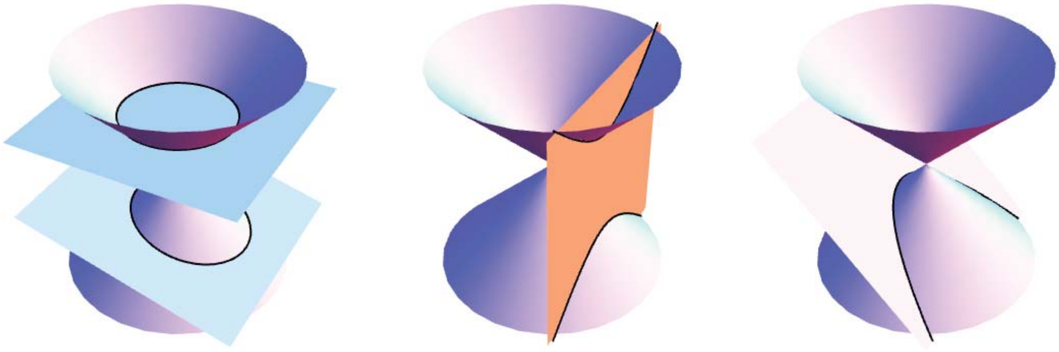

Figure 11: Conic sections.

4 Curves II

4.1 Conics

The simplest planar curve defined in the implicit form is a straight line given by a linear equation $ax + by + c = 0$. Curves defined by a quadratic polynomial equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{6}$$

are called *conic sections*, or *conics*. The name comes from a classical geometrical construction where a curve is obtained as the intersection of a cone with a plane. Ellipses, hyperbolas and parabolas are the three types of conics obtained this way, see Figure 11.

Another geometric construction, called the *focus-directrix construction*, is as follows. Given a line D in the plane, called the *directrix*, and a point F, called the *focus*, a conic is the locus of all points P such that the distance $\operatorname{dist}(P, F)$ from P to F is proportional to the distance $\operatorname{dist}(P, D)$ from P to D. The proportionality constant ε in $\operatorname{dist}(P, F) = \varepsilon \operatorname{dist}(P, D)$ is called the *eccentricity*.

Let us show that the focus-directrix construction gives a curve that satisfies the general equation (6). Suppose that the directrix is the line $D : \alpha x + \beta y + \gamma = 0$ and the focus is the point $F = (x_F, y_Y)$. If $P = (x, y)$ is a general point on the conic, then

$$\operatorname{dist}(P, F) = \sqrt{(x - x_F)^2 + (y - y_F)^2}$$

and

$$\operatorname{dist}(P, D) = \frac{|\alpha x + \beta y + \gamma|}{\sqrt{\alpha^2 + \beta^2}}$$

Squaring both sides in the equation $\operatorname{dist}(P, F) = \varepsilon \operatorname{dist}(P, D)$ and multiplying through by $\alpha^2 + \beta^2$ yields

$$(\alpha^2 + \beta^2)\left((x - x_F)^2 + (y - y_F)^2\right) = \varepsilon^2(\alpha x + \beta y + \gamma)^2 \tag{7}$$

which is quadratic equation in x and y of the form (6).

Let a conic have as a directrix the y-axis and as a focus the point $(k, 0)$. Then Eq. (7) yields

$$(1-\varepsilon^2)x^2 - 2kx + y^2 + k^2 = 0 \tag{8}$$

Assuming that $\varepsilon \neq 1$, Eq. (8) can be written as

$$(x - \frac{k}{1-\varepsilon^2})^2 - \frac{k^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} + \frac{k^2}{1-\varepsilon^2} = 0$$

so that the transformation

$$x' = x - \frac{k}{1-\varepsilon^2}, \quad y' = y$$

yields

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \tag{9}$$

where

$$a = \frac{k\varepsilon}{1-\varepsilon^2}, \quad b^2 = a^2(1-\varepsilon^2).$$

There are two cases:

1. Ellipse. If $\varepsilon < 1$, then $a, b > 0$ and after dropping the primes Eq. (9) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0 \tag{10}$$

   In this coordinate system the origin is the center of the ellipse. The x-axis is the major axis, and the y-axis is the minor axis. The points $(\pm a, 0)$ and $(0, \pm b)$ are referred to as the vertices and the distances a and b as the major and minor radii, respectively. Notice that (10) represents a circle when $a = b$ which can be thought of as the limiting case $\varepsilon \to 0$, $k \to \infty$).

2. Hyperbola. If $\varepsilon > 1$, then $a < 0$ and $b^2 < 0$. Upon setting $b = |b|$, Eq. (9) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{11}$$

   In this coordinate system the origin is the center of the hyperbola. The x-axis is called the transverse axis, and the y-axis is called the semiconjugate or imaginary axis. The points $(\pm a, 0)$ are the vertices and the distances a and b are called the major and minor, or imaginary) radii

Ellipses and hyperbolas defined by different parameters are shown in Figures 12 and 13, respectively.

