Mat-1.3604 Stationary Processes.

Exercise 18.10. 2007 Tikanmäki/Valkeila.
We use the notation from lecture notes: η is a centered orthogonal random measure, ν is the control measure of η, η is defined from $(S, \mathcal{S})$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Usually $S = \mathbb{R}$, when time is continuous, or $S = Z$, when time is discrete.

1. Let ϵ_k, $k \in Z$, be real valued white noise: $\mathbb{E}\epsilon_k = 0$ and $\mathbb{E}\epsilon_k^2 = 1$, and $\alpha \in \mathbb{R}$ with $|\alpha| < 1$. Define process X by

$$X_k = \alpha X_{k-1} + \epsilon_k;$$

   this is so-called autoregressive process of order one, $AR(1)$. What is the moving average representation of X? Since the process X has a moving average representation, it is stationary. Find $C_X(k) = \mathbb{E}\left(X_{k+l}X_l\right)$[1].
2. [Continuation] Find the best linear prediction of X_{n+k} based on the white noise $\epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_{n-m}$, where $m > 0$.
3. Let X_k, $k \in Z$ be a stationary process with mean m. Show that

$$\frac{1}{n}\sum_{k=0}^{n-1} C(k) \to 0 \Leftrightarrow \frac{1}{n}\sum_{k=0}^{n-1} X_k \to m \text{ in } L^2(\mathbb{P});$$

   here $C(k)$ is defined as $C(k) = \mathbb{E}\left[(X_k - m)(X_0 - m)\right]$.
4. Let X be a discrete time real valued stationary process with $\mathbb{E}X_k = 0$ and $X_k \in L^4(\mathbb{P})$. Let $\hat{C}_N(m)$ be the sample (or moment) estimator of the covariance kernel $C(m)$, based on observations $X_0, \ldots, X_N$, $(m < N)$:

$$\hat{C}_N(m) = \frac{1}{N-m}\sum_{k=0}^{N-m-1} X_{m+k}\bar{X}_k.$$

   Show that the condition, as $N \to \infty$,

$$\frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}\left(\left(X_{n+k}\bar{X}_k - C(n)\right)\left(X_n X_0 - C(n)\right)\right) \to 0$$

   is a necessary and sufficient condition for

$$\mathbb{E}|\hat{C}_N(m) - C(m)|^2 \to 0.$$

5. Compute the impulse response of the frequency response $g(\lambda) = 1_{[a,b]}(\lambda)$.

[1]We have changed the convention here. Please see handouts for the week 5 for the explanation

6. Let $h = h(\lambda, s)$ be a measurable function, $h : \mathbb{R} \times [a, b] \to C$ with $-\infty < a < b < \infty$, and measurability is with respect to the product sigma-algebra on $\mathbb{R} \times [a, b]$. Assume that

$$\int_a^b \int_{\mathbb{R}} |h(\lambda, s)|^2 \nu(d\lambda) ds < \infty.$$

Show that then we have the following Fubini type of theorem:

$$\int_a^b \left(\int_{\mathbb{R}} h(\lambda, s) \eta(d\lambda) \right) ds = \int_{\mathbb{R}} \left(\int_a^b h(\lambda, s) ds \right) \eta(d\lambda).$$