

5 Bézier Curves

5.1 Introduction

By allowing the coordinate functions $x(t)$ and $y(t)$ to be arbitrary, we obtain all possible curves $\mathcal{C}(t) = (x(t), y(t))$. However, when building up a geometric modelling system, it is advantageous to work with a restricted class of functions which

- is rich enough to represent a great variety of curves
- are easy to handle in a computer (computation of points and derivatives is efficient)

Perhaps the most widely used class of functions is the *polynomials*. These are very well suited for computer implementations but polynomials cannot represent many relevant curve types such as conic sections exactly. Nevertheless, a circle can be *approximated* in systems using polynomials. If an exact representation of conic sections is desirable, the coordinate functions must be allowed to be *rational*, that is of the form $p(t)/q(t)$ where p and q are polynomials.

An n th-degree power basis curve is defined as

$$\mathcal{C}(t) = \sum_{i=1}^n \mathbf{a}_i t^i, \quad 0 \leq t \leq 1,$$

where the $n + 1$ functions $\{t^i\}$ are called the basis functions, or blending functions, and the $\{\mathbf{a}_i\}$ are the coefficients of the power basis representations. These are vectors $\mathbf{a}_i = (x_i, y_i)$.

Differentiating yields

$$\mathbf{a}_i = \frac{\mathcal{C}^{(i)}(0)}{i!}.$$

Example 5.1. When $n = 1$, $\mathcal{C}(t) = \mathbf{a}_0 + \mathbf{a}_1 t$, $0 \leq t \leq 1$ is a straight line segment between the points \mathbf{a}_0 and $\mathbf{a}_0 + \mathbf{a}_1$. The constant $\mathcal{C}'(t) = \mathbf{a}_1$ gives the direction of the line.

When $n = 2$, $\mathcal{C}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2$, $0 \leq t \leq 1$ is a parabolic arc between the point \mathbf{a}_0 and $\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2$.

The problem with the power basis form in context of geometric modelling is that the relationship of the coefficients \mathbf{a}_i to the actual geometric shape of the curve is not obvious. For instance specification of end conditions at both ends of the curve, not just at the starting point, is desirable in applications. The Bézier method remedies this problem by using another form of polynomials as the basis functions. The works of two French automotive engineers, Pierre Bézier (Renault) and Paul de Faget Casteljau (Citroën) is usually considered as the origin of present day CAD. The idea is to use the Bernstein Polynomial basis (Sergei Bernstein, 1912) to represent curves and surfaces. The procedure is as follows.

5.2 Definition of Bézier Curves and Bernstein Polynomials

Given $n + 1$ control points $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n$, the Bézier curve of degree n is

$$\mathcal{C}(t) = \sum_{i=0}^n \mathbf{b}_i B_{i,n}(t), \quad 0 \leq t \leq 1,$$

where the basis functions are defined as the *Bernstein polynomials*

$$B_{i,n}(t) = \frac{n!}{(n-i)!i!} (1-t)^{n-i} t^i.$$

The low degree Bernstein polynomials are as follows:

$$n = 1: B_0(t) = 1 - t, B_1(t) = t$$

$$n = 2: B_0(t) = (1 - t)^2, B_1(t) = 2(1 - t)t, B_2(t) = t^2$$

$$n = 3: B_0(t) = (1 - t)^3, B_1(t) = 3(1 - t)^2t, B_2(t) = 3(1 - t)t^2, B_3(t) = t^3$$

The basis functions are visualized in Fig. 19.

