Chapter 6

The orthonormal discrete wavelet transform or calculating with wavelets

1. Introduction

In this chapter we study how on can use orthonormal wavelets in numerical computations. It is immediately clear that there are great advantages in having wavelets with compact support, so we will usually make this assumption.

2. Decomposition and reconstruction algorithms

Let $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ be an orthonormal multiresolution for $L^2(\mathbb{R};\mathbb{C})$. Now $V_m = W_{m+1} \oplus W_{m+2} \oplus \ldots \oplus W_k \oplus V_k$, and the question that we shall study here is how one in practice finds this decomposition. Fortunately it turns out to be quite easy.

Proposition 6.1. *Let $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ be a multiresolution for $L^2(\mathbb{R};\mathbb{C})$ and let P_U denote the orthogonal projection onto U. If $f \in V_m$ is given in the* *form*

$$f = \sum_{k\in\mathbb{Z}} c_m(k)\varphi(2^{-m}\bullet - k), \tag{6.1}$$

where $c_m \in \ell^2(\mathbb{Z};\mathbb{C})$, *then*

$$P_{V_{m+1}} f = \sum_{k\in\mathbb{Z}} c_{m+1}(k)\varphi(2^{-m-1}\bullet - k),$$

where

$$c_{m+1}(k) = \sum_{j\in\mathbb{Z}} \overline{\alpha(j-2k)} c_m(j),$$

and

$$P_{W_{m+1}} f = \sum_{k\in\mathbb{Z}} d_{m+1}(k)\psi(2^{-m-1}\bullet - k),$$

where

$$d_{m+1}(k) = \sum_{j\in\mathbb{Z}} (-1)^j \alpha(1-j+2k) c_m(j).$$

Conversely, if

$$f = \sum_{k\in\mathbb{Z}} c_{m+1}(k)\varphi(2^{-m-1}\bullet - k) + \sum_{k\in\mathbb{Z}} d_{m+1}(k)\psi(2^{-m-1}\bullet - k), \tag{6.2}$$

where c_{m+1} *and* $d_{m+1} \in \ell^2(\mathbb{Z};\mathbb{C})$, *then* (6.1) *holds with*

$$c_m(k) = 2\sum_{j\in\mathbb{Z}} \big(\alpha(k-2j)c_{m+1}(j) + (-1)^k \overline{\alpha(1-k+2j)} d_{m+1}(j)\big).$$

Proof. Suppose that f is given in the form (6.1). By definition we have

$$P_{V_{m+1}} f = \sum_{k\in\mathbb{Z}} 2^{m+1} \left\langle f, \varphi(2^{-m-1}\bullet - k)\right\rangle \varphi(2^{-m-1}\bullet - k). \tag{6.3}$$

From (6.1) we get

$$2^{-m-1}\left\langle f, \varphi(2^{-m-1}\bullet - k)\right\rangle = \sum_{j\in\mathbb{Z}} c_m(j) 2^{-m-1} \int_{\mathbb{R}} \varphi(2^m x - j)\overline{\varphi(2^{-m-1}x - k)}\, \mathrm{d}x$$

$$= \sum_{j\in\mathbb{Z}} c_m(j) \int_{\mathbb{R}} \varphi(2x-j)\overline{\varphi(x-k)}\, \mathrm{d}x = \sum_{j\in\mathbb{Z}} c_m(j) \overline{\int_{\mathbb{R}} \varphi(x)\overline{\varphi(2x-(j-2k))}\, \mathrm{d}x},$$

and therefore the desired claim follows from the definition of α in (4.8).

The second claim follows in exactly the same manner when we use (4.19).

If (6.2) holds, then $f \in V_m$ and (6.1) holds with $c_m(k) = 2^{-m}\left\langle f, \varphi(2^{-m}\bullet - k)\right\rangle$. If we insert the expression from (6.2) and change the variable of integration, then we get the desired conclusion. □

Note that coefficients $c_m(k)$ above are not the inner products with the elements in the orthonormal basis, in fact we have

$$c_m(k) = 2^{-\frac{m}{2}} \left\langle f, 2^{-\frac{m}{2}} \varphi(2^{-m} \bullet - k) \right\rangle, \quad k, m \in \mathbb{Z},$$

and hence for example

$$\sum_{k \in \mathbb{Z}} 2^m |c_m(k)|^2 = \|P_{V_m} f\|^2_{L^2(\mathbb{R})}.$$

We can consider the calculations above from a slightly different point of view, that is we look at the sequences only and do not care about whether they arise as coefficients in an expansion with respect to some basis.