When $\varepsilon = 1$, the transformation

$$x' = x - \frac{k}{2}, \quad y' = y$$

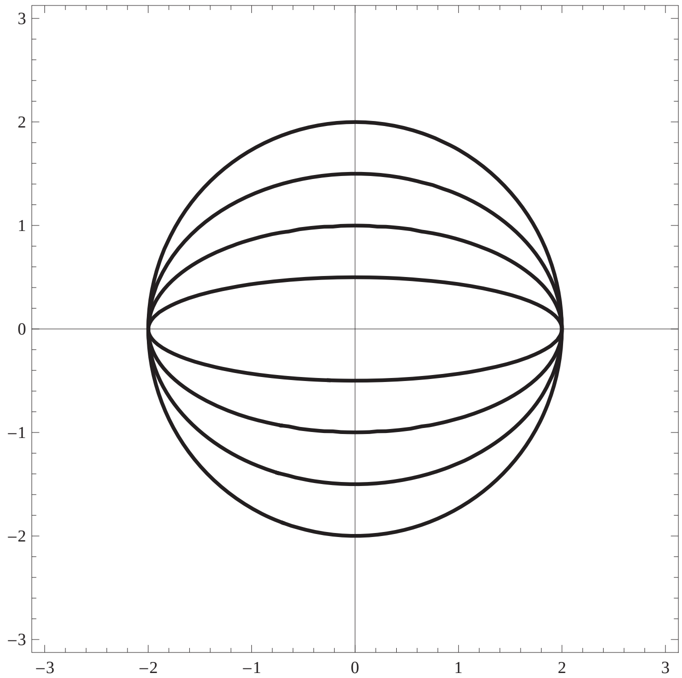

Figure 12: Ellipses defined by different parameters $a = 2$, and $b = 2, 3/2, 1, 1/2$.

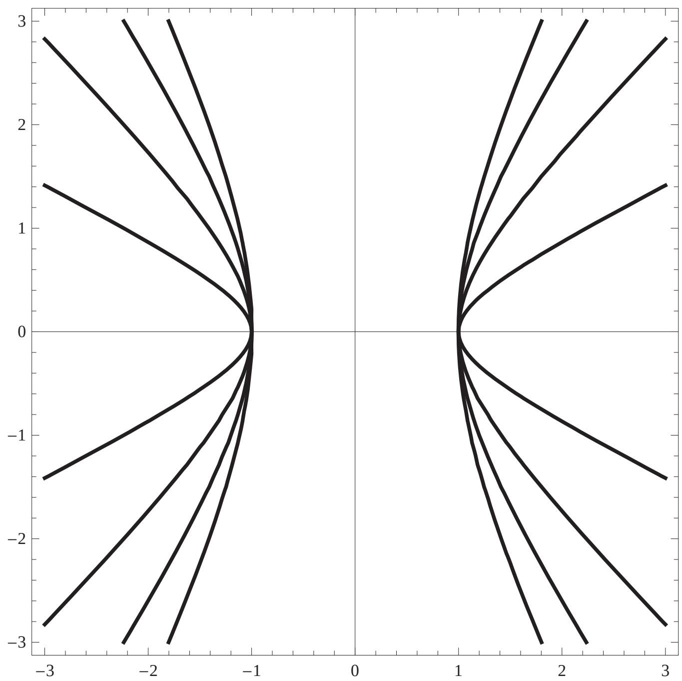

Figure 13: Hyperbolas defined by different parameters $a = 1$, and $b = 2, 3/2, 1, 1/2$.

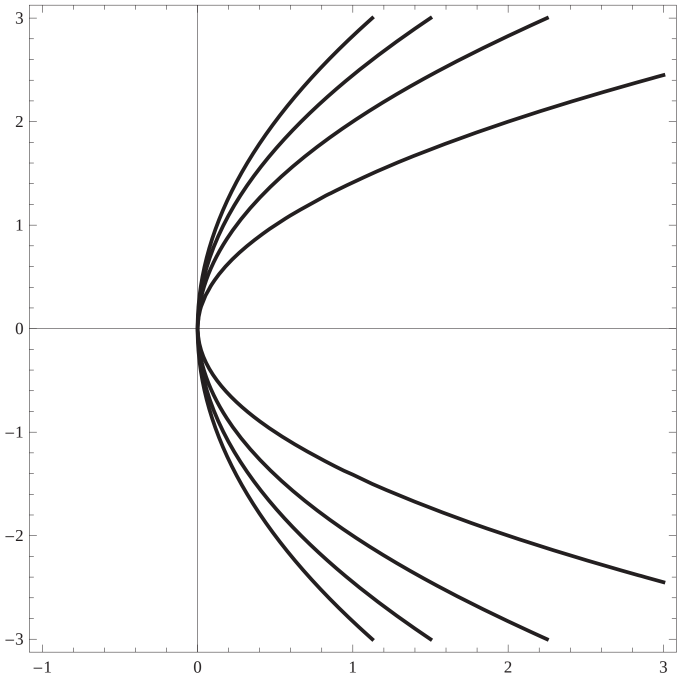

Figure 14: Parabolas defined by different parameters $a = 2, 3/2, 1, 1/2$.

