Chapter 8

The continuous wavelet transform and its discretization

1. The continuous transform

Suppose that $\psi \in L^2(\mathbb{R})$. We define the family of functions $\psi^{a,b}$ by

$$\psi^{a,b}(\underline{x}) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{\underline{x}-b}{a}\right), \quad a \neq 0, \quad b \in \mathbb{R}. \tag{8.1}$$

Often one only considers the case where $a > 0$ and the normalization $\frac{1}{\sqrt{|a|}}$ is used so that $\|\psi^{a,b}\|_{L^2(\mathbb{R})} = \|\psi\|_{L^2(\mathbb{R})}$. Other normalizations can, of course, be used as well. Then we can define

$$(W_\psi f)(a,b) = \left\langle f, \psi^{a,b}\right\rangle = \int_{\mathbb{R}} f(x)\overline{\psi^{a,b}(x)}\,\mathrm{d}x. \tag{8.2}$$

It is clear that $|W_\psi f(a,b)| \leq \|\psi\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}$.

We have the following result:

Theorem 8.1. *Assume $\psi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that*

$$C_\psi \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,\mathrm{d}\omega < \infty.$$

Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_\psi f(a,b)\overline{W_\psi g(a,b)}\frac{1}{a^2}\,\mathrm{d}a\,\mathrm{d}b = C_\psi \langle f, g\rangle,$$

for all f and $g \in L^2(\mathbb{R})$.

This theorem says that in a weak sense, we have (provided, of course, that $\psi \neq 0$)

$$f(\underline{x}) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} W_\psi f(a,b) \psi^{a,b}(\underline{x}) \frac{1}{a^2} \, \mathrm{d}a \, \mathrm{d}b.$$

Proof. First we note that

$$\widehat{\psi^{a,b}}(\underline{\omega}) = \sqrt{|a|} \mathrm{e}^{-\mathrm{i}2\pi b \underline{\omega}} \hat{\psi}(a\underline{\omega}),$$

so that by Plancherel's theorem (2.4) we have

$$W_\psi f(a,b) = \sqrt{|a|} \int_{\mathbb{R}} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}2\pi\omega b} \overline{\hat{\psi}(a\omega)} \, \mathrm{d}\omega = \sqrt{|a|} \widehat{\hat{f}(\bullet)\overline{\hat{\psi}(a\bullet)}}(-b).$$

Thus we get, again using Plancherel's theorem,

$$
\begin{aligned}
(8.3) \quad & \int_{\mathbb{R}} \int_{\mathbb{R}} W_\psi f(a,b) \overline{W_\psi g(a,b)} \frac{1}{a^2} \, \mathrm{d}a \, \mathrm{d}b \\
&= |a| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \widehat{\hat{f}(\bullet)\overline{\hat{\psi}(a\bullet)}}(-b) \overline{\widehat{\hat{g}(\bullet)\overline{\hat{\psi}(a\bullet)}}(-b)} \, \mathrm{d}b \right) \frac{1}{a^2} \, \mathrm{d}a \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \hat{f}(\omega) \overline{\hat{g}(\omega)} |\hat{\psi}(a\omega)|^2 \, \mathrm{d}\omega \right) \frac{1}{|a|} \, \mathrm{d}a \\
&= \int_{\mathbb{R}} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left(\int_{\mathbb{R}} |\hat{\psi}(a\omega|^2 \frac{1}{|a|} \, \mathrm{d}a \right) \mathrm{d}\omega.
\end{aligned}
$$

A simple change of variables now shows that

$$\int_{\mathbb{R}} |\hat{\psi}(a\omega|^2 \frac{1}{|a|} \, \mathrm{d}a = C_\psi,$$

and then the claim follows from equation (8.3). ☐

If one does not want to use negative as well as positive values for the dilation a then one gets almost the same result, provided

$$(8.4) \qquad \int_{-\infty}^{0} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega = \int_{0}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega \stackrel{\mathrm{def}}{=} c_\psi.$$

This is of course the case when ψ is real-valued.

Corollary 8.2. *Assume $\psi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that (8.4) holds with $c_\psi < \infty$. Then*

$$\int_{\mathbb{R}} \int_{0}^{\infty} W_\psi f(a,b) \overline{W_\psi g(a,b)} \frac{1}{a^2} \, \mathrm{d}a \, \mathrm{d}b = c_\psi \langle f, g \rangle,$$

for all f and $g \in L^2(\mathbb{R})$.