Definition 6.2. *Let $\alpha \in \ell^2(\mathbb{Z})$. Then*

$$(T_\alpha c)(\underline{k}) = \sum_{j \in \mathbb{Z}} \overline{\alpha(j - 2\underline{k})} c(j),$$

and

$$(S_\alpha c)(\underline{k}) = 2 \sum_{j \in \mathbb{Z}} \alpha(\underline{k} - 2j) c(j).$$

Thus we see that the operator T_α consists of a convolution with the sequence $\overline{\alpha(-\underline{k})}$ followed by a downsampling or decimation, i.e., one throws out all the odd coefficients. Similarly, S_α is an upsampling, i.e., one inserts zeroes in every second position in the sequence, followed be a convolution with α and multiplication by 2.

Now we have the following result:

Proposition 6.3. *Let $\alpha \in \ell^2(\mathbb{Z})$ be such that $\hat{\alpha} \in L^\infty(\mathbb{T})$. Then the operators T_α and S_α are bounded operators on $\ell^2(\mathbb{Z})$, $T_\alpha^* = \frac{1}{2} S_\alpha$ and*

$$\widehat{T_\alpha c}(\underline{\omega}) = \tfrac{1}{2} \left(\overline{\hat{\alpha}(\tfrac{\omega}{2})} \hat{c}(\tfrac{\omega}{2}) + \overline{\hat{\alpha}(\tfrac{\omega+1}{2})} \hat{c}(\tfrac{\omega+1}{2}) \right)$$
$$\widehat{S_\alpha c}(\underline{\omega}) = 2 \hat{\alpha}(\underline{\omega}) \hat{c}(2\underline{\omega}).$$

The proof is left as an exercise.

When we look at the reconstruction of the sequences we note that it is of course a special case to have the same sequences α and β determining both in the analysis and the synthesis. However, in this case we have the following result.

Theorem 6.4. *Let α and $\beta \in \ell^2(\mathbb{Z})$ be such that $\hat{\alpha}$ and $\hat{\beta} \in L^\infty(\mathbb{T})$. Then*

$$S_\alpha T_\alpha + S_\beta T_\beta = I$$

if and only if the matrix the matrix

$$\begin{pmatrix} \hat{\alpha}(\omega) & \hat{\alpha}(\omega+\frac{1}{2}) \\ \hat{\beta}(\omega) & \hat{\beta}(\omega+\frac{1}{2}) \end{pmatrix} \quad \textit{is unitary for almost every } \omega \in \mathbb{R}.$$

Proof. By proposition 6.3 we know that the Fourier transform of the sequence $(S_\alpha T_\alpha + S_\beta T_\beta)c$ is

$$\hat{\alpha}(\underline{\omega})\left(\overline{\hat{\alpha}(\underline{\omega})}\hat{c}(\underline{\omega}) + \overline{\hat{\alpha}(\underline{\omega}+\tfrac{1}{2})}\hat{c}(\underline{\omega}+\tfrac{1}{2})\right) \\ + \hat{\beta}(\underline{\omega})\left(\overline{\hat{\beta}(\underline{\omega})}\hat{c}(\underline{\omega}) + \overline{\hat{\beta}(\underline{\omega}+\tfrac{1}{2})}\hat{c}(\underline{\omega}+\tfrac{1}{2})\right).$$