yields the parabola

$$y^2 = 4ax, \quad a = \frac{k}{2} > 0 \tag{12}$$

with focus $F = (a, 0)$ and directrix $x = -a$. The parabola has no center, but the origin is its vertex. The x-axis its axis and a is called the focal distance. Parabolas with different focal distances are shown in Figure 14.

We shall omit the proofs of the facts that all sections of a cone are expressible algebraically by the quadratic equation (6) and that any conic can be obtained by a focus directrix-construction.

4.2 Classification of Conics

Consider a conic defined by Eq. (6). If (6) can be factored into a product of two linear terms, then the conic is a union of two lines and the conic is said to be *reducible*. If the factorization is not possible the conic is said to be irreducible. A condition on the coefficients determining whether a conic is reducible is as follows. Let $a \neq 0$ and multiply (6) by a to give

$$a^2x^2 + 2abxy + 2adx = -acy^2 - 2aey - af$$

Completing the square on the left hand side yields

$$(ax + by + d)^2 = \underbrace{(b^2 - ac)}_{A}\, y^2 + \underbrace{2(bd - ae)}_{B}\, y + \underbrace{(d^2 - af)}_{C} \tag{13}$$

Thus, the conic has linear factors if and only if the right hand side of Eq. (13) is a perfect square. This happens if and only if the discriminant is zero, that is

$$B^2 - 4AC = 4(bd - ae)^2 - 4(b^2 - ac)(d^2 - af) = 0 \tag{14}$$

In homogeneous coordinates $(x, y) \hookrightarrow (x/w, y/w, w)$, the conic is given by the equation

$$ax^2 + 2bxy + cy^2 + 2dxw + 2eyw + fw^2 = 0. \tag{15}$$

This can be expressed in the matrix form as

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} x & y & w \end{pmatrix} \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = 0 \tag{16}$$

It is easy to check that condition (14) is equivalent to the condition $D = 0$, where D is called the *discriminant* of the conic and defined as the determinant

$$D = \det(\mathbf{M}). \tag{17}$$

When $D = 0$, we have $Ay^2 + By + c = A(y + \frac{B}{2A})^2$ and two possibilities:

1. When $A = b^2 - ac \geq 0$, the conic has two real linear factors and the conic is a pair of lines.

2. When $A = b^2 - ac < 0$, the conic has two complex linear factors and the conic is a point.

The irreducible conics are

1. hyperbolas when $b^2 - ac > 0$

2. ellipses when $b^2 - ac < 0$

3. parabolas when $b^2 - ac = 0$

Example 4.1. Consider the conic given by $x^2 + 2xy - 3y^2 + 4x - 5 = 0$. We have $a = 1$, $b = 1$, $c = -3$, $d = 2$, $e = 0$, and $f = -5$. Therefore

$$D = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -3 & 0 \\ 2 & 0 & -5 \end{vmatrix} = 32$$

so that the conic is irreducible. Because $b^2 - ac = 4 > 0$, the conic is a hyperbola, see Fig. 15

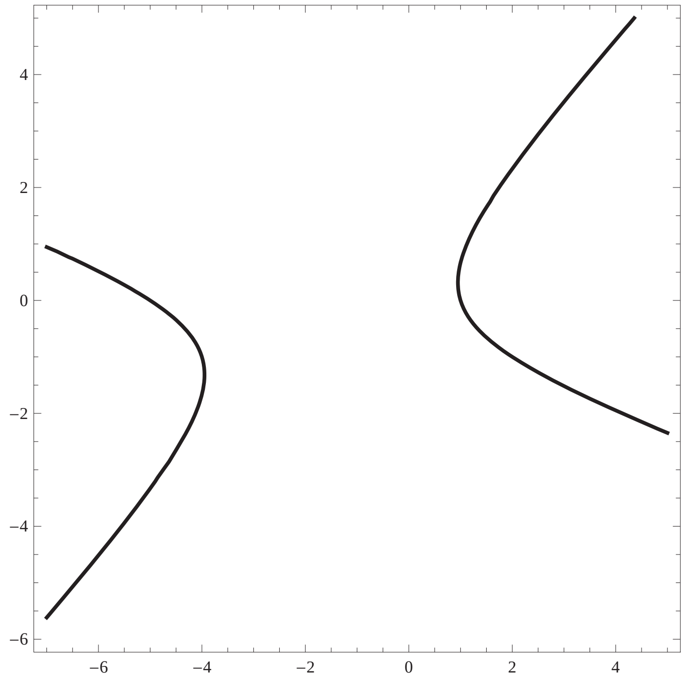

Figure 15: The hyperbola $x^2 + 2xy - 3y^2 + 4x - 5 = 0$.