This result says that in a weak sense we have

$$f = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{0}^{\infty} W_\psi f(a,b) \psi^{a,b}(\underline{x}) \frac{1}{a^2} \, \mathrm{d}a \, \mathrm{d}b.$$

2. Frames of wavelets

The continuous wavelet transform is not necessarily a very practical tool since it is not clear to what extent the integrals can actually be computed. If one uses discretizations, one question to ask is when the inner products $\langle f, \psi_{m,n}\rangle$, where where $\psi_{m,n}(\underline{x}) = a_*^{-\frac{m}{2}}\psi(a_*^{-m}\underline{x} - nb_*)$, really characterize the function f. If the sequence $(\psi_{m,n})_{m,n\in\mathbb{Z}}$ is an orthonormal basis, there are no problems, but if ψ is some quite general function, there is no reason to expect that to be the case. But it turns out to be possible to give relatively simple conditions for this sequence to be a frame, see Definition 3.6. We have the following result:

Theorem 8.3. *Assume a_* and $b_* > 0$ and that $\psi \in L^2(\mathbb{R})$ is such that*

$$0 < \operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)|^2 \le \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)|^2 < \infty, \tag{8.5}$$

and

$$\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \sqrt{\beta\left(\frac{k}{b_*}\right)\beta\left(-\frac{k}{b_*}\right)} < \operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)|^2, \tag{8.6}$$

where

$$\beta(\underline{s}) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)||\hat{\psi}(a_*^m\omega + \underline{s})|.$$

Then the sequence $(\psi_{m,n})_{m,n\in\mathbb{Z}}$ where $\psi_{m,n}(\underline{x}) = a_*^{-\frac{m}{2}}\psi(a_*^{-m}\underline{x} - nb_*)$ is a frame in $L^2(\mathbb{R})$ with frame bounds

$$A = \frac{1}{b_*}\left(\operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)|^2 - \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \sqrt{\beta\left(\frac{k}{b_*}\right)\beta\left(-\frac{k}{b_*}\right)}\right),$$

$$B = \frac{1}{b_*}\left(\operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)|^2 + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \sqrt{\beta\left(\frac{k}{b_*}\right)\beta\left(-\frac{k}{b_*}\right)}\right).$$

Proof. A simple calculation shows that

$$\widehat{\psi_{m,n}}(\underline{\omega}) = a_*^{\frac{m}{2}}\hat{\psi}(a_*^m\underline{\omega})\mathrm{e}^{-\mathrm{i}2\pi a_*^m b_* n\underline{\omega}}.$$

Thus we get for $f \in L^2(\mathbb{R})$, by first using Plancherel's theorem for functions in $L^2(\mathbb{R})$, then writing the intgeral over $\mathbb{R}$ as a sum of integrals, then using

Plancherel's theorem for periodic functions, then expanding the product and again writing the sum of integrals as one integral,

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} |\langle f, \psi_{m,n}\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} |\left\langle \hat f, \widehat{\psi_{m,n}} \right\rangle|^2 \\
&= \sum_{m,n\in\mathbb{Z}} \left| \int_{\mathbb{R}} \hat f(\omega) a_*^{\frac{m}{2}} \overline{\hat\psi(a_*^m \underline{\omega})} e^{i2\pi a_*^m b_* n \underline{\omega}} \, d\omega \right|^2 \\
&= \sum_{m\in\mathbb{Z}} a_*^m \sum_{n\in\mathbb{Z}} \left| \int_0^{\frac{1}{a_*^m b_*}} e^{i2\pi a_*^m b_* n \underline{\omega}} \sum_{j\in\mathbb{Z}} \hat f\big(\omega + \tfrac{j}{a_*^m b_*}\big) \overline{\hat\psi\big(a_*^m \omega + \tfrac{j}{b_*}\big)} \, d\omega \right|^2
\end{aligned}$$

(by Parseval's equality applied to the $\dfrac{1}{a_*^m b_*}$ periodic function

$$\sum_{j\in\mathbb{Z}} \hat f\big(\underline{\omega} + \tfrac{j}{a_*^m b_*}\big) \overline{\hat\psi\big(a_*^m \underline{\omega} + \tfrac{j}{b_*}\big)})$$

