2 Homogeneous Coordinates

The problem of concatenation of a translation with other types of transformations can be avoided by using an alternative coordinate system for which computations are performed by 3×3 matrix multiplications. Indeed, we may write

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{pmatrix}$$

and define a new coordinate system in which the point P with Cartesian coordinates (x, y) is represented by the *homogeneous* or *projective* coordinates $(x, y, 1)$, or any multiple $(\lambda x, \lambda y, \lambda)$, $\lambda \neq 0$. For computations column vectors can be used:

$$P \hat{=} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \hat{=} \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda \end{pmatrix}$$

The set of all homogeneous coordinates (x, y, w) is called the *projective plane* and denoted by $\mathbb{P}^2$.

Example 2.1. $(4, 8, 4)$, $(2, 4, 2)$, and $(-1, -2, -1)$ are all homogeneous coordinates of the point $(1, 2)$ because

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = -1 \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

Example 2.2. The Cartesian coordinates of the point with homogeneous coordinates $(-2, 3, 4)$ are obtained by dividing the coordinates by $w = 4$ to give alternative homogeneous coordinates $(-1/2, 3/4, 1)$. The Cartesian coordinates are therefore $(x, y) = (-1/2, 3/4)$.

Definition 3. A projective transformation of the projective plane is a mapping $\mathsf{L} : \mathbb{P}^2 \to \mathbb{P}^2$ of the form

$$\mathsf{L}(x, y, w) = (ax + by + cw, dx + ey + fw, gx + hy + kw)$$

for some constants $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. A matrix

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

which represents a linear transformation of the projective plane is called a *homogeneous transformation matrix*. When $g = h = 0$ and $k \neq 0$, L corresponds to an affine transformation of the Cartesian plane.

2.1 Transformations in homogeneous coordinates

We will now present the homogeneous transformation matrices for translations, scalings and rotations.

Translations

The homogenous translation matrix $\mathbf{T}(h_x, h_y)$ is

$$\mathbf{T}(h_x, h_y) = \begin{pmatrix} 1 & 0 & h_x \\ 0 & 1 & h_y \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed,

$$\begin{pmatrix} 1 & 0 & h_x \\ 0 & 1 & h_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + h_x \\ y + h_y \\ 1 \end{pmatrix}$$

so that the point (x, y) is translated to $(x + h_x, y + h_y)$.

Example 2.3. In Example 1.2, the translation with $h_x = 1$ and $h_y = 2$ was applied to the quadrilateral with vertices

$$P_1 = (2, 1), \quad P_2 = (3, 2), \quad P_3 = (4, 4), \quad P_4 = (1, 3).$$

represent the four vertices a quadrilateral. The translation can be applied by writing the homogeneous coordinates of the vertices as a 3×4 matrix and multiplying that matrix by the homogeneous translation matrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 & 2 \\ 3 & 4 & 6 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The images have Cartesian coordinates

$$P_1' = (3, 3), \quad P_2' = (4, 4), \quad P_3' = (5, 6), \quad P_4' = (2, 5).$$

Scaling about the Origin

The homogeneous scaling matrix $\mathbf{\Lambda}(\lambda_x, \lambda_y)$ is

$$\mathbf{\Lambda}(\lambda_x, \lambda_y) = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the point $(x, y, 1)$ is mapped to $(\lambda_x x, \lambda_y y, 1)$:

$$\begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_x x \\ \lambda_y y \\ 1. \end{pmatrix}$$

Example 2.4. A scaling about the origin by a factor 2 in the x-direction, and by a factor of 3 in the y-direction, of the triangle with vertices $(1,1)$, $(2,1)$, and $(1,2)$ is determined by

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 \\ 3 & 3 & 6 \\ 1 & 1 & 1 \end{pmatrix}$$

The image is a triangle with vertices $(2,3)$, $(4,3)$, and $(2,6)$.

Rotation about the Origin

The rotation matrix for homogenous coordinates reads

$$\mathbf{A}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where a positive φ corresponds to an anticlockwise rotation.

Example 2.5. A clockwise rotation about the origin through an angle $\pi/4$ of the triangle with vertices $(1,1)$, $(2,1)$, and $(1,2)$ is determined by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \sqrt{2} & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 1 & 1 & 1 \end{pmatrix}$$

The image is a triangle with vertices $(0, 1.414)$, $(0.707, 2.121)$, and $(-0.707, 2.121)$.