Example 4.2. Consider the conic given by $-2x^2 + xy - x - y + 3 = 0$. We have $a = -2$, $b = 1/2$, $c = 0$, $d = -1/2$, $e = -1/2$, and $f = 3$. Therefore

$$D = \begin{vmatrix} -2 & 1/2 & -1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 3 \end{vmatrix} = 0$$

so that the conic is reducible. It has the factorization

$$(x - 1)(-2x + y - 3) = 0$$

corresponding to the two lines $x - 1 = 0$ and $2x - y + 3 = 0$ shown in Fig. 16.

Standard Form

A conic for which the axes or lines of symmetry coincide with coordinate axes are said to be in standard form. The standard forms of irreducible conics are the ones derived in Eqs. (10)–(12). The standard form of a conic can be obtained by applying an orthogonal change of coordinates.

First of all, a non-singular planar transformation does not affect the irreducibility of a conic:

Theorem 1. *Let $\hat{f}$ be the image of the conic $f = \mathbf{x}^T\mathbf{M}\mathbf{x}$ following the application of a non-singular planar transformation with transformation matrix $\mathbf{A}$. Then $\hat{f}$ is irreducible if and only if f is irreducible.*

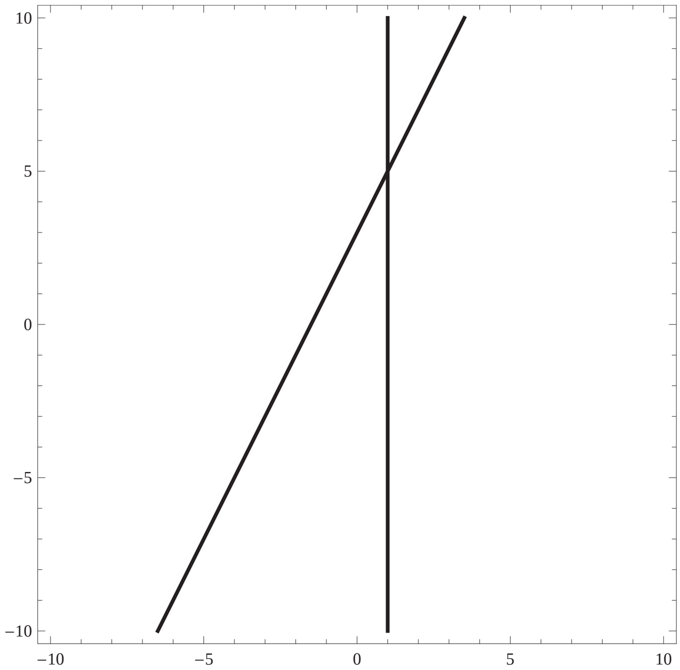

Figure 16: The reducible conic $(x-1)(-2x+y-3)=0$.

Proof. The transformation $\mathbf{x} = \mathbf{A}\hat{\mathbf{x}}$ yields a conic

$$(\mathbf{A}\hat{\mathbf{x}})^T\mathbf{M}(\mathbf{A}\hat{\mathbf{x}}) = \hat{\mathbf{x}}^T\mathbf{A}^T\mathbf{M}\mathbf{A}\hat{\mathbf{x}} = \hat{\mathbf{x}}^T\hat{\mathbf{M}}\hat{\mathbf{x}},$$

where $\hat{\mathbf{M}} = \mathbf{A}^T\mathbf{M}\mathbf{A}$. Consequently,

$$\det(\hat{\mathbf{M}}) = \det(\mathbf{A}^T\mathbf{M}\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{M})\det(\mathbf{A}) = \det(\mathbf{A})^2\det(\mathbf{M}).$$