If this expression is to be (almost everywhere) equal to $\hat{c}(\underline{\omega})$ for all c then one must have

$$\hat{\alpha}(\underline{\omega})\overline{\hat{\alpha}(\underline{\omega})} + \hat{\beta}(\underline{\omega})\overline{\hat{\beta}(\underline{\omega})} \overset{\text{a.e.}}{=} 1,$$

and

$$\hat{\alpha}(\underline{\omega})\overline{\hat{\alpha}(\underline{\omega}+\tfrac{1}{2})} + \hat{\beta}(\underline{\omega})\overline{\hat{\beta}(\underline{\omega}+\tfrac{1}{2})} \overset{\text{a.e.}}{=} 0,$$

and this is easily seen to be equivalent to

$$\begin{pmatrix} \hat{\alpha}(\underline{\omega}) & \hat{\alpha}(\underline{\omega}+\frac{1}{2}) \\ \hat{\beta}(\underline{\omega}) & \hat{\beta}(\underline{\omega}+\frac{1}{2}) \end{pmatrix}\begin{pmatrix} \hat{\alpha}(\underline{\omega}) & \hat{\alpha}(\underline{\omega}+\frac{1}{2}) \\ \hat{\beta}(\underline{\omega}) & \hat{\beta}(\underline{\omega}+\frac{1}{2}) \end{pmatrix}^* \overset{\text{a.e.}}{=} I.$$

This is exactly the claim. □

3. Wavelet packets

From the results above we see that the decomposition of a function into the spaces $W_{m+1} \oplus W_{m+2} \oplus \ldots \oplus W_{m+k} \oplus V_{m+k}$ can be achieved by calculating the sequences

$$T_\beta c, T_\beta T_\alpha c, \ldots T_\beta T_\alpha^j c \ldots T_\beta T_\alpha^{k-1} c, T_\alpha^k c$$

Furthermore, we see that if $f \in V_m$ and the sequence c_m is defined by (6.1), then we have

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{j=1}^{k} 2^{m+j}\|T_\beta T_\alpha^{j-1} c_m\|^2_{\ell^2(\mathbb{Z})} + 2^{m+k}\|T_\alpha^k c_m\|^2_{\ell^2(\mathbb{Z})}.$$

Now there is no reason to consider only these combinations of T_α and T_β, we can split the spaces W_j as well, using the same filters α and β, (or in principle some other ones as well). This is the idea behind wavelet packets. Now one clearly does not necessarily have to split all sequences the same number of times, but on needs some criterion for deciding when to do and when not to do it. For this we need some kind of cost function. One possibility is to take a cost function of "entropy" type, i.e., take

$$\mathcal{K}(c) = -\sum_{j\in\mathbb{Z}} |c(j)|^2 \log(|c(j)|^2).$$

But one has to remember to use normalized sequences. If one wants to find the splitting that minimizes $\mathcal{K}$, given that one can do at most k splittings, one possibility is to proceed as follows:

(1) Form a set S consisting of the 2^k sequences $T_{\gamma_k} T_{\gamma_{k-1}} \dots T_{\gamma_1} c$ where γ_i is either α or β, (together with information about how these sequences have been formed).

(2) Let $\mathcal{K}^*(s) = \mathcal{K}(2^{\frac{k}{2}} s)$ for each $s \in S$.

(3) If $\mathcal{K}^*(T_\alpha T_{\gamma_j} \dots T_{\gamma_1} c)$ and $\mathcal{K}^*(T_\beta T_{\gamma_j} \dots T_{\gamma_1} c)$ have been caculated, then $\mathcal{K}^*(T_{\gamma_j} \dots T_{\gamma_1} c) = \min\{\mathcal{K}^*(T_\alpha T_{\gamma_j} \dots T_{\gamma_1} c) + \mathcal{K}^*(T_\beta T_{\gamma_j} \dots T_{\gamma_1} c),$ $\mathcal{K}((2^{\frac{j}{2}} T_{\gamma_j} \dots T_{\gamma_1} c)\}$ and if $\mathcal{K}(2^{\frac{j}{2}} T_{\gamma_j} \dots T_{\gamma_1} c)$ gives the minimimum, then all sequences of the form $T_{\eta_1} \dots T_{\eta_p} T_{\gamma_j} \dots T_{\gamma_1} c$ where $\eta_i = \alpha$ or β and $1 \le p \le k - j$ are remove from S and $T_{\gamma_j} \dots T_{\gamma_1} c$ is added to S.

(4) Step (3) is repeated until $\mathcal{K}^*(c)$ has been calculated.