$$\begin{aligned}
&= \frac{1}{b_*} \sum_{m\in\mathbb{Z}} \int_0^{\frac{1}{a_*^m b_*}} \left| \sum_{j\in\mathbb{Z}} \hat f\big(\omega + \tfrac{j}{a_*^m b_*}\big) \overline{\hat\psi\big(a_*^m \omega + \tfrac{j}{b_*}\big)} \right|^2 d\omega \\
&= \frac{1}{b_*} \sum_{m\in\mathbb{Z}} \int_0^{\frac{1}{a_*^m b_*}} \sum_{j\in\mathbb{Z}} \hat f\big(\omega + \tfrac{j}{a_*^m b_*}\big) \overline{\hat\psi\big(a_*^m \omega + \tfrac{j}{b_*}\big)} \\
&\qquad \times \sum_{k\in\mathbb{Z}} \overline{\hat f\big(\omega + \tfrac{j+k}{a_*^m b_*}\big)} \hat\psi\big(a_*^m \omega + \tfrac{j+k}{b_*}\big) \, d\omega \\
&= \frac{1}{b_*} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat f(\omega) \overline{\hat f\big(\omega + \tfrac{k}{a_*^m b_*}\big) \hat\psi(a_*^m \omega)} \hat\psi\big(a_*^m \omega + \tfrac{k}{b_*}\big) \, d\omega \\
&= \frac{1}{b_*} \int_{\mathbb{R}} |\hat f(\omega)|^2 \left(\sum_{m\in\mathbb{Z}} |\hat\psi(a_*^m \omega)|^2 \right) d\omega \\
&\qquad + \frac{1}{b_*} \sum_{m\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \int_{\mathbb{R}} \hat f(\omega) \overline{\hat f\big(\omega + \tfrac{k}{a_*^m b_*}\big) \hat\psi(a_*^m \omega)} \hat\psi\big(a_*^m \omega + \tfrac{k}{b_*}\big) \, d\omega
\end{aligned}$$

We have to get some estimates for the second term, and we get by using the Cauchy-Schwarz inequality, a change of variables, and then the Cauchy-Schwarz inequality in the sum over m:

$$\left| \frac{1}{b_*} \sum_{m\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \int_{\mathbb{R}} \hat f(\omega) \overline{\hat f\big(\omega + \tfrac{k}{a_*^m b_*}\big) \hat\psi(a_*^m \omega)} \hat\psi\big(a_*^m \omega + \tfrac{k}{b_*}\big) \, d\omega \right|$$

$$
\begin{aligned}
&\le \frac{1}{b_*}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{m\in\mathbb{Z}}\left(\int_{\mathbb{R}}|\hat f(\omega)|^2|\hat\psi(a_*^m\omega)||\hat\psi(a_*^m\omega+\tfrac{k}{b_*})|\,\mathrm{d}\omega\right)^{\frac12}\\
&\qquad\qquad \times\left(\int_{\mathbb{R}}|\hat f(\omega+\tfrac{k}{a_*^m b_*})|^2|\hat\psi(a_*^m\omega)||\hat\psi(a_*^m\omega+\tfrac{k}{b_*})|\,\mathrm{d}\omega\right)^{\frac12}\\
&= \frac{1}{b_*}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{m\in\mathbb{Z}}\left(\int_{\mathbb{R}}|\hat f(\omega)|^2|\hat\psi(a_*^m\omega)||\hat\psi(a_*^m\omega+\tfrac{k}{b_*})|\,\mathrm{d}\omega\right)^{\frac12}\\
&\times\left(\int_{\mathbb{R}}|\hat f(\eta)|^2|\hat\psi(a_*^m\eta-\tfrac{k}{b_*})||\hat\psi(a_*^m\eta)|\,\mathrm{d}\eta\right)^{\frac12}\\
&\le \frac{1}{b_*}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\left(\int_{\mathbb{R}}|\hat f(\omega)|^2\sum_{m\in\mathbb{Z}}|\hat\psi(a_*^m\omega)||\hat\psi(a_*^m\omega+\tfrac{k}{b_*})|\right)^{\frac12}\\
&\times\left(\int_{\mathbb{R}}|\hat f(\eta)|^2\sum_{m\in\mathbb{Z}}|\hat\psi(a_*^m\eta)||\hat\psi(a_*^m\eta-\tfrac{k}{b_*})|\right)^{\frac12}\\
&\le \frac{1}{b_*}\|f\|^2_{L^2(\mathbb{R})}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sqrt{\beta\left(\frac{k}{b_*}\right)\beta\left(-\frac{k}{b_*}\right)}.
\end{aligned}
$$

When we combine the results we have deduced above, we get the desired conclusion. □

It is clear that for (8.5) to hold we must have $a_*\neq 1$, but it turns out that one can quite easily get sufficient conditions for the assumptions of Theorem 8.3 to hold.

