Mat-1.3604 Stationary Processes.

Exercise 4.10. 2007 Tikanmäki/Valkeila.

1. Let C be a continuous function. Prove that

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{-\frac{t^2}{2\epsilon^2}}}{\epsilon\sqrt{2\pi}}C(t)dt = C(0).$$

2. Find the spectral density of the wide sense stationary process with covariance kernel $C(t) = e^{-|t|}\cos(\alpha t)$.
3. Let η be an orthogonal random measure on $(S, \mathcal{S}_0)$ with control measure ν. Show that for $A, B \in \mathcal{S}_0$
   (i) $(\eta(A), \eta(B))_{L^2(\mathbb{P})} = \nu(A \cap B)$.
   (ii) $||\eta(A) - \eta(B)||^2_{L^2(\mathbb{P})} = \nu(A \triangle B)$.
   (iii) $\eta(A \cup B) = \eta(A) + \eta(B) - \eta(A \cap B)$.
4. [Continuation] If $A \subset B$, then
   (i) $||\eta(A)||_{L^2(\mathbb{P})} \le ||\eta(B)||_{L^2(\mathbb{P})}$.
   (ii) $\eta(B \setminus A) = \eta(B) - \eta(A)$.
5. Let η be an orthogonal random measure on $(S, \mathcal{S}_0)$ with conrol measure ν, and h is an elementary function on $(S, \mathcal{S}_0)$ with

$$h = \sum_{k=1}^{n} c_k 1_{A_k} = \sum_{j=1}^{m} d_j 1_{B_j},$$

   where B_j and A_k are disjoint, $c_k, d_j \in \mathbb{C}$. Show that $I(h)$ is well defined, where

$$I(h) := \sum_{k=1}^{n} c_k \eta(A_k);$$

   i.e. show that $I(h)$ is the same random variable in $L^2(\mathbb{P})$ as $\sum_{j=1}^{d} d_j \eta(B_j)$.
6. Show that the stochastic integral I is linear: for two elementary functions h, g and complex numbers α, β

$$I(\alpha h + \beta g) = \alpha I(h) + \beta I(g).$$