1 Geometric Transformations

1.1 Introduction

We shall denote *points*, that is, elements of the euclidean plane $\mathbb{E}^2$, by regular upper-case letters. Given a length unit, and two orthogonal lines of reference called the x-axis and the y-axis, each point $P \in \mathbb{E}^2$ can be represented by an ordered pair of real numbers (x, y) measuring the perpendicular distance of P from the y- and x-axis, respectively. The axes intersect at the *origin* O, with coordinates $(0, 0)$. For computational purposes it is often advantageous to identify the point P with a column vector $\mathbf{x}$ such that

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \mathrel{\widehat{=}} \overrightarrow{OP} = x\vec{i} + y\vec{j},$$

where $\vec{i}$ and $\vec{j}$ are the unit vectors in the x- and y-coordinate directions, respectively.

If $P_0 = (x_0, y_0) \mathrel{\widehat{=}} \mathbf{x}_0$ is a point in the euclidean plane and $\mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ is a non-zero vector, then the set

$$L = \{P \in E^2 \mid P \mathrel{\widehat{=}} \mathbf{x} = \mathbf{x}_0 + t\mathbf{v},\ t \in \mathbb{R}\}$$

is a line through a point P_0 with a *direction vector* $\mathbf{v}$. The equation

$$(x(t), y(t)) = (x_0 + tv_1, y_0 + tv_2)$$

is referred to as the *parametric form* of the line L, or the *parametrization* of L. Elimination of the parameter t yields a relationship between x and y:

$$ax + by + c = 0, \tag{1}$$

where $a = v_2$, $b = -v_1$ and $c = -x_0v_2 + y_0v_1$. This is known as the *implicit form* of the line. Notice that Equation (1) can be written in terms of the normal vector $\mathbf{n} = (a, b)$ of L as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{x}_0) = 0 \quad \Leftrightarrow \quad \mathbf{x} - \mathbf{x}_0 \perp \mathbf{n}$$

1.2 Geometric Mappings

Definition 1. An *affine transformation* of the plane is a mapping $\mathsf{L} : \mathbb{E}^2 \to \mathbb{E}^2$ of the form

$$\mathsf{L}(x, y) = (ax + by + c, dx + ey + f) \tag{2}$$

with some constants $a, b, c, d, e, f \in \mathbb{R}$. The point $P' = \mathsf{L}(P)$ is the image of P. If $\mathbb{S} \subset \mathbb{E}^2$, then the set of all points $\mathsf{L}(P)$, where $P \in \mathbb{S}$, is called the image of $\mathbb{S}$ and denoted $L(\mathbb{S})$.

The transformation (2) can be written using the matrix notation as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{A} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} c \\ f \end{pmatrix}$$

Example 1.1. Let $\mathsf{L}(x, y) = (x + 3y + 1, 2x + y - 1)$. The images of the points $(3, 1)$ and $(2, 1)$ are $\mathsf{L}(3, 1) = (7, 6)$ and $\mathsf{L}(2, 1) = (6, 4)$.