Let us recall that the quantities $\frac{n!}{(n-i)!i!}$ are called the *binomial coefficients* and denoted by $\binom{n}{i}$. These can be tabulated into a triangular array known as Pascal's triangle:

$$\begin{array}{cccccc} & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & 1 & 2 & 1 \\ & & & & & 1 & 3 & 3 & 1 \\ & & & 1 & 4 & 6 & 4 & 1 \\ & & & & & & & & & \vdots \end{array}$$

In other words, the binomial coefficients have the property

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}. \quad (22)$$

Pascal's rule is

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$$

Example 5.2. A linear Bézier curve is a line segment joining the two control points \mathbf{b}_0 to \mathbf{b}_1 :

$$\mathcal{C}(t) = \mathbf{b}_0(1 - t) + \mathbf{b}_1 t,$$

Since $\mathcal{C}(0) = \mathbf{b}_0$ and $\mathcal{C}(1) = \mathbf{b}_1$, it is said that the curve *interpolates* the first and last control points.

The Bézier form for the line segment passing through the points $(1, 1)$ and $(2, 3)$ is

$$\mathcal{C}(t) = (1 - t)(1, 1) + t(2, 3) = (1 + t, 2t)$$

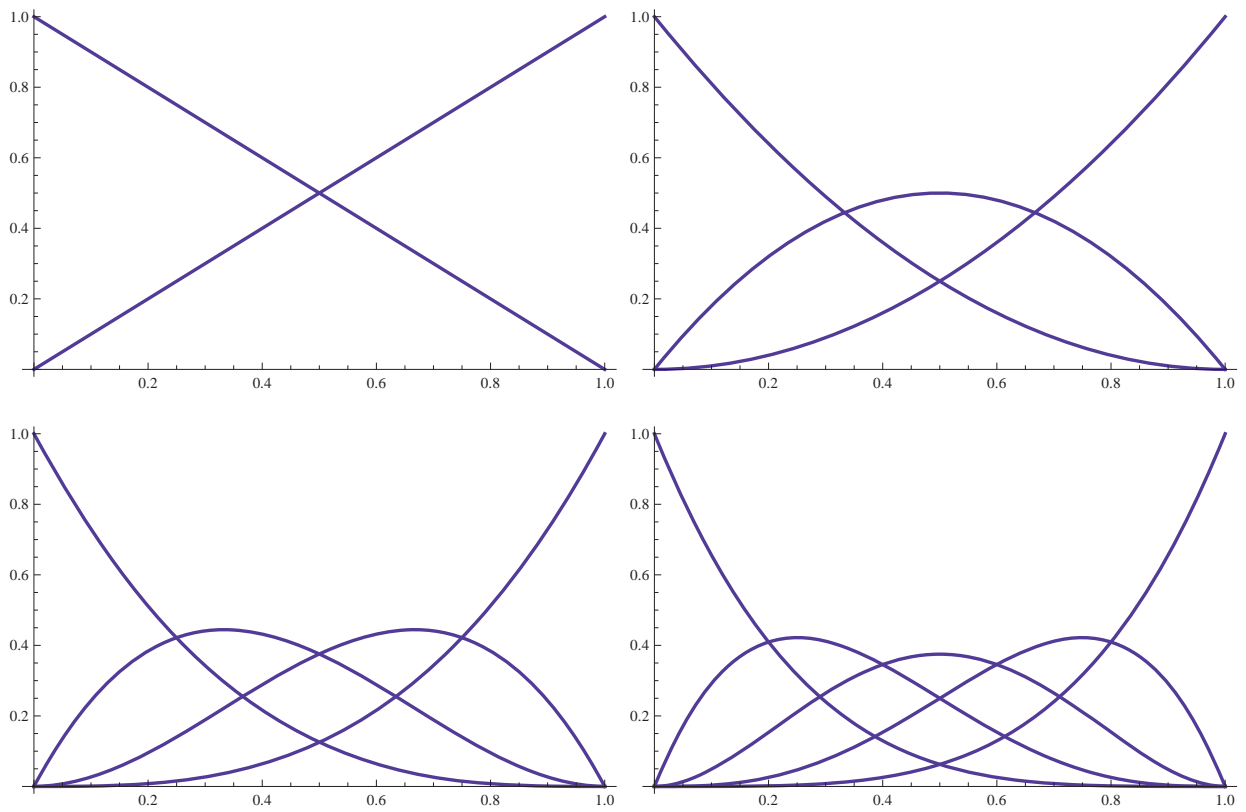


Figure 19: Bernstein polynomials of degree 1, 2, 3, and 4.

