

## 3 Curves I

### 3.1 Introduction. Implicit and Parametric Forms

The two common representations of curves in geometric modelling are by *implicit* and *parametric* forms. In parametric form, the coordinates of each point in the curve are expressed separately as functions of an independent parameter such as  $t$ . A plane curve has the form

$$\mathcal{C}(t) = (x(t), y(t)), \quad t \in [a, b],$$

where the functions  $x(t)$  and  $y(t)$  are called the *coordinate functions*. The interval  $[a, b]$  is arbitrary but is usually normalized to  $[0, 1]$ .

**Example 3.1.** The first quadrant of the unit circle show in Figure 7 is defined by the parametric functions

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \quad t \in [0, \pi/2].$$

The change of independent variable  $u = \tan \frac{t}{2}$  yields the alternative representation

$$\begin{cases} x(u) = \frac{1 - u^2}{1 + u^2} \\ y(u) = \frac{2u}{1 + u^2} \end{cases} \quad u \in [0, 1].$$

Obviously, the parametrization of a curve is not unique.

A curve can also be defined implicitly such that the coordinates  $(x, y)$  of points on the curve satisfy an equation  $f(x, y) = 0$ , for some function  $f$ .

**Example 3.2.** The first quadrant of the unit circle is defined by the equation

$$x^2 + y^2 - 1 = 0, \quad x, y \in [0, 1].$$

An equivalent representation is given by

$$y = \sqrt{1 - x^2}, \quad x \in [0, 1].$$

which also corresponds to a parametric form  $\mathcal{C}(t) = (t, \sqrt{1 - t^2})$ ,  $t \in [0, 1]$ .

Parametric and implicit forms both have their advantages and disadvantages so that geometric modelling utilizes both techniques:

- The parametric form can be extended easily to spatial curves in 3-space by adding a third coordinate function  $z(t)$  as in  $\mathcal{C}(t) = (x(t), y(t), z(t))$ . The implicit form can only be used to specify curves in a plane.

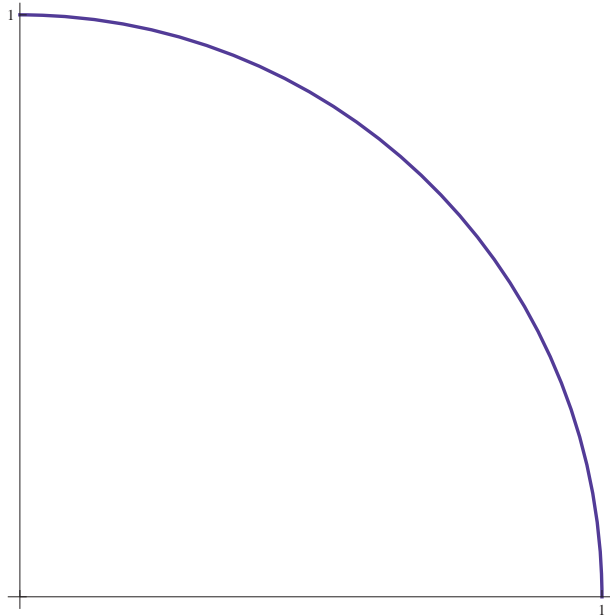


Figure 7: First quadrant of the unit circle.

- Representation of bounded curve segments is straightforward with the parametric form by restricting the parameter interval but non-trivial when the implicit form is used. On the other, the description of unbounded geometries, such as the straight line  $ax + by + c = 0$ , is often more natural in the implicit form.
- Parametric curves have a built-in ‘direction’ from  $\mathcal{C}(a)$  to  $\mathcal{C}(b)$ , when  $t \in [a, b]$ , which enables generation of ordered sequences of points along the curve.
- The complexity of geometric manipulations depend on the method of representation. For instance, computation of a point on a curve is trivial in parametric form but difficult in the implicit form. Conversely, determining whether a given point is on a curve is trivial in the implicit form but difficult in the parametric form.

### 3.2 Parametric Curves

Parametric curves are convenient for visualization since points on the curve are easily computed. A curve  $\mathcal{C}(t) = (x(t), y(t))$ ,  $t \in [a, b]$  is rendered by choosing  $n + 1$  parameter values  $a = t_0 < t_1 < \dots < t_n = b$  and evaluating the points  $(x(t_i), y(t_i))$ . The points are joined by line segments to give a linear approximation to the curve as show in Figure 8. The smoothness of the curve can be controlled by the number of evaluated points.