2.2 Concatenation of Transformations

In homogeneous coordinates, the concatenation of transformations L_1 and L_2, denoted by $\mathsf{L}_1 \circ \mathsf{L}_2$ reduces to matrix multiplication. For instance, the concatenation of a rotation $\mathbf{A}_\varphi$ about the origin and a translation $\mathbf{T}(h_x, h_y)$ has the homogeneous transformation matrix

$$\mathbf{T}(h_x, h_y)\mathbf{A}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & h_x \\ 0 & 1 & h_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & h_x \\ \sin\varphi & \cos\varphi & h_y \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2.6. An anticlockwise rotation of $\pi/2$ about the origin followed by scaling $\lambda_x = 2$, $\lambda_y = 3$ has the transformation matrix

$$\mathbf{\Lambda}(2,3)\mathbf{A}_{\pi/2} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 4. The identity transformation of the plane I satisfies $\mathsf{I} \circ \mathsf{L} = \mathsf{L} \circ \mathsf{I} = \mathsf{L}$ for all planar transformations L. The transformation matrix of the identity transformation is the identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 5. The inverse transformation of a transformation L is denoted by L^{-1} and is such that $\mathsf{L} \circ \mathsf{L}^{-1} = \mathsf{I} = \mathsf{L}^{-1} \circ \mathsf{L}$.

The definition of the matrix inverse implies the following

Lemma 1. If $\mathbf{A}$ is the homogeneous transformation matrix of L, then L^{-1} exists if and only if $\mathbf{A}^{-1}$ exists. Moreover, $\mathbf{A}^{-1}$ is the transformation matrix of L^{-1}.

An invertible transformation $\mathsf{L} : \mathbb{P}^2 \to \mathbb{P}^2$ is called a *non-singular* transformation. According to Lemma 1, a transformation is non-singular if and only if its transformation matrix is non-singular. Matrix algebra can be used to determine the inverse transformation matrix of concatenated transformations.

Example 2.7. If $\mathbf{A}$ and $\mathbf{B}$ are non-singular, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. Recall that for the scaling $\mathbf{\Lambda}(\lambda_x, \lambda_y)$ and the rotation $\mathbf{A}_\varphi$ we have that $\mathbf{\Lambda}(\lambda_x, \lambda_y)^{-1} = \mathbf{\Lambda}(1/\lambda_x, 1/\lambda_y)$ and $\mathbf{A}_\varphi^{-1} = \mathbf{A}_{-\varphi}$. Application to the transformation $\mathbf{\Lambda}(2,3)\mathbf{A}_{\pi/2}$ of Example 2.6 yields

$$\begin{aligned} (\mathbf{\Lambda}(2,3)\mathbf{A}_{\pi/2})^{-1} &= \mathbf{A}_{-\pi/2}\mathbf{\Lambda}(1/2, 1/3) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

This is in agreement with the inverse of the matrix of Example 3.6:

$$\begin{pmatrix} 0 & -2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Rotation about an Arbitrary Point

An anticlockwise rotation through an angle φ about a point (x_0, y_0) is obtained by

1. mapping (x_0, y_0) to the origin
2. rotating through an angle φ about the origin

3. mapping the origin to (x_0, y_0)

The associted homogeneous rotation matrix is

$$\begin{aligned}
\mathbf{A}_\varphi(x_0, y_0) &= \mathbf{T}(x_0, y_0)\mathbf{A}_\varphi\mathbf{T}(-x_0, -y_0) \\
&= \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & \sin(\varphi)y_0 - (\cos(\varphi) - 1)x_0 \\ \sin(\varphi) & \cos(\varphi) & -\sin(\varphi)x_0 - (\cos(\varphi) - 1)y_0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Example 2.8. A triangle with vertices $P_1 = (1, 1)$, $P_2 = (2, 1)$, and $P_3 = (1, 2)$ is rotated clockwise about $(2, 1)$ through an angle $\pi/4$. The corresponding transformation matrix is

$$\begin{aligned}
\mathbf{A}_{\pi/4}(2, 1) &= \mathbf{T}(2, 1)\mathbf{A}_{-\pi/4}\mathbf{T}(-2, -1) \\
&= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 - \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 + \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Application to the vertices yields

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 - \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 + \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 - \frac{1}{\sqrt{2}} & 2 & 2 \\ 1 + \frac{1}{\sqrt{2}} & 1 & 1 + \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$$

The effect of the rotation is shown in Figure 5.