We have

$$\det(\hat{\mathbf{M}}) = 0 \quad \Leftrightarrow \quad \det(\mathbf{M}) = 0$$

because $\mathbf{A}$ is non-singular. □

If the change of coordinates is orthogonal we have the following

Theorem 2. *Let $\hat{f}$ be the image of the conic f following the application of an orthogonal change of coordinates. Then*

$$b^2 - ac = \hat{b}^2 - \hat{a}\hat{c}$$
$$a + c = \hat{a} + \hat{c}$$

Proof. Substituting

$$\begin{aligned} x &= \cos(\varphi)\hat{x} - \sin(\varphi)\hat{y} + h_x\hat{w} \\ y &= \sin(\varphi)\hat{x} + \cos(\varphi)\hat{y} + h_y\hat{w} \\ w &= \hat{w} \end{aligned}$$

into (15) yields

$$\hat{a}\hat{x}^2 + 2\hat{b}\hat{x}\hat{y} + \hat{c}\hat{y}^2 + 2\hat{d}\hat{x}\hat{w} + 2\hat{e}\hat{y}\hat{w} + \hat{f}\hat{w}^2 \tag{18}$$

where

$$\begin{aligned}
\hat{a} &= a\cos^2\varphi + 2b\cos\phi\sin\varphi + c\sin^2\varphi \\
\hat{b} &= b(\cos^2\varphi - \sin^2\varphi) + (c-a)\sin\varphi\cos\varphi \\
\hat{c} &= a\sin^2\varphi - 2b\cos\varphi\sin\varphi + c\cos^2\varphi
\end{aligned}$$

The claims follow after simplification. □

Recall that if $\mathbf{A}$ is the transformation matrix of an orthogonal change of coordinates, then $\det(\mathbf{A}) = 1$. Since $\det(\hat{\mathbf{M}}) = \det(\mathbf{M})$ and $\hat{b}^2 - \hat{a}\hat{c} = b^2 - ac$ the type of conic is unaffected by the change of coordinates. Notice that if Eq. (6) is multiplied by λ, then $a + c$, $b^2 - ac$, and D become $\lambda(a + c)$, $\lambda^2(b^2 - ac)$, and $\lambda^3 D$. The ratios $(a + c) : \sqrt{b^2 - ac} : \sqrt[3]{D}$ are *absolute invariants* and remain the same in any Cartesian coordinate system.

Theorem 3. *An irreducible conic can be mapped to standard form by using an orthogonal coordinate transform.*

Proof. Consider the transformation used in the previous theorem. The cross $\hat{x}\hat{y}$ term becomes eliminated if the angle is chosen such that

$$\hat{b} = b\cos(2\varphi) + \frac{1}{2}(c-a)\sin(2\varphi) = 0$$

The angle satisfies

$$\tan(2\varphi) = \frac{2b}{a-c}$$

when $a \neq c$ and $\varphi = \pi/4$ when $a = c$. If $\hat{a}, \hat{c} \neq 0$, then (18) can be put into the form

$$\hat{a}\left(\hat{x} + \frac{\hat{d}}{\hat{a}}\hat{w}\right)^2 + \hat{c}\left(\hat{y} + \frac{\hat{e}}{\hat{c}}\hat{w}\right)^2 + \left(\hat{f} - \frac{\hat{d}^2}{\hat{a}} - \frac{\hat{e}^2}{\hat{c}}\right)\hat{w}^2 = 0$$

so that a translation $\mathbf{T}(-\frac{\hat{d}}{\hat{a}}, h_y = -\frac{\hat{e}}{\hat{c}})$ yields

$$\hat{a}\hat{x}^2 + \hat{c}\hat{y}^2 + \left(\hat{f} - \frac{\hat{d}^2}{\hat{a}} - \frac{\hat{e}^2}{\hat{c}}\right)\hat{w}^2 = 0$$

which is the standard form of a hyperbola or ellipse in Eqs. (11) and (10).

If $\hat{a} = 0$, $\hat{c} \neq 0$, then (18) takes the form

$$\hat{c}\left(\hat{y} + \frac{\hat{e}}{\hat{c}}\hat{w}\right)^2 + 2\hat{d}\hat{x}\hat{w} + \hat{f}\hat{w}^2 = 0$$

and translation $\mathbf{T}(\frac{\hat{f}}{2\hat{d}}, -\frac{\hat{e}}{\hat{c}})$ yields

$$\hat{c}\hat{y}^2 + 2\hat{d}\hat{x}\hat{w} = 0$$

which corresponds to a standard form for the parabola in Eq. (12).

If $\hat{a} = \hat{c} = 0$ the conic is reducible. □

Example 4.3. In order to determine the standard form for the ellipse

$$13x^2 - 8xy + 7y^2 - 42x + 36y = 0$$

we note that $a = 13$, $b = -4$, $c = 7$. The required rotation angle satisfies $\tan(2\varphi) = -4/3$ so that $\cos\varphi = 2/\sqrt{5}$, $\sin\varphi = -1/\sqrt{5}$. The rotation

$$x = 2/\sqrt{5}\hat{x} + 1/\sqrt{5}\hat{y}$$
$$y = -1/\sqrt{5}\hat{x} + 2/\sqrt{5}\hat{y}$$

yields after simplification

$$5\hat{y}^2 + 6\sqrt{5}\hat{y} + 15\hat{x}^2 - 24\sqrt{5}\hat{x} = 0$$

Then completing the squares and making a translation $\hat{x} \to \hat{x} + 12/15\sqrt{5}$, $\hat{y} \to \hat{y} - 3/5\sqrt{5}$ gives

$$15\hat{x}^2 + 5\hat{y}^2 = 57$$

The ellipse and its standard form are shown in Fig. 17.

