \) is \( C^{p-k_i} \) (or greater) at \( t = t_i \) and \( C^\infty \) elsewhere.

- **Invariance under affine transformations:** If \( T \) is an affine transformation, then
  \[ T(\sum_{i=0}^{n} b_i N_{i,p}(t)) = \sum_{i=0}^{n} T(b_i) N_{i,p}(t). \]
Open B-splines

A general B-spline curve does not interpolate the first and last control points \( b_0 \) and \( b_n \). For curves of degree \( p \), endpoint interpolation and tangency with the control polygon holds for open, or clamped B-splines. For these the knots satisfy

\[
t_0 = \cdots = t_p \quad \text{and} \quad t_{m-p} = \cdots = t_m
\]

Namely, the local control property gives for \( t_p \)

\[
\mathcal{B}(t_p) = \sum_{i=0}^{p} b_i N_{i,p}(t_p),
\]

where, for \( 0 \leq i \leq p \),

\[
N_{i,p}(t_p) = \frac{t_{i+1}+p-t_p}{t_{i+1}+p-t_i} N_{i,p-1}(t_p) + \frac{t_{i+1}+p-t_p}{t_{i+1}+p-t_{i+1}} N_{i+1,p-1}(t_p) = \frac{t_{i+1}+p-t_p}{t_{i+1}+p-t_{i+1}} N_{i+1,p-1}(t_p)
\]

since \( t_0 = \cdots = t_p \). A similar reasoning allows as to write

\[
N_{i,p}(t_p) = \frac{t_{i+1}+p-t_p}{t_{i+1}+p-t_i} \frac{t_{i+2}+p-t_p}{t_{i+2}+p-t_{i+1}} \cdots \frac{t_{i+p}+p-t_p}{t_{i+p}+p-t_{i+p}} N_{i+p,0}(t_p)
\]

and finally

\[
N_{i,p}(t_p) = \frac{t_{i+1}+p-t_p}{t_{i+1}+p-t_i} \frac{t_{i+2}+p-t_p}{t_{i+2}+p-t_{i+1}} \cdots \frac{t_{i+p}+p-t_p}{t_{i+p}+p-t_{i+p}} N_{i+p,0}(t_p)
\]

Because \( N_{i+p,0}(t_p) > 0 \) for \( i > 0 \), it follows that \( N_{i,p}(t_p) = 0 \) for \( i > 0 \). When \( i = 0 \), we have

\[
N_{0,p}(t) = \frac{t_1+p-t_p}{t_1+p-t_1} \frac{t_2+p-t_p}{t_2+p-t_1} \cdots \frac{t_p+p-t_p}{t_p+p-t_1} N_{p,0}(t_p) = 1
\]

so that

\[
\mathcal{B}(t_p) = \sum_{i=0}^{p} b_i N_{i,p}(t_p) = b_0.
\]

Similar arguments show that \( \mathcal{B}(t_{m-p}) = b_n \).

Open B-splines also satisfy

\[
\mathcal{B}'(t_p) = \frac{p}{t_{p+1}-t_1} (b_1 - b_0) \quad \text{and} \quad \mathcal{B}'(t_{m-p}) = \frac{p}{t_{m-1}-t_{m-p-1}} (b_n - b_{n-1})
\]

so that the control polygon determines the tangent directions of an open B-spline curve at the endpoints. Thus, the properties of open B-splines are very similar to those of Bézier curves.
7.3 NURBS

The NURBS curve of degree \( p \) with control points \( b_0, \ldots, b_n \), weights \( w_0, \ldots, w_n \), and knot vector \( t_0, \ldots, t_m \) is the curve

\[
B(t) = \frac{\sum_{i=0}^{n} w_i b_i N_{i,p}(t)}{\sum_{i=0}^{n} w_i N_{i,p}(t)}
\]

where \( N_{i,p}(t) \) are the B-spline basis functions defined on the specified knot vector. The curve may also be written as

\[
B(t) = \sum_{i=0}^{n} b_i R_{i,p}(t),
\]

where

\[
R_{i,p}(t) = \frac{w_i N_{i,p}(t)}{\sum_{i=0}^{n} w_i N_{i,p}(t)}
\]

are the rational B-spline basis functions. Let \( b_i = (x_i, y_i, z_i) \) and define the homogeneous control points \( b^w_i \) by

\[
\begin{align*}
b^w_i &= (w_i x, w_i y, w_i z, w_i), \quad \text{when} \ w_i \neq 0 \\
b^w_i &= (x, y, z, 0), \quad \text{when} \ w_i = 0
\end{align*}
\]

In homogeneous coordinates, the NURBS curve has the form

\[
B(t) = \sum_{i=0}^{n} b^w_i N_{i,n}(t).
\]

**Example 7.3.** A NURBS representation of a circle is obtained by taking knot vector \( \{0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1\} \), control points \( (1,0), (1,1), (-1,1), (-1,0), (-1,-1), (1,-1), (1,0) \), and weights \( 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1 \). The circle is shown in Fig. 28.

![Figure 28: An example of a NURBS circle.](image-url)