Chapter 7

Subdivision interpolation

Suppose that the values of some function is known for integer arguments. If one wants to estimate the values of the function at other points, then one of the simplest methods is to use linear interpolation. One way of expressing this procedure is to say that one calculates the values of the function at the half-integer points $\frac{1}{2}m$ with m odd by the formula $f(2^{-1}m) = \frac{1}{2}\big(f(\frac{1}{2}(m+1)) + f(\frac{1}{2}(m-1))\big)$. Then one calculates the values at the points $\frac{1}{4}m$ (with m odd) by the formula $f(2^{-2}m) = \frac{1}{2}\big(f(2^{-1}\frac{1}{2}(m+1)) + f(2^{-1}\frac{1}{2}(m-1))\big)$, and proceeds in this manner. In this chapter some generalizations of this procedure will be considered.

1. Subdivision interpolation

In a subdivision cardinal interpolation scheme one calculates new values for the function f by the formula

$$f(2^{-j-1}m) = 2\sum_{k\in\mathbb{Z}} \gamma(m-2k) f(2^{-j}k), \quad m \in \mathbb{Z}, \quad j \geq 0, \tag{7.1}$$

where the sequence $(\gamma(k))_{k\in\mathbb{Z}}$ is the mask that determines interpolation procedure. (The normalizing factor 2 is for convenience introduced here so that it does not appear in the equations one obtains after taking Fourier transforms.) It is clear that if the restriction $F = f_{|\mathbb{Z}}$ of f to the integers is known, then one finds from (7.1) the values of f at the half-integer values $\mathbb{Z} + \frac{1}{2}$ (by taking $j = 0$), then the values at $\mathbb{Z} + \frac{1}{4}$ and $\mathbb{Z} + \frac{3}{4}$ (by taking $j = 1$), and so on. Another way of formulating these calculations is to say that

$$f_{|2^{-j-1}\mathbb{Z}} = S_\gamma f_{|2^{-j}\mathbb{Z}},$$

where S_γ is the operator defined in Definition 6.2.

If the values of f are not given on $\mathbb{Z}$ but on some other set of evenly spaced points, one can use a simple transformation of the argument to reduce the problem to the one considered here.

If $m = 2p$ in (7.1) is even, then one has

$$f(2^{-j}p) = 2\gamma(0)f(2^{-j}p) + 2\sum_{\substack{k\in\mathbb{Z}\\ k\neq p}} \gamma\big(2(p-k)\big)f(2^{-j}k).$$

Since we are studying an interpolation and not a refinement scheme (that is, we do not want to change values of f already calculated) we have to require that

$$\gamma(2k) = \frac{1}{2}\delta_{0,k}, \quad k \in \mathbb{Z}, \tag{7.2}$$

(where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise).

Proposition 7.1. *Suppose that $\gamma \in \ell^1(\mathbb{Z})$and that the function $\Phi \in \mathcal{C}(\mathbb{R})$ satisfies the interpolation condition*

$$\Phi(k) = \delta_{0,k}, \quad k \in \mathbb{Z}. \tag{7.3}$$

Then

$$\Phi(\underline{x}) = 2\sum_{k\in\mathbb{Z}} \gamma(k)\Phi(2\underline{x} - k), \tag{7.4}$$

if and only if $\Phi_{|2^{-j-1}\mathbb{Z}} = S_\gamma \Phi_{|2^{-j}\mathbb{Z}}$ for all $j \geq 0$, that is

$$\Phi(2^{-j-1}m) = 2\sum_{k\in\mathbb{Z}} \gamma(m-2k)\Phi(2^{-j}m), \quad m \in \mathbb{Z}, \tag{7.5}$$

and Φ is thus the fundamental interpolation function for the scheme determined by γ.

Proof. Since Φ is continuous, equation (7.4) is equivalent to the conditiona that for all $j \geq 0$ we have

$$\Phi(2^{-j-1}m) = 2\sum_{k\in\mathbb{Z}} \gamma(k)\Phi(2^{-j}m - k), \quad m \in \mathbb{Z}, \tag{7.6}$$

Next we observe that when $j = 0$ it follows from (7.3) that both (7.5) and (7.6) say that $\Phi(2^{-1}m) = \gamma(m)$ and thus they are equivalent for $j = 0$ Next we show by induction that this equivalence holds for all $j \geq 0$.

Suppose now that (7.5) and (7.6) hold when $j = n-1$ for some $n \geq 1$. Then we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \gamma(m-2k)\Phi(2^{-n}k) &\overset{(7.6)}{=} \sum_{k\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \gamma(m-2k)\gamma(p)\Phi(2^{-n+1}k-p)\\ &= \sum_{p\in\mathbb{Z}} \gamma(p) \sum_{k\in\mathbb{Z}} \gamma(m-2k)\Phi(2^{-(n-1)}(k-2^{n-1}p))\\ &\overset{k-2^{n-1}p=r}{=} \sum_{p\in\mathbb{Z}} \gamma(p)\sum_{r\in\mathbb{Z}} \gamma(m-2^n p+2r)\Phi(2^{-(n-1)}r)\\ &\overset{(7.5)}{=} \sum_{p\in\mathbb{Z}} \gamma(p)\Phi(2^{-n}m-p).\end{aligned}$$

Thus we conclude that (7.5) holds for $j = n$ if and only if (7.6) holds for $j = n$ and this is what we had to prove. □

The simplest way to get a fundamental interpolation function Φ satisfying equation (7.4) is to start with an orthonomal scaling function.

