7 B-splines

A piecewise polynomial curve on the interval $[a, b]$ has a B-spline basis representation with similarities with Bézier curves. The representation is based on

- The degree p so that degree of each segment of the curve $\leq p$
- The *knot vector* $\mathcal{T} = \{t_0, \ldots, t_m\}$ which is a non-decreasing sequence of parameter values, that is $t_i \leq t_{i+1}$, $i = 0, \ldots, m-1$
- Control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$

7.1 B-spline Basis Functions and Curves

The ith B-spline basis function of degree p is denoted by $N_{i,p}(t)$ and defined recursively as

$$N_{i,0}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t)$$

where $i = 0, \ldots, n$ and $p \geq 1$.

Some observations regarding the definition of the basis functions:

- For $p > 0$, $N_{i,p}$ is expressed in terms of two basis functions of degree $p - 1$
- The interval $[t_i, t_{i+1})$ is referred to as the ith knot span. It may have zero length since the knots need not to be distinct.
- If there are repeated knots, then a division $0/0$ may occur. This is taken to be zero.

Example 7.1. Let $\mathcal{T} = \{0, 0, 0, 1, 1, 1\}$ and $p = 2$. The B-spline basis functions of degree 0 are

$$N_{0,0} = N_{1,0} = 0 \quad \text{everywhere}$$

$$N_{2,0} = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{3,0} = N_{4,0} = 0 \quad \text{everywhere}$$

The B-spline basis functions of degrees 1 and 2 can be written as

$$N_{0,1} = \frac{t-0}{0-0}N_{0,0} + \frac{0-t}{0-0}N_{1,0} = 0 \quad \text{everywhere}$$

$$N_{1,1} = \frac{t-0}{0-0}N_{1,0} + \frac{1-t}{1-0}N_{2,0} = \begin{cases} 1-t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{2,1} = \frac{t-0}{1-0}N_{2,0} + \frac{1-t}{1-1}N_{3,0} = \begin{cases} t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{3,1} = \frac{t-1}{1-1}N_{3,0} + \frac{1-t}{1-1}N_{4,0} = 0 \quad \text{everywhere}$$

and

$$N_{0,2} = \frac{t-0}{0-0}N_{0,1} + \frac{1-t}{1-0}N_{1,1} = \begin{cases} (1-t)^2, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{1,2} = \frac{t-0}{1-0}N_{1,1} + \frac{1-t}{1-0}N_{2,1} = \begin{cases} 2t(1-t), & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{2,2} = \frac{t-0}{1-0}N_{2,1} + \frac{1-t}{1-1}N_{3,1} = \begin{cases} t^2, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that in the previous example the $N_{i,2}$ on the interval $[0,1]$ are the quadratic Bernstein polynomials. More generally, the B-spline basis representation with a knot vector

$$\mathcal{T} = \{\underbrace{0,\ldots,0}_{p+1},\underbrace{1,\ldots,1}_{p+1}\}$$

corresponds to the Bernstein basis.

A B-spline curve of degree p with control points $\mathbf{b}_0,\ldots,\mathbf{b}_n$ is defined on the interval $[a,b] = [t_p, t_{m-p}]$ as

$$\mathcal{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i N_{i,p}(t)$$

Example 7.2. Let $p = 2$, and $\mathcal{T} = \{0,0,0,1,2,3,3,4,4,4\}$. The quadratic basis functions are shown in Fig. 26 and the B-spline curve corresponding to the control points $(0,1)$, $(1,1)$, $(3,4)$, $(4,2)$, $(5,3)$, $(6,4)$, $(7,3)$ is shown in Fig. 27.