Reflection in an Arbitrary Line

A reflection in an arbitrary line $L : ax + by + c = 0$ can be determined by transforming the line to one of the axes, reflecting in that axis, and then inverting the first transformation. Let us compute the transformation assuming that $b \neq 0$ so that the line intersects the y-axis at the point $(0, -c/b)$, and has the slope $-a/b = \tan\varphi$. Application of a translation $h_x = 0, h_y = -c/b$ to the origin followed up by a rotation through an angle $-\varphi$ maps the line to the x-axis so that reflection in the x-axis can be applied. The transformation matrix

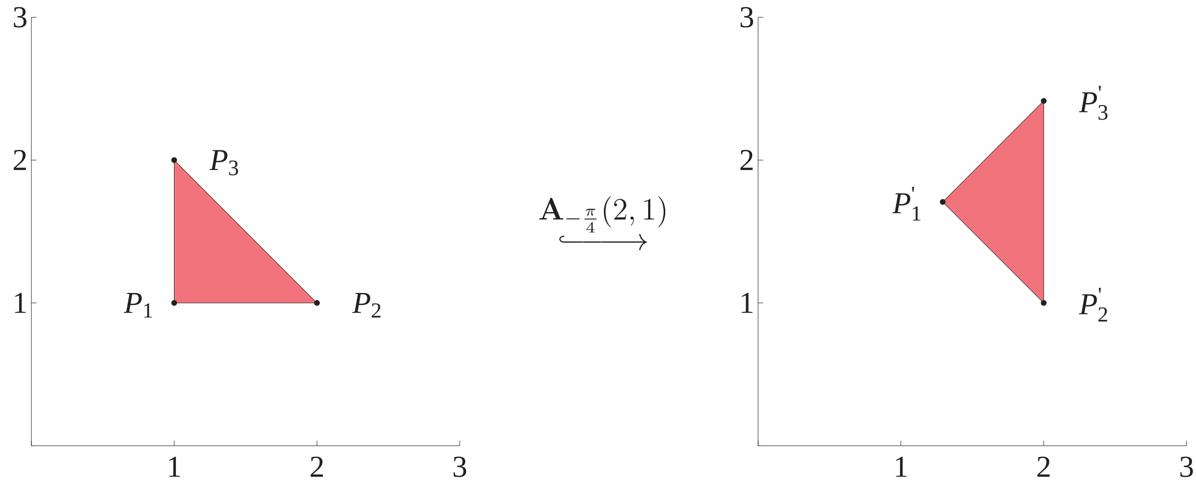

Figure 5: Rotation of a triangle about its vertex P_2 through an angle $\pi/4$.

becomes

$$[\mathbf{A}_{-\varphi}\mathbf{T}(0,-c/b)]^{-1}\,\mathbf{R}_x\mathbf{A}_{-\varphi}\mathbf{T}(0,-c/b)$$

$$= \mathbf{T}(0,c/b)\mathbf{A}_{\varphi}\mathbf{R}_x\mathbf{A}_{-\varphi}\mathbf{T}(0,-c/b)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{c}{b} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\dfrac{c}{b} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2(\varphi)-\sin^2(\varphi) & 2\cos(\varphi)\sin(\varphi) & \frac{2c\cos(\varphi)\sin(\varphi)}{b} \\ 2\cos(\varphi)\sin(\varphi) & \sin^2(\varphi)-\cos^2(\varphi) & \frac{c\left(\sin^2(\varphi)-\cos^2(\varphi)\right)}{b} - \frac{c}{b} \\ 0 & 0 & 1 \end{pmatrix}$$

If $\tan\varphi = -a/b$, then

$$\sin\varphi = \pm\frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\varphi = \mp\frac{b}{\sqrt{a^2+b^2}}$$

so that the transformation matrix may be written as

$$\begin{pmatrix} \dfrac{b^2-a^2}{a^2+b^2} & -\dfrac{2ab}{a^2+b^2} & -\dfrac{2ac}{a^2+b^2} \\ -\dfrac{2ab}{a^2+b^2} & \dfrac{a^2-b^2}{a^2+b^2} & -\dfrac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}$$