4.3 Parametrization of a Conic

In addition to the implicit form (6), conics can also be expressed in a parametric form

$$(x(t), y(t)) = \left(\frac{a_0 + a_1 t + a_2 t^2}{c_0 + c_1 t + c_2 t^2}, \frac{b_0 + b_1 t + b_2 t^2}{c_0 + c_1 t + c_2 t^2} \right). \quad (19)$$

Example 4.4. Consider a parametric curve defined by $x = t^2 - t$, $y = 2t^2 + t - 2$. Then

$$2x - y = 2 - 3t \quad \Rightarrow \quad t = (-2x + y + 2)/3$$

and

$$x = (-2x + y + 2)^2/9 - (-2x + y + 2)/3.$$

Expanding and simplifying yields a conic in an implicit form

$$4x^2 - 4xy + y^2 - 11x + y - 2 = 0.$$

A more general method of conversion is based on the following

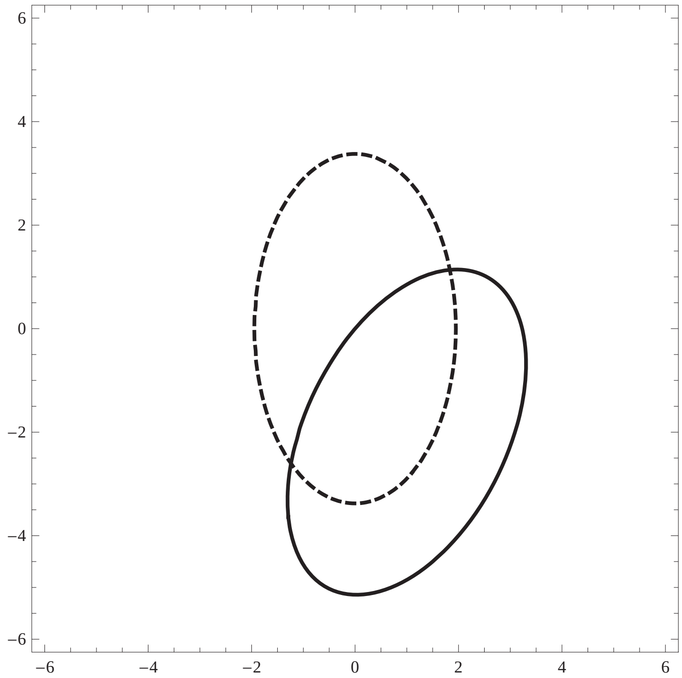

Figure 17: The ellipse $13x^2-8xy+7y^2-42x+36y=0$ and its standard form $15\hat{x}^2+5\hat{y}^2=57$.

Lemma 2. The two quadratic equations

$$\begin{cases} a_1t^2+b_1t+c_1=0, & a_1\neq 0 \\ a_2t^2+b_2t+c_2=0, & a_2\neq 0 \end{cases}$$

have a common root if and only if

$$(a_2c_1-a_1c_2)^2-(a_1b_2-a_2b_1)(b_1c_2-b_2c_1)=0. \tag{20}$$

Proof. Suppose $\tilde{t}$ is a common root of both equations. Then elimination of the quadratic terms yields

$$(a_2b_1-a_1b_2)\tilde{t}+a_2c_1-a_1c_2=0$$

and the elimination of the constant terms yields

$$(a_1c_2-a_2c_1)\tilde{t}^2+(b_1c_2-b_2c_1)\tilde{t}=0$$

In other words,

$$\tilde{t}=\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2} \quad \text{and} \quad \tilde{t}=\frac{b_1c_2-b_2c_1}{a_1c_2-a_2c_1} \tag{21}$$

Therefore

$$(a_2c_1-a_1c_2)(a_1c_2-a_2c_1)-(a_2b_1-a_1b_2)(b_1c_2-b_2c_1)=0$$

Conversely it can be shown that if (20) holds, then at least one of the expressions in (21) is a common root to both equations. We skip the details. □

We can now show that the parametric curve (19) is a conic. Multiplication by the denominator gives the quadratic equations in t:

$$\begin{aligned} a_0 - c_0 x + (-c_1 x + a_1)t + (a_2 - c_2 x)t^2 = 0 \\ b_0 - c_0 y + (-c_1 y + b_1)t + (b_2 - c_2 y)t^2 = 0 \end{aligned}$$