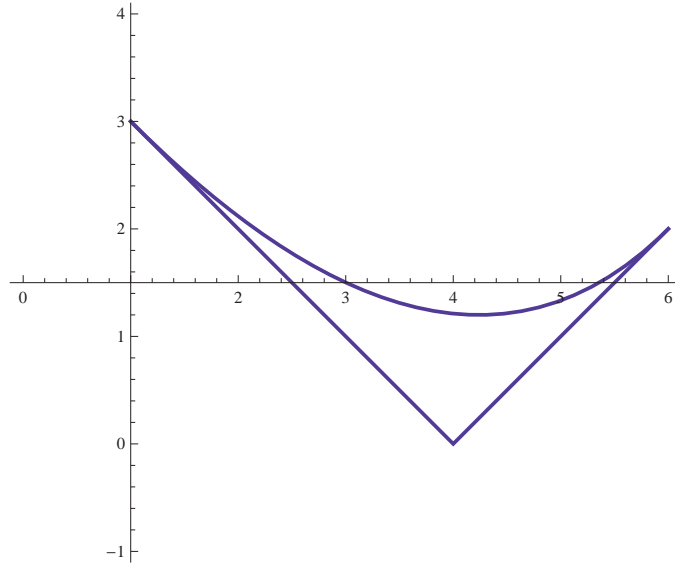


Figure 20: Quadratic Bézier curve and control polygon.

Example 5.3. The quadratic Bézier curve associated with the three control points \mathbf{b}_0 , \mathbf{b}_1 , and \mathbf{b}_2 is

$$\mathcal{C}(t) = (1-t)^2\mathbf{b}_0 + 2(1-t)t\mathbf{b}_1 + t^2\mathbf{b}_2$$

The starting point of the curve is $\mathcal{C}(0) = \mathbf{b}_0$ and the endpoint is \mathbf{b}_2 . The triangle $\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ obtained by joining the control points with line segments in their prescribed order, is called the *control polygon*.

The parametric form of the quadratic Bézier curve with control points $(1, 3)$, $(4, 0)$, and $(6, 2)$ is

$$\begin{aligned} x(t) &= (1-t)^2 \cdot 1 + 2(1-t) \cdot 4 + t^2 \cdot 6, \\ y(t) &= (1-t)^2 \cdot 3 + 2(1-t) \cdot 0 + t^2 \cdot 2. \end{aligned}$$

The curve and the control polygon are shown in Fig. 20.

Example 5.4. The cubic Bézier curve associated with the four control points \mathbf{b}_0 , \mathbf{b}_1 , \mathbf{b}_2 , and \mathbf{b}_3 is

$$\mathcal{C}(t) = (1-t)^3\mathbf{b}_0 + 3(1-t)^2t\mathbf{b}_1 + 3(1-t)t^2\mathbf{b}_2 + t^3\mathbf{b}_3$$

Cubic Bézier curves can exhibit loops and sharp corners called cusps as shown in Fig. 21.

5.3 Properties of Bernstein Polynomials and Bézier Curves

The Bernstein basis polynomials have these properties:

- Non-negativity: $B_{i,n} \geq 0$, $t \in [0, 1]$ for any i, n .