A B-spline curve of degree p can have any number of control points provided that the knot vector is specified accordingly. Each basis function $N_{i,p}$ is determined by the $p+2$ knots $t_i,\ldots,t_{i+p+1}$. If $n+1$ control points (and basis functions) are required, then $n+p+2$ knots must be specified. Therefore the number of knots $m+1$ must equal the number of control points plus the degree plus one. In other words,

$$m = n + p + 1.$$

A knot may be repeated in the knot vector. The number of times a knot value occurs is called the *multiplicity of the knot.*

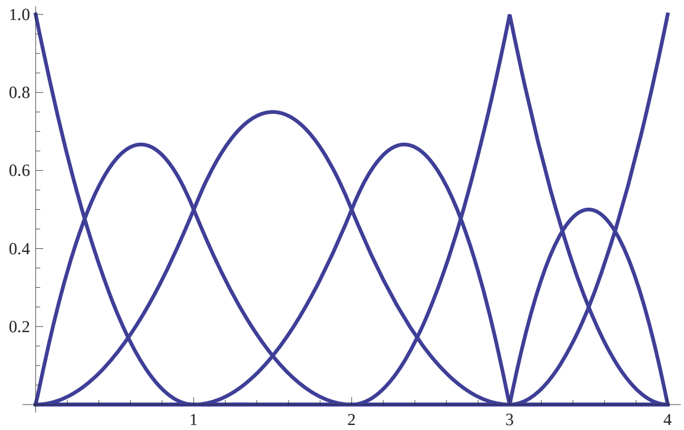

Figure 26: B-spline basis of degree $p = 2$ for the knot vector $\mathcal{T} = \{0, 0, 0, 1, 2, 3, 3, 4, 4, 4\}$.

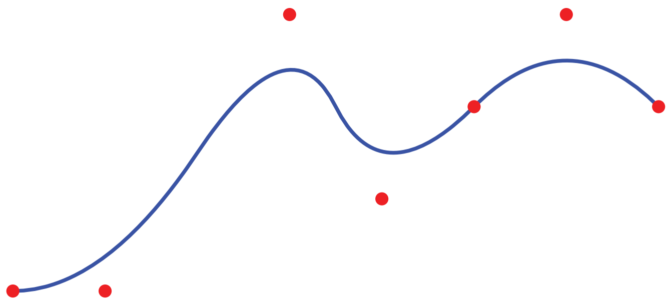

Figure 27: B-spline curve of Example 7.2

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7.2 Properties of B-spline Basis Functions and Curves

Theorem 6. *The B-spline basis functions have the following properties*

- *Positivity:* $N_{i,p}(t) > 0$, $t \in (t_i, t_{i+p+1})$
- *Local support:* $N_{i,p}(t) = 0$, $t \notin (t_i, t_{i+p+1})$
- *Piecewise polynomial:* $N_{i,p}(t)$ *are piecewise polynomial functions of degree* p.
- *Partition of unity:* $\sum_{i=r-p}^{r} N_{i,p}(t) = 1$, $t \in [t_r, t_{r+1})$
- *Continuity: If the interior knot* t_i *has multiplicity* k_i*, then* $N_{i,p}(t)$ *is* C^{p-k_i} *at* $t = t_i$ *and* C^∞ *elsewhere.*

Proof. (Partial) The first three properties can be proved by using induction. The step $p = 0$ is clear from the definition of the basis. Suppose now that the basis functions $N_{i,0}(t), \ldots, N_{i,p}(t)$ satisfy the properties and consider

$$N_{i,p+1}(t) = \frac{t - t_i}{t_{i+p+1} - t_i} N_{i,p}(t) + \frac{t_{i+p+2} - t}{t_{i+p+2} - t_{i+1}} N_{i+1,p}(t),$$

Suppose $t \notin (t_i, t_{i+p+2})$. Then $N_{i,p}(t) = N_{i+1,p}(t) = 0$ because they are assumed to satisfy the local support property. Hence, $N_{i,p+1}(t) = 0$. Suppose now that $t \in (t_i, t_{i+p+2})$. Then

$$\frac{t - t_i}{t_{i+p+1} - t_i}, \frac{t_{i+p+2} - t}{t_{i+p+2} - t_{i+1}} > 0$$

Moreover, we have

$$\begin{aligned} N_{i,p}(t) > 0, \ N_{i+1,p} \geq 0, \quad &\text{when } t \in (t_i, t_{i+p+1}) \\ N_{i,p}(t) \geq 0, \ N_{i+1,p} > 0, \quad &\text{when } t \in (t_{i+1}, t_{i+p+2}) \end{aligned}$$

which imply $N_{i,p+1}(t) > 0$ in both cases.