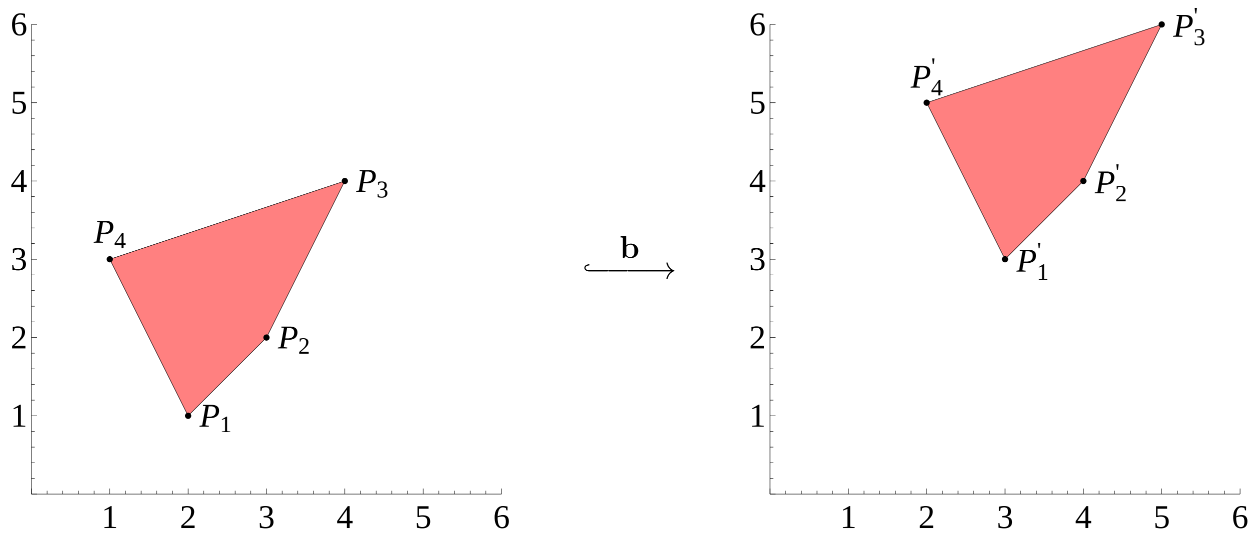

Figure 1: Translation of a quadrilateral.

Translations

A *translation* is transformation obtained by adding a constant amount to each coordinate so that $P' = (x + h_x, y + h_y)$. In the matrix language, a translation is given by $\mathbf{A} = \mathbf{I}$ and a *translation vector* $\mathbf{b} = \begin{pmatrix} h_x & h_y \end{pmatrix}^T$.

Example 1.2. Let

$$P_1 = (2, 1), \quad P_2 = (3, 2), \quad P_3 = (4, 4), \quad P_4 = (1, 3)$$

represent the four vertices a quadrilateral. Application of the translation $\mathbf{b} = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ yields images of the vertices

$$P_1' = (3, 3), \quad P_2' = (4, 4), \quad P_3' = (5, 6), \quad P_4' = (2, 5)$$

The translated quadrilateral is shown in Figure 1.

Scaling

A scaling about the origin is determined by scaling factors $\lambda_x, \lambda_y \in \mathbb{R}$ such that $P' = (\lambda_x x, \lambda_y y)$. A scaling can be represented using the scaling transformation matrix

$$\mathbf{\Lambda}(\lambda_x, \lambda_y) = \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{pmatrix}$$

Example 1.3. Application of the scaling transformation $\mathbf{\Lambda}(\frac{3}{2}, \frac{3}{4})$ to the quadrilateral of Example 1.2 can be realised by representing the coordinates of the vertices as the 2×4 matrix

$$\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

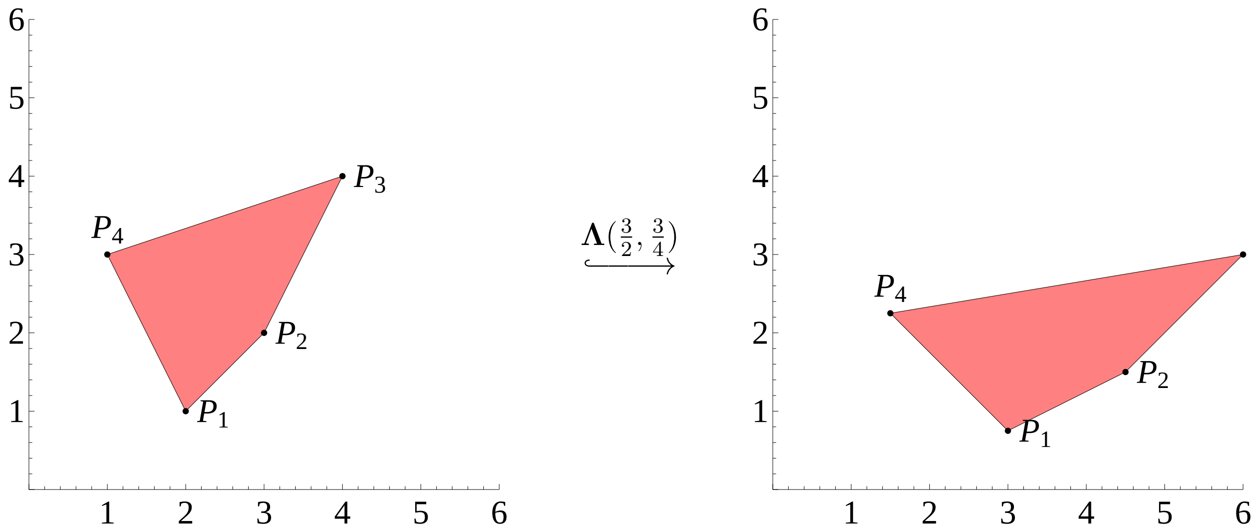

Figure 2: Effect of scaling on a quadrilateral.

and multiplying by the scaling transformation matrix

$$\begin{pmatrix} \mathbf{x}'_1 & \mathbf{x}'_2 & \mathbf{x}'_3 & \mathbf{x}'_4 \end{pmatrix} = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.75 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 3. & 4.5 & 6. & 1.5 \\ 0.75 & 1.5 & 3. & 2.25 \end{pmatrix}$$

The effect of the scaling to the quadrilateral is visualized in Figure 2.