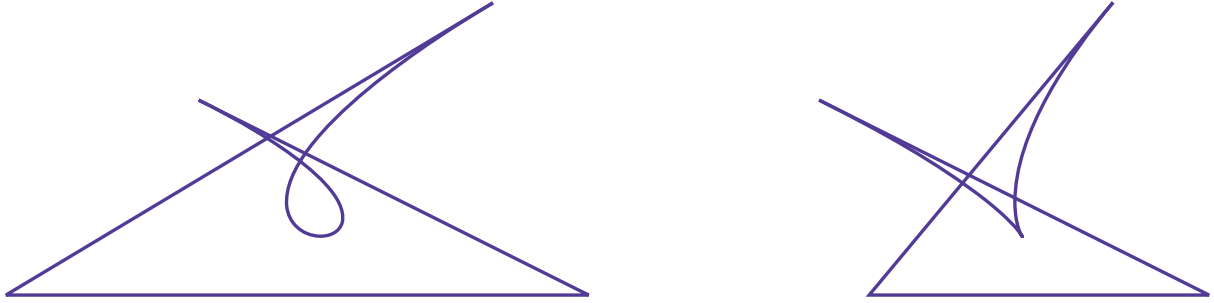


Figure 21: Cubic Bézier curves with a loop and a cusp.

- Symmetry: $B_{n-i,n}(t) = B_{i,n}(1-t)$ for $i = 0, \dots, n$.
- Partition of Unity: $\sum_{i=0}^n B_{i,n}(t) = 1$.
- Recursion: $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$ for $i = 0, \dots, n$, where $B_{-1,n-1}(t) = 0$ and $B_{n,n-1}(t) = 0$.

The non-negativity and partition of unity properties lead to two important properties of Bézier curves which are the *convex hull property* and invariance under transformations. The symmetry property implies that a symmetric control polygon leads to a symmetric curve. The recursion property gives rise to efficient evaluation and rendering algorithms.

The non-negativity and symmetry properties are rather obvious from the definition of the Bernstein polynomials. To show the partition of unity property, we apply Pascal's rule to $((1-t) + t)^n = 1$ which gives

$$((1-t) + t)^n = \sum_{i=0}^n \binom{n}{i} (1-t)^{n-i} t^i = \sum_{i=0}^n B_{i,n}(t) = 1.$$

The recursion property follows from the property of binomial coefficients (22). By definition

$$B_{i,n-1}(t) = \binom{n-1}{i} (1-t)^{n-1-i} t^i$$

$$B_{i-1,n-1}(t) = \binom{n-1}{i-1} (1-t)^{n-i} t^{i-1}$$

so that

$$\begin{aligned}
(1-t)B_{i,n-1}(t) - tB_{i-1,n-1}(t) &= \binom{n-1}{i}(1-t)^{n-i}t^i + \binom{n-1}{i-1}(1-t)^{n-i}t^i \\
&= \left[\binom{n-1}{i} + \binom{n-1}{i-1} \right] (1-t)^{n-i}t^i \\
&= \binom{n}{i}(1-t)^{n-i}t^i \\
&= B_{i,n}(t).
\end{aligned}$$

Convex Hull

Given a set of points $X = \{\mathbf{x}_0, \dots, \mathbf{x}_n\}$ the *convex hull* of X , denoted by $CH(X)$ is the set of points

$$CH(X) = \{a_0\mathbf{x}_0 + \dots + a_n\mathbf{x}_n \mid a_0 + \dots + a_n = 1, a_i \geq 0\}.$$

The convex hull can be thought of as the polygon which is formed when an elastic band is placed around the set of point and permitted to shrink around the points. The definition is valid also in 3-space and intuitively the elastic band can be thought to be replaced by an elastic balloon.

Properties of Bézier Curves

A Bézier curve $\mathcal{C}(t)$ with control points $\mathbf{b}_0, \dots, \mathbf{b}_n$ has the following properties

- Endpoint interpolation property: $\mathcal{C}(0) = \mathbf{b}_0$ and $\mathcal{C}(1) = \mathbf{b}_n$
- Endpoint tangent property: $\mathcal{C}'(0) = n(\mathbf{b}_1 - \mathbf{b}_0)$ and $\mathcal{C}'(1) = n(\mathbf{b}_n - \mathbf{b}_{n-1})$
- Convex hull property: $\mathcal{C}(t) \in CH(\mathbf{b}_0, \dots, \mathbf{b}_n)$ for all $t \in [0, 1]$
- Invariance under affine transformations: If L is an affine transformation, then

$$L \left(\sum_{i=0}^n \mathbf{b}_i B_{i,n}(t) \right) = \sum_{i=0}^n L(\mathbf{b}_i) B_{i,n}(t).$$

- Variation Diminishing: For a planar Bézier curve the number of intersections of a given line and the curve is less than or equal the number of intersections of that line with the control polygon.