Since the sum of piecewise polynomials is a piecewise polynomial and the product of a polynomial and a piecewise polynomial is a piecewise polynomial, $N_{i,p+1}$ is a piecewise polynomial.

The partition of unity property can be also shown by using induction. We skip the details.

The continuity property follows from the formula for the derivative of B-spline basis functions (proof by induction):

$$N'_{i,p}(t) = \frac{p}{t_{i+p} - t_i} N_{i,p-1}(t) - \frac{p}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t) \tag{24}$$

Namely, if $N_{i,p-1}$ are C^{p-1-k_i}, then so is $N'_{i,p}$. Thus, $N_{i,p}$ is C^{p-k_i}. Since $N_{i,1}$ are C^0, the continuity property follows by induction. □

The properties of B-spline basis functions yield the following properties of B-spline curves.

Theorem 7. *A B-spline curve $\mathcal{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i N_{i,p}(t)$ of degree p associated to the knot vector $\{t_0, \ldots, t_m\}$ satisfies*

- *Local control: Each segment of the curve depends on $p+1$ control points. If $t \in [t_r, t_{r+1})$ with $p \leq r \leq m - p - 1$, then*

$$\mathcal{B}(t) = \sum_{i=r-p}^{r} \mathbf{b}_i N_{i,p}(t)$$

- *Convex Hull property: If $t \in [t_r, t_{r+1})$ ($p \leq r \leq m - p - 1$), then*

$$\mathcal{B}(t) = CH(\mathbf{b}_{r-d}, \ldots, \mathbf{b}_r)$$

- *Continuity: if k_i is the multiplicity of the breakpoint $t = t_i$ then $\mathcal{B}(t)$ is C^{p-k_i} (or greater) at $t = t_i$ and C^∞ elsewhere.*

- *Invariance under affine transformations: If T is an affine transformation, then*

$$\mathsf{T}\Big(\sum_{i=0}^{n} \mathbf{b}_i N_{i,p}(t)\Big) = \sum_{i=0}^{n} \mathsf{T}(\mathbf{b}_i) N_{i,p}(t).$$

Open B-splines

A general B-spline curve does not interpolate the first and last control points $\mathbf{b}_0$ and $\mathbf{b}_n$. For curves of degree p, endpoint interpolation and tangency with the control polygon holds for open, or clamped B-splines. For these the knots satisfy

$$t_0 = \cdots = t_p \quad \text{and} \quad t_{m-p} = \cdots = t_m$$

Namely, the local control property gives for t_p

$$\mathcal{B}(t_p) = \sum_{i=0}^{p} \mathbf{b}_i N_{i,p}(t_p),$$

where, for $0 \leq i \leq p$,

$$N_{i,p}(t_p) = \frac{t_p - t_i}{t_{i+p} - t_i} N_{i,p-1}(t_p) + \frac{t_{i+1+p} - t_p}{t_{i+1+p} - t_{i+1}} N_{i+1,p-1}(t_p) = \frac{t_{i+1+p} - t_p}{t_{i+1+p} - t_{i+1}} N_{i+1,p-1}(t_p)$$

since $t_0 = \cdots = t_p$. A similar reasoning allows as to write

$$N_{i,p}(t_p) = \frac{t_{i+1+p} - t_p}{t_{i+1+p} - t_{i+1}} \frac{t_{i+2+p} - t_p}{t_{i+2+p} - t_{i+2}} N_{i+2,p-2}(t_p)$$

and finally

$$N_{i,p}(t_p) = \frac{t_{i+1+p} - t_p}{t_{i+1+p} - t_{i+1}} \frac{t_{i+2+p} - t_p}{t_{i+2+p} - t_{i+2}} \cdots \frac{t_{i+p+p} - t_p}{t_{i+p+p} - t_{i+p}} N_{i+p,0}(t_p)$$

