Mat-1.3604 Stationary Processes.

Exercise 11.10. 2007 Tikanmäki/Valkeila.
We use the notation from lecture notes: η is a centered orthogonal random measure, ν is the control measure of η, η is defined from $(S, \mathcal{S})$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

1. Prove the following general fact: Let $f(t)$, $t \in [a, b]$ take values in a Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$. Show that f is continuous at point t_0 if and only if there exists a limit
$$\lim_{s,t \to t_0} \langle f(s), f(t) \rangle.$$
[You can use this fact in the following.]
2. Let X_t, $t \in \mathbb{R}$ be a continuous stationary process. Show that it is k times differentiable in $L^2(\mathbb{P})$ at every point t if and only if
$$\int_{\infty}^{\infty} \lambda^{2k} \nu(d\lambda) < \infty.$$
3. [Continuation] Denote by $X^{(k)}$ the k^{th} derivative of the stationary process X. Find its spectral representation and show that $X^{(k)}$ is stationary.
4. Let N_t, $t \geq 0$ be the standard Poisson process: the increments are independent and $N_{t+s} - N_t$ has a Poisson distribution with parameter s. Show that N is stochastically differentiable at every point $t > 0$ with derivative $= 0$, but N is not differentiable in $L^2(\mathbb{P})$.
5. Let $X = (X_k)_{k \in Z}$ be a discrete time stationary process with covariance kernel $C(k)_{k \in Z}$. Show that
$$\frac{1}{N} \sum_{k=0}^{N-1} C(k) \to \nu(\{0\}).$$
6. Assume that X is L^2 continuous centered stationary process. Find the spectral representation of process $\int_0^T X_s ds$. Show that
$$\frac{1}{T} \int_0^T X_t dt \to \eta(\{0\})$$
in $L^2(\mathbb{P})$.