For a fixed t these hold if and only if $B^2 - AC = 0$ where

$$\begin{aligned} B &= (b_2 - c_2 y)(a_0 - c_0 x) - (a_2 - c_2 x)(b_0 - c_0 y) \\ A &= -(b_2 - c_2 y)(-c_1 x + a_1) + (a_2 - c_2 x)(-c_1 y + b_1) \\ C &= (-c_1 x + a_1)(b_0 - c_0 y) - (c_1 y + b_1)(a_0 - c_0 x) \end{aligned}$$

Since every point (x, y) of the curve (19) satisfies the equation $B^2 - AC = 0$ which is a quadratic polynomial equation in x and y of the form (6), the curve is a conic.

Example 4.5. The implicit equation of the curve

$$(x(t), y(t)) = \left(\frac{18 - 19t + 8t^2}{-2 - 5t + 4t^2}, \frac{-6 + 7t + 3t^2}{-2 - 5t + 4t^2} \right)$$

is given by $B^2 - AC = 0$ where

$$\begin{aligned} B &= 102 - 18x - 88y \\ A &= 113 - 43x - 36y \\ C &= -12 - 44x - 128y \end{aligned}$$

Expanding and simplifying gives the equation

$$-2x^2 - 5xy + 4y^2 + x - 5y + 15 = 0.$$

Intersections of a Conic with a Line

The intersection of a conic with a line can be found by elimination of variables and solution of a quadratic equation.

Example 4.6. The intersection of the conic $-2x^2 - 5xy + 4y^2 + x - 5y + 15 = 0$ and the line $x - y + 2 = 0$ can be found by substituting $y = 2 + x$ into the conic equation to give

$$3x^2 - 2x + 21 = 0 \quad \Leftrightarrow \quad x = -7/3 \text{ or } x = 3$$

This gives the points of intersection $(-7/3, -1/3)$ and $(3, 5)$, see Fig. 18.

If a parametric form $x = t + 1$, $y = t + 3$ of the line is used. These can be substituted to the conic equation to give

$$3t^2 + 4t - 20 - 0 \quad \Leftrightarrow \quad t = -10/3 \text{ or } t = 2.$$

Substitution to the parametric form of the line yields the same points of intersection.

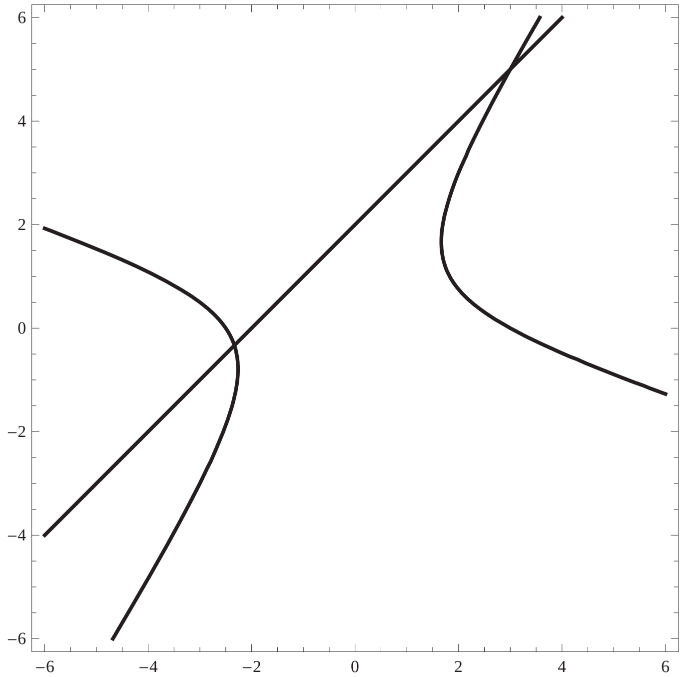

Figure 18: Intersection of the conic $-2x^2 - 5xy + 4y^2 + x - 5y + 15 = 0$ and the line $x - y + 2 = 0$.

Example 4.7. The intersection of the parametrized conic

$$(x(t), y(t)) = \left(\frac{18 - 19t + 8t^2}{-2 - 5t + 4t^2}, \frac{-6 + 7t + 3t^2}{-2 - 5t + 4t^2} \right)$$

with the line $x - y + 2 = 0$ can be found by substituting

$$x = \frac{18 - 19t + 8t^2}{-2 - 5t + 4t^2}, \quad y = \frac{-6 + 7t + 3t^2}{-2 - 5t + 4t^2}$$

into the equation of the line to give

$$x - y + 2 = \frac{13t^2 - 36t + 20}{4t^2 - 5t - 2} = 0$$

The solutions to $13t^2 - 36t + 20 = 0$ are $t = 10/13, t = 2$. Substituting for t in the parametrization of the conic yields the points $(-7/3, -1/3)$ and $(3, 5)$.