Proposition 7.2. *Assume that $\varphi \in L^2(\mathbb{R})$ is such that $(\varphi(\bullet - k))_{k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$ such that*

$$\varphi(\underline{x}) = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\underline{x}-k),$$

where $\alpha \in \ell^2(\mathbb{Z};\mathbb{R})$. Then

$$\Phi(\underline{x}) \overset{\text{def}}{=} \int_{\mathbb{R}} \Phi(\underline{x}+t)\overline{\Phi(t)}\,\mathrm{d}t,$$

satisfies (7.4) and (7.3) with

$$\gamma(\underline{k}) = \sum_{j\in\mathbb{Z}} \alpha(j)\overline{\alpha(\underline{k}+j)}.$$

Proof. It is clear from the definition of orthonormality that (7.3) holds so it remains to establish (7.4). A straightforward calculation gives

$$\begin{aligned}\Phi(\underline{x}) &= \int_{\mathbb{R}} \varphi(\underline{x}+t)\overline{\varphi(t)}\,\mathrm{d}t = 4\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \alpha(j)\varphi(2\underline{x}+2t-j)\sum_{k\in\mathbb{Z}} \overline{\alpha(k)\varphi(2t-k)}\,\mathrm{d}t\\ &= 4\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \alpha(j)\overline{\alpha(k)}\int_{\mathbb{R}} \varphi(2\underline{x}+2t-j)\varphi(2t-k)\,\mathrm{d}t\\ &= 2\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \alpha(j)\overline{\alpha(k)}\Phi(2\underline{x}+k-j) = 2\sum_{p\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \alpha(k)\overline{\alpha(k+p)}\Phi(2\underline{x}-p),\end{aligned}$$

and this gives the claim.

□

2. Calculating projections for multiresolutions

Suppose we have a multiresolution $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$, we know the values of a function f at the points $2^m k$, and we want to calculate an approximation of the projection of f onto the space V_m. Since we do not know the function f exactly we cannot get the exact projection. But the idea we present here is to interpolate the function f, using a subdivision scheme, and then calculate the exact projection of the interpolated function. If Φ is the fundamental interpolation function of some scheme, that is, Φ satisfied the assumptions of Proposition 7.1, then the interpolating function will be

$$I_\Phi(f_{|2^m\mathbb{Z}})(\underline{x}) = \sum_{n\in\mathbb{Z}} f(2^m n)\Phi(2^{-m}\underline{x} - n). \tag{7.7}$$

What we want to calculate is the coefficients $c_m(\underline{k})$ in the expression $I_\Phi(f_{|2^m\mathbb{Z}})(\underline{x}) = \sum_{k\in\mathbb{Z}} c_m(k)\varphi(2^{-m}\underline{x} - k)$. We get the following result:

Proposition 7.3. *Suppose that* $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ *is an orthonormal multiresolution with filter* α *and suppose that the function* $\Phi \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ *satisfies the dilation equation* (7.4) *for some sequence* γ *and that the interpolation condition* (7.3) *holds. Then the sequence* C_m *(defined by* $C_m(\underline{k}) = 2^{\frac{=}{m}2}\int_{\mathbb{R}} I_\Phi(f_{|2^m\mathbb{Z}})(x)\overline{\varphi(2^{-m}x - \underline{k})}\,\mathrm{d}x$ *is given by*

$$C_m(\underline{k}) = 2^{\frac{m}{2}}\sum_{j\in\mathbb{Z}} f_{|2^m\mathbb{Z}}(\underline{k} - j)\rho(j) \tag{7.8}$$

where the sequence $\rho(\underline{k}) = \int_{\mathbb{R}} \Phi(x)\varphi(x - \underline{k})$ *satisfies*

$$\rho(\underline{k}) = 2\sum_{n\in\mathbb{Z}} \eta(n)\rho(2\underline{k} - n), \tag{7.9}$$

where $\eta(\underline{k}) = \sum_{j\in\mathbb{Z}} \gamma(j)\overline{\alpha(j - \underline{k})}$.

It is of course clear that one in addition to (7.9) needs a normalization condition and intuitively it is clear that if we calculate ρ from (7.9), then we should require $\sum_{k\in\mathbb{Z}} \rho(k) = 1$, but we do not here go into the details about which assumptions, if any, are needed for this to be follow from (7.8).

Proof. Clearly we have by equation (7.7) and some straightforward calculations

$$C_m(\underline{k}) = 2^{-\frac{m}{2}}\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} f_{|2^m\mathbb{Z}}(n)\Phi(2^{-m}x - n)\varphi(2^{-m}x - k)\,\mathrm{d}x$$
$$= 2^{\frac{m}{2}}\sum_{n\in\mathbb{Z}} f_{|2^m\mathbb{Z}}(n)\int_{\mathbb{R}}\Phi(x)\overline{\varphi(x - (k - n))}\,\mathrm{d}x,$$

so that we have (7.8).

Since $\rho(\underline{k}) = \int_{\mathbb{R}} \Phi(x)\varphi(x-\underline{k})$ we get from (4.9) and (7.4) that

$$\begin{aligned}
\rho(\underline{k}) = 4\int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \gamma(j)\Phi(2x-j) \sum_{n\in\mathbb{Z}} \overline{\alpha(n)\varphi(2(x-k)-n)}\,\mathrm{d}x \\
= 4\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \Phi(2x-j)\overline{\varphi(2x-2k-n)}\,\mathrm{d}x \\
\overset{2x-j=t}{=} 2\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \Phi(t)\overline{\varphi(t-(2k+n-j)}\,\mathrm{d}t \\
= \sum_{r\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \gamma(j)\overline{\alpha(j-r)} \right) \rho(2k-r),
\end{aligned}$$

and we have the desired conclusion. □