Corollary 8.4. *Assume that $\psi\in L^2(\mathbb{R};\mathbb{R})$ is real-valued, not the zero function, $\hat\psi$ is continuous and satisfies*

$$
\sup_{\omega\in\mathbb{R}\setminus\{0\}}\frac{|\hat\psi(\omega)|(1+|\omega|^\gamma)}{|\omega|^\alpha}<\infty,
$$

for some constants $\gamma>\alpha+1>1$. Then the assumptions of Theorem 8.3 hold, provided a_* has been chosen sufficiently close to 1 and then $b_*>0$ has been chosen to be sufficiently small.

Proof. Since ψ is real-valued we have $\hat\psi(-\omega)=\overline{\hat\psi(\omega)}$ and since $\hat\psi$ is continuous and not identically zero there are positive numbers ω_0, δ and τ so that

$|\hat{\psi}(\omega)| \geq \delta$ when $\omega_0 \leq |\omega| \leq \omega_0 + \tau$. It follows that if $0 < |a_* - 1| < \frac{\tau}{\omega_0}$ then

$$\operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_*^m\omega)|^2 \geq \delta^2 > 0,$$

because if for example $1 < a_* < 1 + \frac{\tau}{\omega_0}$ and $a_*^m|\omega| < \omega_0$ then $a_*^{m+1}|\omega| \leq a_*\omega_0 < \omega_0 + \tau$ and therefore there is for every $\omega \neq 0$ an index m_* such that $\omega_0 \leq |a_*^{m_*}\omega| \leq \omega_0 + \tau$ and we have the first inequality in (8.5).

Next we derive some useful inequalities. Clearly

$$\frac{|x|^\alpha}{1+|x|^\gamma}\frac{|x+s|^\alpha}{1+|x+s|^\gamma} \leq \frac{|x|^\alpha}{|x+s|^{\gamma-\alpha}},$$

and if $|s| > 2|x|$ then $|x+s| \geq |s| - |x| \geq |s| - \frac{1}{2}|s| = |\frac{s}{2}|$ so that we get

$$\frac{|x|^\alpha}{1+|x|^\gamma}\frac{|x+s|^\alpha}{1+|x+s|^\gamma} \leq \frac{|x|^\alpha}{|\frac{s}{2}|^{\gamma-\alpha}}, \quad |s| \geq 2|x|. \tag{8.7}$$

In the same way we get the following crude estimate

$$\begin{aligned}\frac{|x|^\alpha}{1+|x|^\gamma}\frac{|x+s|^\alpha}{1+|x+s|^\gamma} &\leq \frac{1}{|x|^{\gamma-\alpha}}\frac{1}{|x+s|^{\gamma-\alpha}} \leq \frac{1}{|x|^{\gamma-\alpha}|\frac{s}{2}|^{\gamma-\alpha}} \\ &= \frac{1}{|x|^{\frac{\gamma-\alpha-1}{2}}|\frac{s}{2}|^{\frac{\gamma-\alpha+1}{2}}}\cdot\frac{1}{|x|^{\frac{\gamma-\alpha+1}{2}}|\frac{s}{2}|^{\frac{\gamma-\alpha-1}{2}}} \leq \frac{1}{|x|^{\frac{\gamma-\alpha-1}{2}}|\frac{s}{2}|^{\frac{\gamma-\alpha+1}{2}}}, \\ &\qquad |s| \geq 2|x| \geq 2.\end{aligned} \tag{8.8}$$

because $\gamma - \alpha - 1 > 0$. On the other hand we have for the same reason

$$\begin{aligned}\frac{|x|^\alpha}{1+|x|^\gamma}\frac{|x+s|^\alpha}{1+|x+s|^\gamma} &\leq \frac{1}{|x|^{\gamma-\alpha}} = \frac{1}{|x|^{\frac{\gamma-\alpha-1}{2}}|x|^{\frac{\gamma-\alpha+1}{2}}} \\ &\leq \frac{1}{|x|^{\frac{\gamma-\alpha-1}{2}}|\frac{s}{2}|^{\frac{\gamma-\alpha+1}{2}}}, \quad |s| \leq 2|x|.\end{aligned} \tag{8.9}$$

If $a_* > 0$ and $a_* \neq 1$ and we may without loss of generality assume that $a_* > 1$ because in the sums involving a_* we may replace m by $-m$ which is the same as replacing a_* by $\frac{1}{a_*}$.

