Mat-1.3604 Stationary Processes.

Exercise 27.9. 2007 *Tikanmäki/Valkeila.*

1. Show that the function

$$R(s,t) = e^{-|t-s|} \tag{1}$$

is non-negative definite, $s, t \in \mathbb{R}$.

2. Let X be a weakly stationary process with mean $EX_t = m$ and covariance (1). Show that
   - ★ $\mathbb{E}\left(\frac{1}{T}\int_0^T X_s ds\right) = m$.
   - ★ $L^2(\mathbb{P}) - \lim_{T\to\infty} \frac{1}{T}\int_0^T X_s ds = m$.

3. If $X_t, t \in [a,b]$ is L^2 continuous, then

$$\frac{d}{dt}\int_a^t X_s ds = X_t,$$

where everything is understood in L^2: integral and differential.

4. Let X be L^2 differentiable. Show that

$$\mathbb{E}\left(\dot{X}_s\ X_t\right) = \frac{d}{ds}C(s,t).$$

5. Let X be L^2 differentiabe on $\mathbb{R}$ with continuous derivative in $L^2(\mathbb{P})$. Show that

$$X_t - X_s = \int_s^t \dot{X}_u\ du$$

in $L^2(\mathbb{P})$.

6. Find the Karhunen-Loéve expansion on the interval $[0,1]$ for a process with covariance function $C(s,t) = st$.