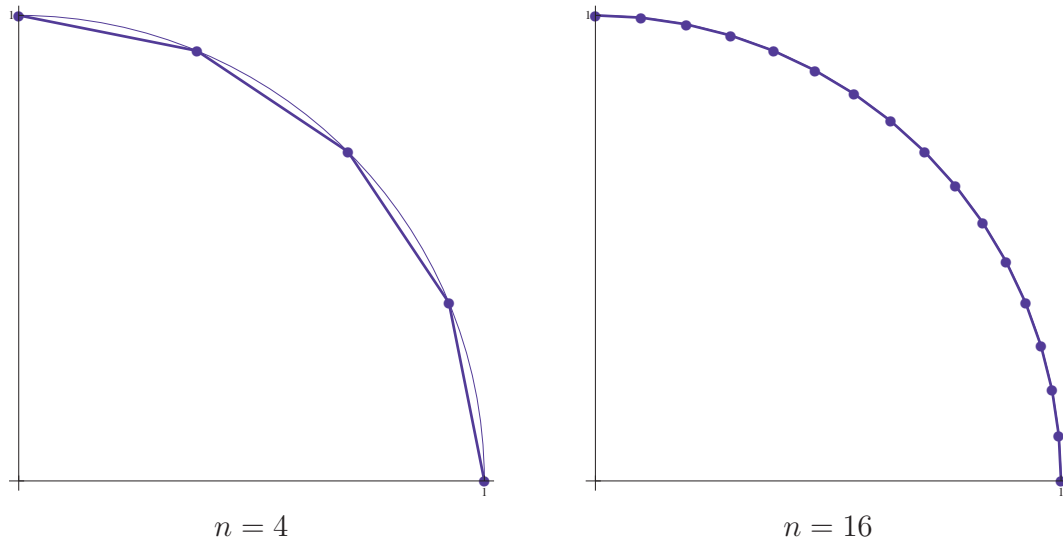


Figure 8: Piecewise linear approximations of the circular arc.

### Tangent and Normal to the Curve

**Definition 6.** Let  $\mathcal{C}(t) = (x(t), y(t))$  be a curve defined on  $[a, b]$ . Then  $\mathcal{C}(t)$  is said to be  $C^k$ -continuous, or shortly  $C^k$ , if the coordinate functions are  $k$  times continuously differentiable on  $[a, b]$ . If infinitely many derivatives exist, then  $\mathcal{C}(t)$  is said to be  $C^\infty$ .

If  $\mathcal{C}(t)$  is a  $C^1$  curve defined on an interval  $[a, b]$ , the function

$$v(t) = |\mathcal{C}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

is called the *speed* of the curve  $\mathcal{C}(t)$ . If  $v(t) \neq 0$  for all  $t \in [a, b]$ , then  $\mathcal{C}(t)$  is *regular*. If  $v(t) = 1$  for all  $t \in [a, b]$ , then  $\mathcal{C}(t)$  is referred to as a *unit speed curve* and the corresponding parametrization is called *uniform*.

Let  $\mathcal{C}(t)$  be a regular curve, and suppose that

$$P = (x(t), y(t)) \quad \text{and} \quad Q = (x(t + \Delta t), y(t + \Delta t))$$

respectively. Then the vector

$$\lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{PQ}}{\Delta t} = (x'(t), y'(t)) = \mathcal{C}'(t)$$

can be interpreted as the *tangent vector*. The unit vector

$$\boldsymbol{\tau}(t) = \frac{\mathcal{C}'(t)}{|\mathcal{C}'(t)|} = \frac{\mathcal{C}'(t)}{v(t)}$$

is the *unit tangent vector*.

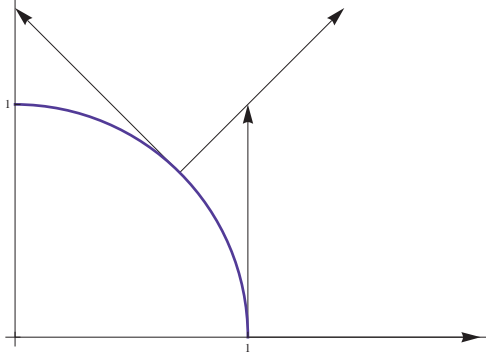


Figure 9: Tangent and normal vectors of the unit quarter circle.

A *unit normal vector* can be obtained by rotating the unit tangent vector through an angle  $\pi/2$ :

$$\boldsymbol{\nu}(t) = \left( \frac{-y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right)$$

**Example 3.3.** Let  $\mathcal{C}(t) = (\cos t, \sin t)$ . Then the tangent vector is  $\mathcal{C}'(t) = (-\sin t, \cos t)$  and the normal vector is  $\boldsymbol{\nu}(t) = (-\cos t, -\sin t)$ . These unit vectors are shown for the parameter values  $t = 0$  and  $t = \pi/4$  in Figure 9.

### Arclength. Reparametrization

Consider a regular curve  $\mathcal{C}(t) = (x(t), y(t))$ ,  $t \in [a, b]$  and a partitioning  $a = t_0 < t_1 < \dots < t_n = b$  of the interval  $[a, b]$ . Then the length of the broken line joining the points  $\mathcal{C}(t_i)$  approximates the length of the curve  $\mathcal{C}$  on the interval  $[a, b]$ :

$$L_n = \sum_{k=0}^{n-1} \sqrt{(x(t_{k+1}) - x(t_k))^2 + (y(t_{k+1}) - y(t_k))^2}. \quad (4)$$