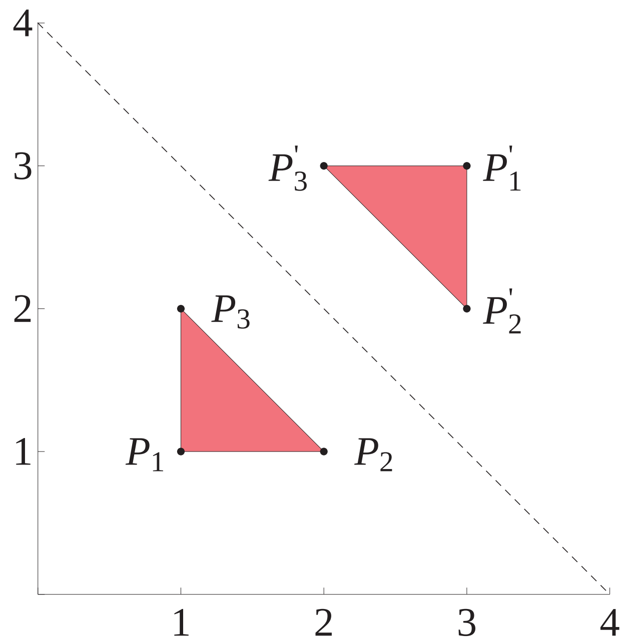

Figure 6: Reflection of a triangle about in the line $x + y = 4$.

Since the homogeneous coordinates are used, the matrix can be rescaled to a simpler form

$$\mathbf{R}(a, b, c) = \begin{pmatrix} b^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 & -2bc \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \tag{3}$$

Example 2.9. The homogeneous transformation matrix that reflects for reflection in the line $x + y - 4 = 0$ reads

$$\mathbf{R}(1, 1, -4) = \begin{pmatrix} 0 & -2 & 8 \\ -2 & 0 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$

The vertices of the triangle in Example 2.8 become mapped as

$$\begin{pmatrix} 0 & -2 & 8 \\ -2 & 0 & 8 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 4 \\ 6 & 4 & 6 \\ 2 & 2 & 2 \end{pmatrix}$$

The Cartesian coordinates of the vertices of the reflected triangle are $P_1' = (3, 3)$, $P_2' = (3, 2)$, and $P_3' = (2, 3)$. The situation is visualized in Figure 6.

2.3 Exercises

1. Write down three sets of homogeneous coordinates of $(-3, 1)$.

2. A point has Cartesian coordinates $(2, -4)$ and homogeneous coordinates $(-2, \eta, -1)$ and $(4, -8, w)$. What are the values of η and w?

3. A square has vertices $P_1 = (1, 1)$, $P_2 = (2, 1)$, $P_3 = (2, 2)$, and $P_4 = (1, 2)$. Rotate the square about P_2 through an angle $\pi/4$.

4. [Instancing] Construct a model of the front of a house with a door and few windows by applying suitable scaling and translation transformations of the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. Finalize the design by inserting a door knob using the point primitive.

5. Determine the transformation matrix of a reflection in the line $3x - 2y + 2$. Express first using the formula (3) and then as concatenation of suitable transformations used in the passage to (3). Visualize the reflection of a triangle of Example 2.8.

6. [Orthogonal Change of Coordinates] Consider a Cartesian coordinate system with origin O and coordinates x, y. Another coordinate system with origin $O' = (x_0, y_0)$ and coordinates x', y' is obtained by applying a rotation through an angle φ followed by the translation $\mathbf{T}(x_0, y_0)$ to the origin and axes of the first system:

$$
\begin{aligned}
x &= x' \cos\varphi - y' \sin\varphi + x_0 \\
y &= y' \sin\varphi + y' \cos\varphi + y_0
\end{aligned}
$$

   a) Determine the homogeneous transformation matrix $\mathbf{Q}$ of the change of coordinates and show that $\det \mathbf{Q} = 1$.

   b) Determine the inverse change of coordinate transformation.

   c) Show that the change of coordinates preserves the angle between pair of lines.

   d) Show that the equations of the x'- and y'-axes in the xy-system are

$$
\begin{aligned}
(x - x_0)\sin\varphi - (y - y_0)\cos\varphi = 0 \\
(x - x_0)\cos\varphi + (y - y_0)\sin\varphi = 0
\end{aligned}
$$