The endpoint interpolation property is rather obvious from the definition of a Bézier curve. The endpoint tangent property will be proved later when derivation of Bézier curves is considered.

The convex hull property follows because every point in the curve has the form

$$a_0\mathbf{b}_0 + \dots + a_n\mathbf{b}_n \quad \text{with} \quad a_i = B_{i,n}(t)$$

where $a_i \geq 0$ and $\sum_{i=0}^n a_i = 1$ because of the non-negativity of the partition of unity properties of Bernstein polynomials.

To verify affine invariance consider an affine transformation

$$\mathbf{L}(x, y) = (ax + by + c, dx + ey + f).$$

and let $\mathcal{C}(t)$ be a Bézier curve of degree n with control points $\mathbf{b}_i = (x_i, y_i)$, $i = 0, \dots, n$. Then, by the partition of unity,

$$\begin{aligned} \mathbf{L}(\mathcal{C}(t)) &= \left(a \sum_i x_i B_{i,n}(t) + b \sum_i y_i B_{i,n}(t) + c, d \sum_i x_i B_{i,n}(t) + e \sum_i y_i B_{i,n}(t) + f \right) \\ &= \left(\sum_i (ax_i + by_i + c) B_{i,n}(t), \sum_i (dx_i + ey_i + f) B_{i,n}(t) \right) \\ &= \sum_i ((ax_i + by_i + c), (dx_i + ey_i + f)) B_{i,n}(t) \\ &= \sum_i \mathbf{L}(\mathbf{b}_i) B_{i,n}(t) \end{aligned}$$

where all the sums are taken from $i = 0$ to $i = n$.

The proof of the variation diminishing property is beyond the scope of this course.

5.4 Exercises

1. Weierstrass theorem states that any continuous function can be approximated by a polynomial to within any prescribed tolerance ε . The Bernstein polynomial approximation $B(t)$ of degree n of a function $f(t)$ over the interval $[0, 1]$ is defined as

$$p_n(t) = \sum_{i=0}^n f(t_i) B_{i,n}(t), \quad t_i = \frac{i}{n}.$$

According to the Weierstrass theorem for any ε there is an index n such that

$$|f(t) - p_n(t)| < \varepsilon.$$

- a) Plot the Bernstein approximations of different degrees for the function $f(t) = \sin(\pi t)$.
- b) For each approximation, plot the error function

$$e_n(t) = |f(t) - p_n(t)|$$

and estimate the value of n for which the maximum error over the interval $[0, 1]$ is less than 0.01.

2. Plot the cubic Bézier curve and the control polygon defined by control points

- a) $(0, 1)$, $(2, 5)$, $(4, 6)$, and $(8, 1)$.
- b) $(1, 1)$, $(3.4, 1.8)$, $(6, 6.5)$, and $(9, 1)$.

Do the curves satisfy the convex hull property?

3. Let a Bezier curve be defined by the control points $(0, 0)$, $(2, 1)$, $(3, -1)$, and $(1, -2)$. Perform the following transformations to the curve:
 - a) a translation of 4 units in the x -direction and 2 units in the y -direction
 - b) a rotation about the origin through an angle of 90° in the anti-clockwise direction
 - c) a reflection in the line $y = x$
4. [Cusp] Visualize the Bézier curve with control points $(2, 2)$, $(6, 0)$, $(2, 0)$, and $(6, 2)$.