Application of the mean value theorem yields

$$L_n = \sum_{k=0}^{n-1} \sqrt{x'(\xi_k)^2 + y'(\eta_k)^2} (t_{k+1} - t_k) \quad (5)$$

for some  $\xi_k, \eta_k \in (t_k, t_{k+1})$ . It can be shown that the limit of (5) when the partitioning is refined such that

$$h = \max_{k=0, \dots, n-1} (t_{k+1} - t_k)$$

tends to zero and  $n$  tends to infinity, is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_a^b v(t) dt$$

The arclength function from  $(x(t_0), y(t_0))$  to  $(x(t), y(t))$  is

$$s(t; t_0) = \int_{t_0}^t v(u) \, du.$$

**Example 3.4.**

- (a) The speed of the quarter circle  $\mathcal{C}(t) = (\cos \frac{\pi}{2}t, \sin \frac{\pi}{2}t)$ , for  $t \in [0, 1]$  is  $v(t) = \pi/2$ . The arclength function from  $t_0 = 0$  is

$$s(t; 0) = \int_0^t \frac{\pi}{2} \, du = \frac{\pi}{2}t.$$

The length of the curve is  $L = s(1; 0) = \pi/2$ .

- (b) The speed of

$$\mathcal{C}(t) = \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \quad t \in [0, 1]$$

is

$$v(t) = \frac{2}{1+t^2}.$$

The arclength function from  $t_0 = 0$  is

$$s(t; 0) = \int_0^t \frac{2}{1+u^2} \, du = 2 \arctan t$$

The length is  $L = s(1; 0) = 2\pi/4 = \pi/2$ .

- (c) The speed of

$$\mathcal{C}(t) = (t, \sqrt{1-t^2}) \quad t \in [0, 1]$$

is

$$v(t) = \frac{1}{\sqrt{1-t^2}}.$$

The arclength function from  $t_0 = 0$  is

$$s(t; 0) = \int_0^t \frac{1}{\sqrt{1-u^2}} \, du = \arcsin t$$

The length is  $L = s(1; 0) = \pi/2$ .

The different arclength functions are shown in Figure 10.

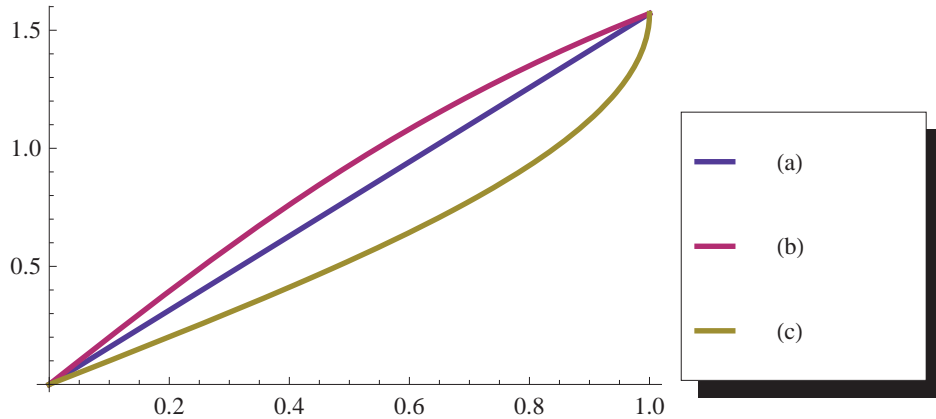


Figure 10: Comparison of arclength functions for different parametrizations of the quarter circle.

### 3.3 Exercises

1. A wheel with radius  $R$  rolls along the  $x$ -axis without friction. The velocity of the centroid of the wheel is  $v_0$ . A point  $P$  at the wheel rim is located at the origin when  $t = 0$ . The trajectory of  $P$  is the *cycloid*

$$\mathcal{C}(t) = (v_0 t - R \sin \varphi(t), R - R \cos \varphi(t))$$

where  $R\varphi(t) = v_0 t$ . Plot the cycloid (specify some values for  $R$  and  $v_0$ ) determine the speed  $v(t) = |\mathcal{C}'(t)|$  of  $P$ . What are the maximum and minimum values of  $v(t)$  and at what points are they attained?

2. A point moves in the plane such that the position at time  $t$  is  $P(t) = (x(t), y(t))$ , where  $x(t) = \sin f(t)$ ,  $y(t) = 1 - \cos f(t)$ , and where further

$$f(t) = \begin{cases} t^2(3-t)^2 & t \in [0, 3] \\ 0 & \text{otherwise} \end{cases}$$

When is the speed of  $P$  at maximum? Plot the trajectory of  $P$ .

3. Suppose  $\mathcal{C}(t)$  is a regular curve with unit normal  $\boldsymbol{\nu}(t)$ . The curve  $\mathcal{O}_d(t) = \mathcal{C}(t) + d\boldsymbol{\nu}(t)$  is called the *offset* of  $\mathcal{C}(t)$  at a distance  $d$ . Determine the offset of the curve

$$\mathcal{C}(t) = (1 - 3t + 3t^2, 3t^2 - 2t^3)$$

at a distance  $d$ . Plot the curve and its offset at a distance  $d = 1$ .

4. Write a routine that determines an approximation of the arclength of a curve using the formula (4).