Reflections

The horizontal and vertical flip or mirror effect used in computer graphics packages can be executed by applying a transformation called a *reflection*. It is easy to see that the reflections in the x- and y-axis are the transformations $\mathsf{L}(x,y) = (x,-y)$ and $\mathsf{L}(x,y) = (-x,y)$, respectively. These can be obtained by multiplying the coordinate vector $\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^T$ by the reflection matrices

$$\mathbf{R}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{R}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 1.4. Applying the reflection $\mathbf{R}_y$ to the quadrilateral of Example 1.2, gives the points

$$\begin{pmatrix} \mathbf{x}'_1 & \mathbf{x}'_2 & \mathbf{x}'_3 & \mathbf{x}'_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -4 & -1 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

The effect of the mirroring is shown in Figure 3.

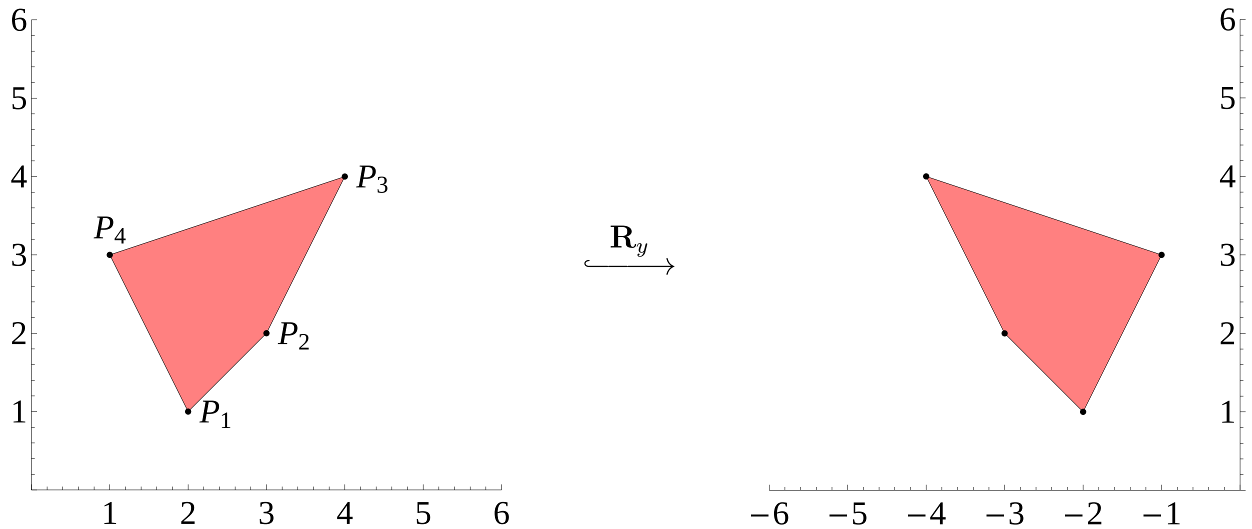

Figure 3: Reflection of a quadrilateral in the y-axis.

Rotations

A rotation about the origin through and angle φ maps the point P to a point Q such that the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OP}$ equals φ. If P makes an angle θ with the x-axis and is a distance r from the origin, then $P = (x, y) = (r\cos\theta, r\sin\theta)$. Similarly we have that $Q = (x', y') = (r\cos(\theta+\varphi), r\sin(\theta+\varphi))$. Trigonometric identities

$$\begin{aligned}\cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi \\ \sin(\theta+\varphi) &= \sin\theta\cos\varphi + \cos\theta\sin\varphi\end{aligned}$$

yield

$$\begin{aligned}x' &= x\cos\varphi - y\sin\varphi \\ y' &= x\sin\varphi + y\cos\varphi\end{aligned}$$

so that the transformation can be executed by multiplication with the *rotation matrix*

$$\mathbf{A}_\varphi = \begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}$$

Example 1.5. Applying the rotation $\mathbf{A}_{\pi/2}$ to the quadrilateral of Example 1.2, gives the points

$$\begin{pmatrix}\mathbf{x}'_1 & \mathbf{x}'_2 & \mathbf{x}'_3 & \mathbf{x}'_4\end{pmatrix} = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3\end{pmatrix} = \begin{pmatrix}-1 & -2 & -4 & -3 \\ 2 & 3 & 4 & 1\end{pmatrix}.$$

The effect of the rotation is shown in Figure 4.