Because $N_{i+p,0}(t_p) > 0$ for $i > 0$, it follows that $N_{i,p}(t_p) = 0$ for $i > 0$. When $i = 0$, we have

$$N_{0,p}(t) = \frac{t_{1+p} - t_p}{t_{1+p} - t_1} \frac{t_{2+p} - t_p}{t_{2+p} - t_2} \cdots \frac{t_{p+p} - t_p}{t_{p+p} - t_p} N_{p,0}(t_p) = 1$$

so that

$$\mathcal{B}(t_p) = \sum_{i=0}^{p} \mathbf{b}_i N_{i,p}(t_p) = \mathbf{b}_0.$$

Similar arguments show that $\mathcal{B}(t_{m-p}) = \mathbf{b}_n$.

Open B-splines also satisfy

$$\mathcal{B}'(t_p) = \frac{p}{t_{p+1} - t_1}(\mathbf{b}_1 - \mathbf{b}_0) \quad \text{and} \quad \mathcal{B}'(t_{m-p}) = \frac{p}{t_{m-1} - t_{m-p-1}}(\mathbf{b}_n - \mathbf{b}_{n-1})$$

so that the control polygon determines the tangent directions of an open B-spline curve at the endpoints. Thus, the properties of open B-splines are very similar to those of Bézier curves.

7.3 NURBS

The NURBS curve of degree p with control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$, weights $w_0, \ldots, w_n$, and knot vector $t_0, \ldots, t_m$ is the curve

$$\mathcal{B}(t) = \frac{\sum_{i=0}^{n} w_i \mathbf{b}_i N_{i,p}(t)}{\sum_{i=0}^{n} w_i N_{i,p}(t)}$$

where $N_{i,p}(t)$ are the B-spline basis functions defined on the specified knot vector. The curve may also be written as

$$\mathcal{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i R_{i,p}(t),$$

where

$$R_{i,p}(t) = \frac{w_i N_{i,p}(t)}{\sum_{i=0}^{n} w_i N_{i,p}(t)}$$

are the rational B-spline basis functions. Let $\mathbf{b}_i = (x_i, y_i, z_i)$ and define the homogeneous control points $\mathbf{b}_i^w$ by

$$\begin{cases} \mathbf{b}_i^w = (w_i x, w_i y, w_i z, w_i), & \text{when } w_i \neq 0 \\ \mathbf{b}_i^w = (x, y, z, 0), & \text{when } w_i = 0 \end{cases}$$

In homogeneous coordinates, the NURBS curve has the form

$$\mathcal{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i^w N_{i,n}(t).$$

Example 7.3. A NURBS representation of a circle is obtained by taking knot vector $\{0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1\}$, control points (1,0), (1,1), (-1,1), (-1,0), (-1,-1), (1,-1), (1,0), and weights 1, $\frac{1}{2}$, $\frac{1}{2}$, 1, $\frac{1}{2}$, $\frac{1}{2}$, 1. The circle is shown in Fig. 28.

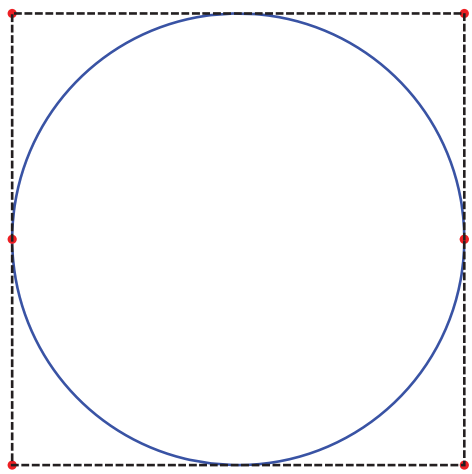

Figure 28: An example of a NURBS circle.