It follows from the assumptions that there is a constant C such that

$$|\hat{\psi}(\omega)| \leq C\frac{|\omega|^\alpha}{1+|\omega|^\gamma}, \quad \omega \in \mathbb{R}. \tag{8.10}$$

Since $\hat{\psi}(0) = 0$ we have $\sum_{m\in\mathbb{Z}}|\hat{\psi}(a_*^m\omega)|^2 = 0$ if $\omega = 0$. Let $\omega \neq 0$ and let m_0 be such that $a_*^{m_0}|\omega| \leq 1$ but $a_*^{m_0+1}|\omega| > 1$, i.e., $a_*^{m_1}|\omega| > 1$ where

$m_1 = m_0 + 1$. Then

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \left|\hat{\psi}(a_*^m\omega)\right|^2 &\le C^2 \sum_{m=-\infty}^{m_0} \frac{|a_*^m\omega|^{2\alpha}}{1+|a_*^m\omega|^{2\gamma}} + C^2 \sum_{m=m_1}^{\infty} \frac{|a_*^m\omega|^{2\alpha}}{1+|a_*^m\omega|^{2\gamma}} \\ &\le C^2 \sum_{m=-\infty}^{m_0} |a_*^m\omega|^{2\alpha} + C^2 \sum_{m=m_1}^{\infty} |a_*^m\omega|^{2(\alpha-\gamma)} \\ &= C^2 \frac{|a_*^{m_0}\omega|^{2\alpha}}{1-(\frac{1}{a_*})^{2\alpha}} + C^2 \frac{|a_*^{m_1}\omega|^{2(\alpha-\gamma)}}{1-a_*{}^{2(\alpha-\gamma)}} \le C^2 \frac{1}{1-a_*^{-2\alpha}} + C^2 \frac{1}{1-a_*^{2(\alpha-\gamma)}}.\end{aligned}$$

by the formula for the sum of a geometric series and because $|a_*^{m_0}\omega| \le 1$, $|a_*^{m_1}\omega| \ge 1$, $2\alpha > 0$ ja $2(\alpha-\gamma) < 0$. This gives the second inequality in (8.5).

If now $|s| \ge 2$ then we have by (8.7)–(8.10) and the fact that $a_*^{m_0}|\omega| \le 1$ and $a_*^{m_1}|\omega| > 1$ that

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} \left|\hat{\psi}(a_*^m\omega)\right| \left|\hat{\psi}(a_*^m\omega + \underline{s})\right| \\ &\le C^2 \sum_{m=-\infty}^{m_0} |a_*^m\omega|^{\alpha} \left|\tfrac{s}{2}\right|^{-\gamma+\alpha} + C^2 \sum_{m=m_1}^{\infty} |a_*^m\omega|^{-\frac{\gamma-\alpha-1}{2}} \left|\tfrac{s}{2}\right|^{-\frac{\gamma-\alpha+1}{2}} \\ &= C^2 \frac{|a_*^{m_0}\omega|^{\alpha}}{1-a_*^{-\alpha}} \left|\tfrac{s}{2}\right|^{-\gamma+\alpha} + C^2 \frac{|a_*^{m_1}\omega|^{-\frac{\gamma-\alpha-1}{2}}}{1-a_*^{-\frac{\gamma-\alpha-1}{2}}} \left|\tfrac{s}{2}\right|^{-\frac{\gamma-\alpha+1}{2}}.\end{aligned}$$

From this we see that there is a constant $C_1 = C^2 2^{\frac{\gamma-\alpha+1}{2}} \left(\frac{1}{1-a_*^{-\alpha}} + \frac{1}{1-a_*^{-\frac{\gamma-\alpha-1}{2}}}\right)$ such that

$$\beta(s) \le C_1 |s|^{-\frac{\gamma-\alpha+1}{2}}, \quad |s| \ge 2.$$

Then

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \sqrt{\beta\left(\frac{k}{b_*}\right)\beta\left(-\frac{k}{b_*}\right)} \le 2C_1 b_*^{\frac{\gamma-\alpha+1}{2}} \sum_{k=1}^{\infty} k^{-\frac{\gamma-\alpha+1}{2}},$$

when $0 < b_* \le \frac{1}{2}$. Thus we see that (8.6) holds provided

$$2C_1 b_*^{\frac{\gamma-\alpha+1}{2}} \sum_{k=1}^{\infty} k^{-\frac{\gamma-\alpha+1}{2}} < \delta^2,$$

which is possible if b_* is sufficiently small. □