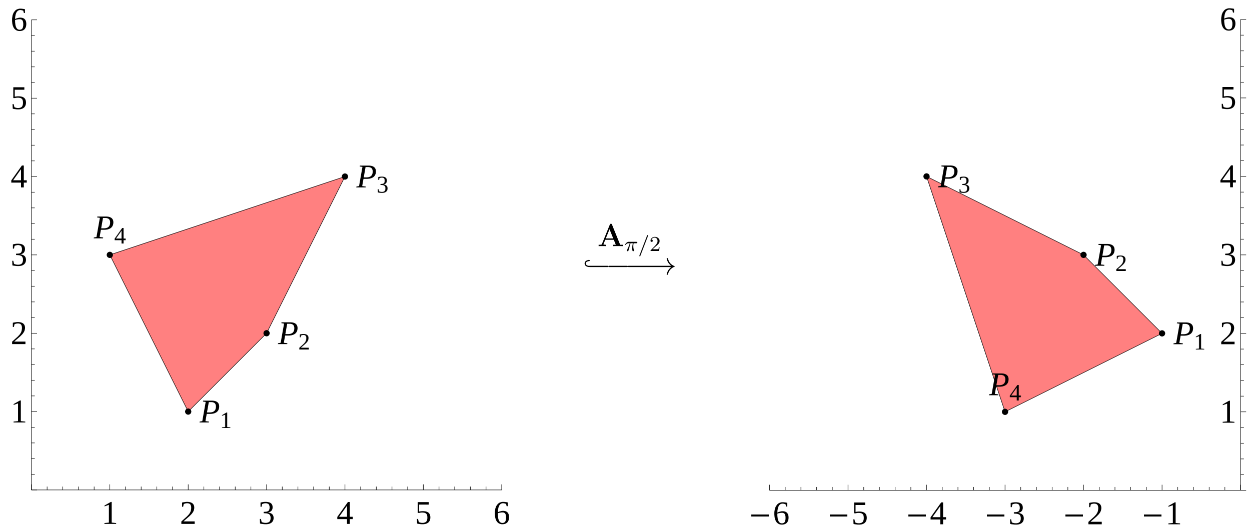

Figure 4: Rotation of a quadrilateral about the origin through $\pi/2$ degrees.

Inverse Transformation

Definition 2. The transformation which leaves all points of the plane unchanged is called the *identity transformation* I. The *inverse transformation* of L, denoted by L^{-1}, is the transformation such that $\mathsf{L}^{-1}(\mathsf{L}(P)) = P$ and $\mathsf{L}(\mathsf{L}^{-1}(P)) = P$.

Example 1.6. A translation T determined by the vector $\mathbf{b} = \begin{pmatrix} h_x & h_y \end{pmatrix}^T$ maps a point $P = (x, y)$ to $P' = (x + h_x, y + h_y)$. The transformation T^{-1} required to map P' back to P corresponds to the vector $\mathbf{b} = \begin{pmatrix} -h_x & -h_y \end{pmatrix}^T$.

The process of following one transformation by another one is called *concatenation* or *composition*. Concatenation of translations with other types of transformations require combination of vector addition with matrix multiplication. However, if homogeneous coordinates are introduced, all transformations can be represented by matrices so that concatenation and inversion of transformations can be performed by matrix multiplication and inversion. This technique is the subject of the next section.

1.3 Exercises

1. Consider the affine transformation
$$\mathbf{f}(x, y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$
How does the transformation of a quadrilateral with vertices $(1, 1)$, $(3, 1)$, $(3, 2)$ and $(1, 2)$ look like?

2. Apply the translation corresponding to $\mathbf{b} = \begin{pmatrix} 1 & -3 \end{pmatrix}^T$ to the quadrilateral of the previous exercise.

3. Determine the inverse transformation of the translation in the previous exercise. Verify that it returns the quadrilateral to its original position.

4. Apply the reflection $\mathbf{R}_x$ to the quadrilateral of exercise 1.

5. Show that the inverse of $\mathbf{R}_x$ is $\mathbf{R}_x$ and that the inverse of $\mathbf{R}_y$ is $\mathbf{R}_y$.

6. Verify that for the rotation matrix $\mathbf{A}_\varphi$, it holds that $\mathbf{A}_\varphi^{-1} = \mathbf{A}_{-\varphi}$.