The parametric form of an irreducible conic can be found by considering a family of lines $y = t(x - x_0) + y_0$ through a point (x_0, y_0) on the conic. Substitution of $y = t(x - x_0) + y_0$ to the equation of the conic yields a polynomial equation $p(x) = 0$ which is quadratic in x and t. Because $x = x_0$ is a root of $p(x)$, it can be factored as $p(x) = (x - x_0)(r(t)x - s(t))$ with some quadratic polynomials $r(t)$ and $s(t)$. The x-coordinate of the other point of intersection is given by $x(t) = s(t)/r(t)$ and the y-coordinate is given by substituting $x = s(t)/r(t)$ in $y = t(x - x_0) + y_0$.

Example 4.8. A parametrization of the hyperbola $-2x^2 - 5xy + 4y^2 + x - 5y + 15 = 0$ can be found by considering lines through the point $(3, 5)$ which is on the conic. Substituting $y = t(x-3) + 5$ yields

$$4(t(x-3)+5)^2 - 5x(t(x-3)+5) - 5(t(x-3)+5) - 2x^2 + x + 15 = 0$$

which can be factored as

$$(x-3)\left(4t^2x - 12t^2 - 5tx + 35t - 2x - 30\right) = 0$$

Solving for x gives $x = 3$ and

$$x(t) = \frac{12t^2 - 35t + 30}{4t^2 - 5t - 2}$$

and substitution for x in $y = t(x-3) + 5$ gives

$$y(t) = \frac{10 - 11t}{-4t^2 + 5t + 2}$$

4.4 Exercises

1. Verify that the discriminant condition $\det M = 0$ is equivalent to the condition (14).

2. For the conics below determine whether the conic is irreducible or reducible. If it is irreducible, then deterine whether it is an ellipse, a hyperbola, or a parabola. If the conic is reducible then determine the linear factors

   a) $4x^2 - 3xy + y^2 - x + 2y + 7 = 0$
   b) $3x^2 - xy - 2y^2 + 6x + 4y = 0$
   c) $2xy + 3y = 5$

3. Determine the standard form of the following conics:

   a) $13x^2 - 10xy + 13y^2 - 12\sqrt{2}x + 60\sqrt{2}y + 72 = 0$
   b) $6x^2 + 12xy + 6y^2 - 35\sqrt{2}x - 37\sqrt{2}y + 118 = 0$
   b) $11x^2 - 6x\sqrt{3}y - 6x\sqrt{3} + y^2 + 2y - 63 = 0$

4. [Translational invariants] Show that a translation does not alter the values a, b, and c.

5. Convert the following conics from parametric to implicit form:

   a) $(t^2 - 1, t + 2)$
   b) $(2t^2 + t - 1, t^2 - 3t + 3)$
   c) $\left(-\frac{4t^2 + t + 1}{4t^2 - 2t + 1}, -\frac{3t^2}{4t^2 - 2t + 1}\right)$

6. Find the points of intersection of the following conics and lines:

   a) $9x^2 - xy + y^2 - 4x + 2y + 1 = 0$ and $(x(t), y(t)) = (2t - 3, -3t + 4)$
   b) $3x^2 - 2xy + y^2 - 5x + 6y - 16 = 0$ and $x + 3y - 6 = 0$
   c) $(x(t), y(t)) = (t^2 + 1, t - 1)$ and $-3x - 2y + 4 = 0$

7. Determine a parametrization of the conic $x^2 - 2xy + 4y^2 + 2x + y + 1 = 0$ by considering lines through the point $(0, -1)$.

8. Convert the following conics from implicit to parametric form:

   a) $x^2 + 2y^2 - 2xy + 2y = 0$
   b) $x^2 + 2xy - y^2 - 1 = 0$
   c) $2x^2 - y^2 + 4x - 2y = 0$

9. [Radar] Show that a ray of light emanating from the focus $F = (a, 0)$ of the parabola $y^2 = 4ax$ becomes reflected parallel to the x-axis. Steps:

   a) Determine the tangent at a point $P = (x, y)$ on the parabola.
   b) Determine the angle between the line FP and the tangent.
   c) Show that the angle between the tangent and x-axis is the same as in b)

   Visualize the parabola and few reflected rays.

10. [Hyperbolic Navigation] Let F_1 and F_2 be given points, and let d_1 and d_2 be the distance of a point P from F_1 and F_2 respectively. Show that the locus of all points $P = (x, y)$ such that $d_1 - d_2$ is constant is a hyperbola.